

# Beam based measurements of relative RF phase

Project report

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## Abstract

A beam-based relative phase alignment technique is proposed for multi RF systems. Algorithms for the measurements of the synchronous phase and bunch length based on the beam profile's instantaneous phase are developed. Synchronous phase behaviour in multi RF systems is classified in the general case; and the exact solution in the special case of the harmonic number  $n = 1, 2$  is presented.

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# 1 Introduction

The purpose of this report is to propose a method for a beam based alignment of a double harmonic RF system. The equation

$$\sin(\phi_{s0}) = \sin(\phi_s) + \alpha \sin(n\phi_s + \Phi_2) \quad (1)$$

connects its synchronous phase with the synchronous phase of a single RF system. Here,  $\alpha = \frac{V_2}{V_1}$  is the voltage ratio of the cavities,  $n$  is the harmonic number,  $\Phi_2$  is the phase difference between the cavities,  $\phi_s$  is the synchronous phase of the double harmonic system, and  $\phi_{s0}$  is the synchronous phase of the corresponding single harmonic system.

The experiment described here measures  $\phi_s$  and is set by choosing:

1. A certain time in the ramp corresponding to the synchrotron frequency  $\omega_{RF}$ . We assume that  $\phi_{s0}$  is not measured.
2. The harmonic number  $n \in \mathbb{N}$ .
3. The set of values  $\alpha_i > 0$ . We assume the voltage setting in both cavities is controlled with a great precision, that is  $\alpha_i$  are known.
4. A number of samples of  $\Phi_2$  that span the entire range  $[0, 2\pi)$ .

Since the purpose of this experiment is to calculate the phase misalignment, we introduce it as:

$$\Phi_2 = \tilde{\Phi}_2 + C(\omega_{RF}). \quad (2)$$

The quantity  $\Phi_2$  is the actual phase difference that was programmed instead of  $\tilde{\Phi}_2$  due to the misalignment  $C(\omega_{RF})$ . For simplicity we will write  $C(\omega_{RF}) \equiv C$  remembering that its value varies with  $\omega_{RF}$ . In the next section we propose the method for the phase alignment. In section 3 we discuss the results of the proposed phase alignment method at CERN's PSB. In section 4 we propose algorithms for the measurements of the synchronous phase and the bunch length that were developed and implemented during this project. The appendix is reserved for mathematical derivations and some general results of eq. 1.

## 2 Phase alignment method description

First of all, we assume a small value of  $\alpha < 1$  in order not to destroy the beam and keep  $\phi_s$  single-valued. The exact constraints on  $\alpha$  are derived in appendix B.3. For our considerations an explicit relation  $\phi_s(\Phi_2, n, \phi_{s0}, \alpha)$  would be mostly desirable however it is not available, what is discussed in appendix A.

The method relies on the link between  $\Phi_2^{\max/\min}$  and  $\Phi_2^{\alpha\pm}$  defined as:

1.  $\Phi_2^{\max/\min}$  are the 2 solutions of  $\frac{\partial \phi_s}{\partial \Phi_2} = 0$ .
2.  $\Phi_2^{\alpha\pm}$  are the 2 solutions of  $\frac{\partial \phi_s}{\partial \alpha} = 0$ .

In appendix B.2 we show that they satisfy

$$-\sin(\phi_{s0}) = \sin\left(\frac{\Phi_2^{\max} + \frac{\pi}{2}}{n}\right) + \alpha \quad (3)$$

$$-\sin(\phi_{s0}) = \sin\left(\frac{\Phi_2^{\min} - \frac{\pi}{2}}{n}\right) - \alpha \quad (4)$$

$$-\sin(\phi_{s0}) = \sin\left(\frac{\Phi_2^{\alpha+}}{n}\right) \quad (5)$$

$$-\sin(\phi_{s0}) = \sin\left(\frac{\Phi_2^{\alpha-} - \pi}{n}\right) \quad (6)$$

We now combine (for example) the equations 3 and 5 to get

$$\sin\left(\frac{\Phi_2^{\max} + \frac{\pi}{2}}{n}\right) + \alpha = \sin\left(\frac{\Phi_2^{\alpha+}}{n}\right) \quad (7)$$

Thus we managed to eliminate  $\phi_{s0}$  and  $\phi_s$  from the equation. Substituting 2, we finally get

$$\sin\left(\frac{\tilde{\Phi}_2^{\max} + C + \frac{\pi}{2}}{n}\right) + \alpha = \sin\left(\frac{\tilde{\Phi}_2^{\alpha+} + C}{n}\right) \quad (8)$$

Finding a value for  $C$  that satisfies this relation solves the phase alignment problem. In the next section we discuss how to measure  $\tilde{\Phi}_2^{\alpha+}$  and  $\tilde{\Phi}_2^{\max}$  giving the CERN's PSB as an example.

### 3 Alignment of Ferrite and FINEMET<sup>TM</sup> cavities at PSB, ring 4.

An experiment was set by choosing a time during the ramp  $t = 400\mu s$ , a harmonic number  $n = 1$ , and range of values of  $\alpha_i \in (0.25, 0.4375, 0.625)$  with  $V_1 = 8kV$ . Then the synchronous phase  $\phi_s$  was measured for a few samples  $\Phi_2 \in [0, 2\pi)$  as shown in Figure 1.

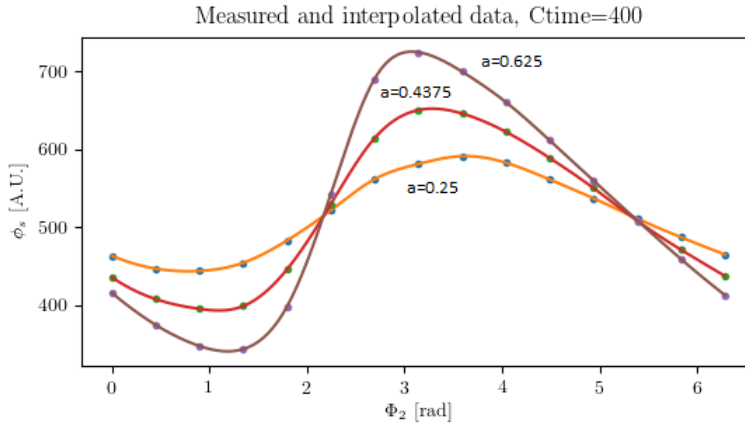


Figure 1: Measurement of the synchronous phase

By interpolation the phase difference  $\tilde{\Phi}_2^{\max}(\alpha_i)$  where the synchronous phase has its maximum for the chosen  $\alpha_i$  was found. (Figure 2)

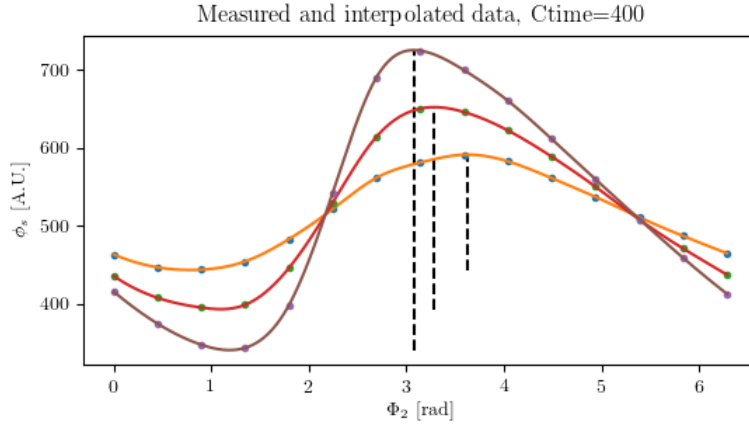


Figure 2: Finding the maxima.

Next, as shown in Figure 3, the point  $\tilde{\Phi}_2^{\alpha+}$ , which is the same for all  $\alpha_i$  was found by minimising the sum of squared differences of  $\phi_s(\alpha_i, \tilde{\Phi}_2)$ , i.e.

$$\tilde{\Phi}_2^{\alpha\pm} = \operatorname{argmin} \sum_{i>j} \left( \tilde{\phi}_s(\alpha_i, \tilde{\Phi}_2) - \tilde{\phi}_s(\alpha_j, \tilde{\Phi}_2) \right)^2 \quad (9)$$

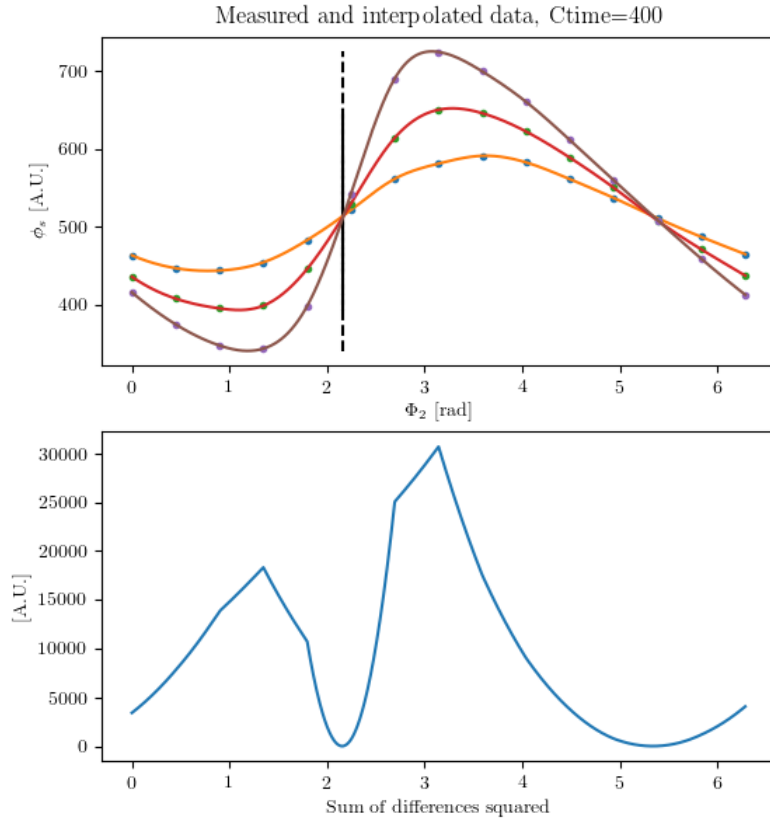


Figure 3: Finding the point  $\Phi_2^{\alpha+}$ . To insure that the point  $\Phi_2^{\alpha+}$  is selected instead of the second solution,  $\Phi_2^{\alpha-}$ , it is sufficient to look for the minima of the bottom plot "to the left" of  $\Phi_2^{\max}$

Since both  $\tilde{\Phi}_2^\alpha$  and  $\tilde{\Phi}_2^{\max}(\alpha_i)$  were known, the constant  $C$  for the system had to be evaluated. Defining the function

$$f(\alpha_i, \lambda) = \sin\left(\frac{\tilde{\Phi}_2^{\max}(\alpha_i) + \lambda - \frac{\pi}{2}}{n}\right) - \sin\left(\frac{\tilde{\Phi}_2^\alpha + \lambda}{n}\right) + \alpha_i \quad (10)$$

it is clear that  $\forall \alpha_i f(\alpha_i, C) = 0$ . Therefore (Figure 4)

$$C = \operatorname{argmin}_{i>j} \sum (f(\alpha_i, \lambda) - f(\alpha_j, \lambda))^2 \quad (11)$$

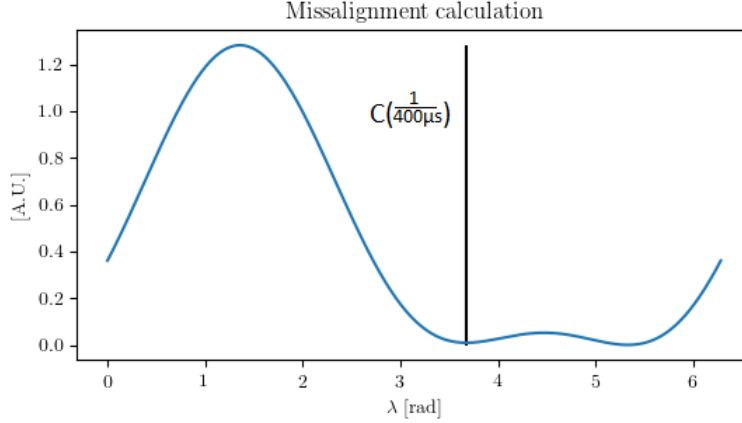


Figure 4: Finding the misalignment. It is necessary to consistently choose the same minimum throughout different measurements, in this case the minimum "to the right" of the peak.

This measurement was repeated twice per a time in the ramp for a total of 8 different times  $t \in [400, 450, \dots, 750]\mu s$  resulting in the Figure 5.

## 4 Instantaneous phase applications for synchronous phase and bunch length measurements

In order to complete this project, algorithms for measuring the synchronous phase and beam length from the longitudinal beam profile,  $u$ , were developed. Here we present them. A brief introduction to the Hilbert transform,  $\mathcal{H}(u)$  or  $\hat{u}$ , and the instantaneous phase,  $\theta$ , is given in appendices C.2.1 and C.3.

In appendix C.3 we also show that for any signal with mean 0, solving

$$\theta = \pm \frac{\pi}{2} \quad (12)$$

gives the zeros of the signal and that if the signal is also (nearly) symmetric,

$$\theta = 0 \quad (13)$$

gives an extremum of the signal. In case of the beam profile, the latter is directly linked to  $\phi_s$ , as shown in figure 7a. However, in order to measure  $\lambda$  we need to find the 'endpoints'

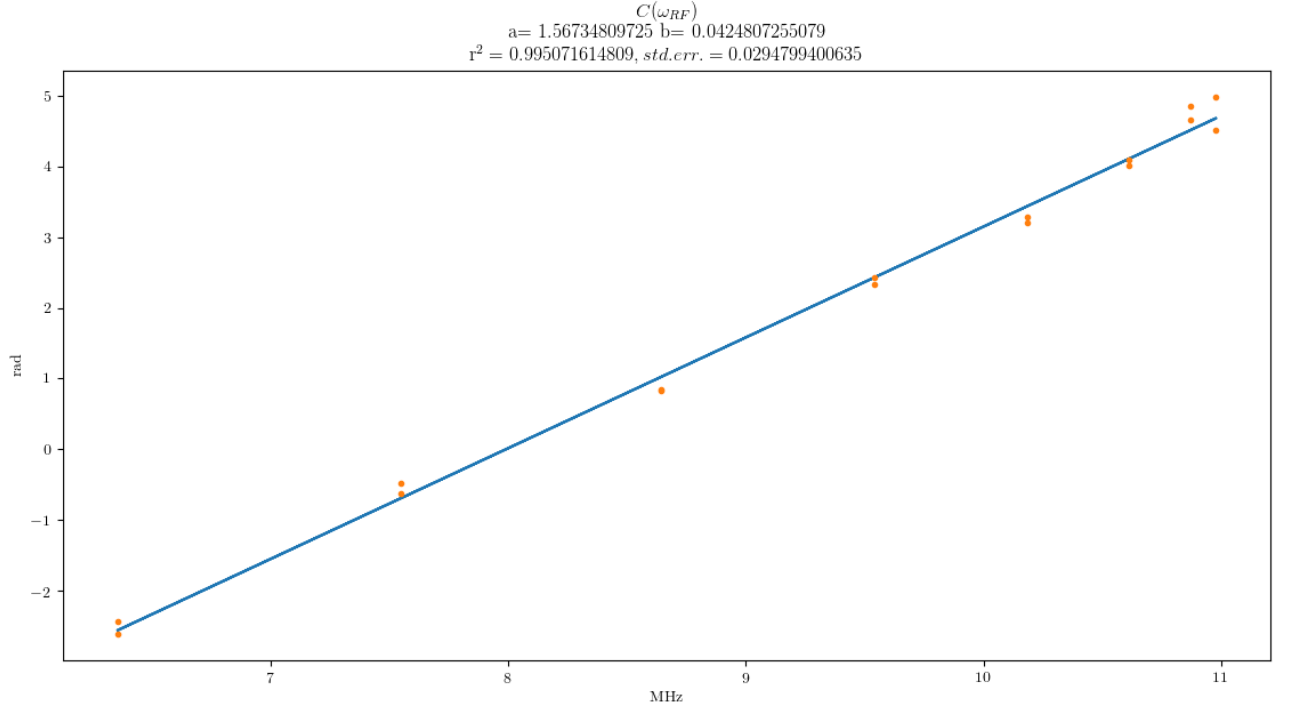


Figure 5: Combining multiple measures of  $C$  at different times in the ramp.

of the profile and not the points, where the signal is equal to its mean. Therefore, we introduce an offset  $u_0$ , so that

$$\tilde{u} = u + u_0 \quad (14)$$

The offset is chosen so that the profile  $\tilde{u}$  never crosses the horizontal axis. This altered version of the signal preserves the synchronous phase measurement, i.e. the solution to

$$\tilde{\theta} = 0 \quad (15)$$

is the same as  $\theta = 0$ . The bunch length can then be calculated from

$$\lambda = \operatorname{argmax}(\tilde{\theta}) - \operatorname{argmin}(\tilde{\theta}) \quad (16)$$

While the synchronous phase is invariant under the choice of  $u_0$ , the measurement of the bunch length will vary with it. In this experiment, we chose  $u_0 = -\min(u) + 0.1 \max(u)$ . In figures 7a, 7b and 7c we show the results for different types of signals.

The advantages of using this algorithm include:

- It is robust and noise-fluctuations resistant
- It is computationally cheap and easy to implement.
- The profile does not need to be filtered.

On the downside however:

- If the profile is largely asymmetric, the synchronous phase algorithm is inaccurate.

## Summary

We managed to propose a valid method for the relative phase alignment of the multi RF systems along with the algorithms that give access to the misalignment  $C(\omega_{RF})$  directly from the raw data. Further discussion is provided in a lengthy appendix.

## Acknowledgements

I would like to offer my sincere gratitude to my supervisors Simon Albright and Alexandre Lasheen and to my colleagues who supported me throughout this project.

## A The explicit expression for $\phi_s$

We can substitute  $z = e^{i\phi_{s0}}$  into 1 to get the following:

$$\sin(\phi_{s0}) = \frac{e^{i\phi_s} - e^{-i\phi_s}}{2i} + \alpha \left( \frac{e^{in\phi_s} - e^{-in\phi_s}}{2i} \cos(\Phi_2) + \frac{e^{in\phi_s} + e^{-in\phi_s}}{2} \sin(\Phi_2) \right) \quad (17)$$

$$2i \sin(\phi_{s0}) = \frac{e^{2i\phi_s} - 1}{e^{i\phi_s}} + \alpha \left( \frac{e^{2in\phi_s} - 1}{e^{in\phi_s}} \cos(\Phi_2) + \frac{e^{2in\phi_s} + 1}{e^{in\phi_s}} i \sin(\Phi_2) \right) \quad (18)$$

$$2ie^{in\phi_s} \sin(\phi_{s0}) = e^{i(n-1)\phi_s} (e^{2i\phi_s} - 1) + \alpha \left( (e^{2in\phi_s} - 1) \cos(\Phi_2) + (e^{2in\phi_s} + 1) i \sin(\Phi_2) \right) \quad (19)$$

$$2iz^n \sin(\phi_{s0}) = z^{n-1} (z^2 - 1) + \alpha \left( (z^{2n} - 1) \cos(\Phi_2) + (z^{2n} + 1) i \sin(\Phi_2) \right) \quad (20)$$

$$2iz^n \sin(\phi_{s0}) = z^{n-1} (z^2 - 1) + \alpha (z^{2n} i e^{i\Phi_2} + i e^{-i\Phi_2}) \quad (21)$$

$$\boxed{0 = \alpha z^{2n} e^{i\Phi_2} - iz^{n+1} - 2z^n \sin(\phi_{s0}) + iz^{n-1} + \alpha e^{-i\Phi_2}}$$

Note that this equation can be solved exactly for  $n \leq 2$ . For higher  $n$  the polynomial is of order  $o(n) > 5$  and therefore solving it is non-trivial. If we know the  $2n$  roots of this polynomial,  $z_i \in \{z_1, z_2, z_3, \dots, z_{2n}\}$ , then  $\phi_s$  is defined piecewise as the complex argument of each  $z_i$ , i.e.

$$\phi_s = \begin{cases} \arg(z_1) & \text{if condition}_1 \\ \arg(z_2) & \text{if condition}_2 \\ \vdots & \\ \arg(z_{2n}) & \text{if condition}_{2n} \end{cases} \quad (22)$$

Summarising, we managed to solve the problem exactly for  $n = 1, 2$  only.

## B Parametric expression for $\phi_s$

The proposed parametrisation allows us to plot exactly  $\phi_s(n, \alpha, \phi_{s0}, \Phi_2)$ . It is very efficient computationally. It gives access to the Fourier expansion of  $\phi_s$  which then allows us to calculate the Hilbert transform of  $\phi_s$ . In appendix B.4 we show that this has potential applications in modelling  $\lambda$ .

### B.1 Parametrisation

By rewriting 1 we get

$$\frac{\sin(\phi_{s0}) - \sin(\phi_s)}{\alpha} = \sin(n\phi_s + \Phi_2) \quad (23)$$

We choose to substitute

$$t = \frac{\sin(\phi_{s0}) - \sin(\phi_s)}{\alpha} \quad (24)$$



so that also

$$t = \sin(n\phi_s + \Phi_2) \quad (25)$$

From 24 we get

$$\sin(\phi_s) = \sin(\phi_{s0}) - \alpha t \quad (26)$$

This can be written as<sup>1</sup>

$$\phi_s = \sin^{-1}(\sin(\phi_{s0}) - \alpha t) \quad (27)$$

From 25 we get:

$$n\phi_s + \Phi_2 = \begin{cases} \sin^{-1}(t) \\ \pi - \sin^{-1}(t) \end{cases} \quad (28)$$

Therefore we split  $\Phi_2$  into two parts.

$$\Phi_2^+ = \sin^{-1}(t) - n \sin^{-1}(\sin(\phi_{s0}) - \alpha t) \quad (29)$$

$$\Phi_2^- = \pi - \sin^{-1}(t) - n \sin^{-1}(\sin(\phi_{s0}) - \alpha t) \quad (30)$$

At this point it is worth mentioning that the domain of this parametrisation is  $\alpha \neq 0$  and

$$t \in \left[ \frac{\sin(\phi_{s0}) - 1}{\alpha}, \frac{\sin(\phi_{s0}) + 1}{\alpha} \right] \cap [-1, 1] = \left[ \max \left\{ \frac{\sin(\phi_{s0}) - 1}{\alpha}, -1 \right\}, \min \left\{ \frac{\sin(\phi_{s0}) + 1}{\alpha}, 1 \right\} \right] \quad (31)$$

## B.2 Finding $\Phi_2^{\alpha\pm}$ and $\Phi_2^{\max/\min}$

We set the parametrisation so that  $t = 0$  corresponds to  $\Phi_2^{\alpha\pm}$  and  $t = \mp 1$  to  $\Phi_2^{\max/\min}$ . This can be verified from eq. 1 by implicit differentiation<sup>2</sup>.

At  $t = 0$ :

$$\Phi_2^{\alpha+} = -n\phi_{s0} \quad (32)$$

$$\Phi_2^{\alpha-} = \pi - n\phi_{s0} \quad (33)$$

At  $t = 1$ :

$$\Phi_2^{\min} = \frac{\pi}{2} - n \sin^{-1}(\sin(\phi_{s0}) - \alpha) \quad (34)$$

At  $t = -1$ :

$$\Phi_2^{\max} = -\frac{\pi}{2} - n \sin^{-1}(\sin(\phi_{s0}) + \alpha) \quad (35)$$

The  $\pm$  sign in  $\Phi_2^{\alpha\pm}$  reflects whether the point lies in the  $\Phi_2^+$  part or  $\Phi_2^-$  part. The classification of  $\Phi_2^{\max/\min}$  follows from the fact that  $\phi_s(t)$  is a strictly decreasing function in  $t$ , i.e.  $\phi_s(1) < \phi_s(-1)$ .

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<sup>1</sup>The second solution  $\phi_{s2}(t) = \pi - \sin^{-1}(\sin(\phi_{s0}) - \alpha t)$  is not considered, since we require  $\phi_s$  to be a function. Also,  $\phi_{s2} = \pi - \phi_s$  and its easy to translate the properties between the first and the second solution.

<sup>2</sup>For example, for  $\Phi_2^{\max/\min}$  we have to solve  $0 = \cos(\phi_s) \frac{\partial \phi_s}{\partial \Phi_2} + \alpha \cos(n\phi_s + \Phi_2) \left( n \frac{\partial \phi_s}{\partial \Phi_2} + 1 \right)$  and then set  $\frac{\partial \phi_s}{\partial \Phi_2} = 0$  to obtain the results.

### B.3 Beam behaviour classification

For both extrema to exist we require  $t = \pm 1$  to be in the domain defined in 31. That is

$$\frac{\sin(\phi_{s0}) - 1}{\alpha} \leq -1 \wedge \frac{\sin(\phi_{s0}) + 1}{\alpha} \geq 1 \quad (36)$$

$$\alpha < 1 - |\sin(\phi_{s0})| \quad (37)$$

This condition reassures  $\phi_s$  to be continuous. Physically, this means that **if the system does not satisfy 37, then the beam will be lost for certain values of  $\Phi_2$** . Moreover since we are considering  $\phi_s$  to be a function, we need to ensure that it is never multivalued. Since  $\phi_s$  is continuous in  $\Phi_2$  (provided 37 satisfied), it can only be multivalued if at some point  $\frac{\partial \phi_2}{\partial \Phi_2} = \pm \infty$ . It can be shown by parametric differentiation<sup>3</sup> that this implies

$$\alpha \leq \sqrt{\frac{\sin^2(\phi_{s0})}{1 - n^2} + \frac{1}{n^2}} \quad (38)$$

Summarising,  $\phi_s$  is multivalued if the system does not satisfy 38. Different types of behaviour of the system are shown in figure 6. We also see that for  $n = 0$  and  $n = 1$ ,  $\phi_s$  can never be multivalued.

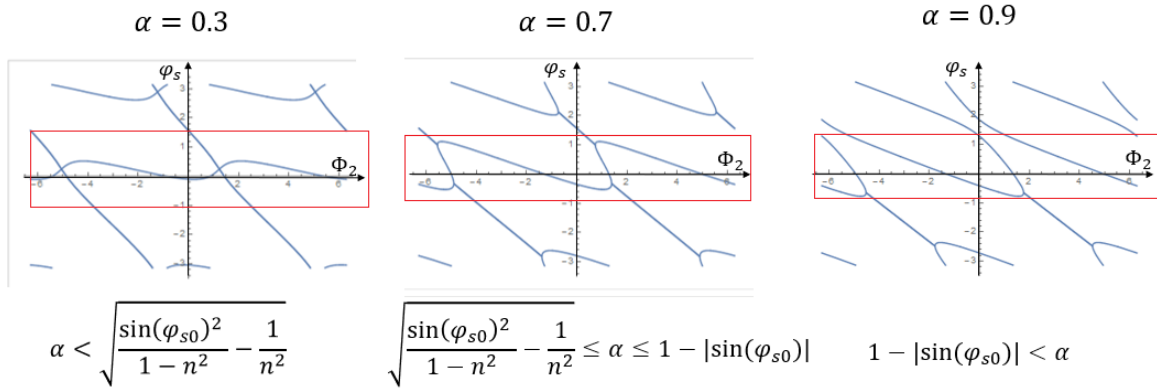


Figure 6: Comparison between different regimes of  $\phi_s$  using the analytic solution case for  $n = 2$ . For  $\phi_s = 0.2$  the corresponding critical values are  $\alpha \approx 0.51$  from eq. 38 and  $\alpha \approx 0.80$  from eq. 37.

### B.4 Fourier series

For any periodic signal  $y(x)$  parametrised in  $t$  over one period we have due to the change of variables

$$y(x) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{imx} \int_0^{2\pi} y(x) e^{-imx} dx = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{imx} \int_{t_{min}}^{t_{max}} y(t) e^{-imx(t)} \frac{dx(t)}{dt} dt \quad (39)$$

<sup>3</sup>Solving  $\frac{d\Phi_2}{dt} / \frac{d\phi_s}{dt} = 0$  which results in a quadratic equation which then has solutions if its discriminant,  $\Delta > 0$ .

Applying this to our problem, since the function  $\Phi_2(t)$  has two branches,  $\Phi_2^+$  and  $\Phi_2^-$ , we have to integrate them separately and sum. Also, we have to invert the integration limits of the second branch<sup>4</sup>. This results in the following formula:

$$\phi_s(\Phi_2) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im\Phi_2} \int_{t_{min}}^{t_{max}} \phi_s(t) \left( e^{-im\Phi_2^+} \frac{d\Phi_2^+}{dt} - e^{-im\Phi_2^-} \frac{d\Phi_2^-}{dt} \right) dt = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} A_m e^{im\Phi_2} \quad (40)$$

Finally

$$\mathcal{H}(u) = -i \sum_{m=-\infty}^{\infty} \text{sgn}(m) A_m e^{imt} \quad (41)$$

The potential applications of this result is that the Hilbert transform of  $\phi_s(\Phi_2)$  can be used to model the measurement of the bunch length,  $\lambda$ . This claim is supported in figures 8a and 8b, however we will not discuss this further.

## C Hilbert transform

### C.1 Prerequisites

#### C.1.1 Convolution integral

Convolution is a mathematical operation that takes two functions  $f(t)$  and  $g(t)$  and produces a third function:

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t - \tau) d\tau$$

We note that convolution is a *bilinear* operator.

#### C.1.2 Cauchy principal value

Cauchy principal value associates a value to certain improper integrals that would otherwise be undefined. For example:

$$\int_{-\infty}^{\infty} \frac{1}{x} dx = \lim_{\epsilon \rightarrow 0} \int_{-1/\epsilon}^{-\epsilon} \frac{1}{x} dx + \lim_{\delta \rightarrow 0} \int_{\delta}^{1/\delta} \frac{1}{x} dx = \lim_{\epsilon \rightarrow 0} \log(\epsilon^2) - \lim_{\delta \rightarrow 0} \log(\delta^2) = \text{undef.} \quad (42)$$

We notice that if  $\epsilon = \delta$  then we can associate a value to this integral, since the last two limits cancel out. That is what we call the Cauchy principal value (p.v.)

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{1}{x} dx = 0 \quad (43)$$

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<sup>4</sup>As we run the parameter  $t \in [-1, 1]$ , at the beginning  $\Phi_2^+(-1) = \Phi_2^-(-1) = \Phi_2^{\max}$ . Then each part traces a curve "in the opposite sense" until they meet at  $t = 1$ , that is  $\Phi_2^{\min}$ .

## C.2 Introduction to the Hilbert transform

### C.2.1 Definitions

Hilbert transform is a linear operator defined by

$$\hat{u}(t) \equiv \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{u(\tau)}{t - \tau} d\tau$$

It can be thought of as either

- *Historically*, the solution to the problem: Given a function  $u(t)$  find  $v(t)$  such that  $s_a(t) = u(t) + iv(t)$  is an analytic function in the upper half of the complex plane. It is  $v(t) = \hat{u}(t)$ .
- *In terms of convolution*,  $\hat{u} = \frac{1}{\pi t} * u$ . The convolution is understood here in the p.v. sense.
- *In terms of Fourier transform*, if we take the Fourier transform of the definition of the Hilbert transform we obtain

$$\hat{u} = \frac{1}{\pi t} * u \quad (44)$$

$$\mathcal{F}(\hat{u}) = \mathcal{F}\left(\frac{1}{\pi t} * u\right) = \mathcal{F}\left(\frac{1}{\pi t}\right) \mathcal{F}(u) = -i \operatorname{sgn}(\omega) \mathcal{F}(u) \quad (45)$$

$$\hat{u}(t) = \mathcal{F}^{-1}(-i \operatorname{sgn}(\omega) \mathcal{F}(u)) \quad (46)$$

The last definition is the easiest to grasp conceptually. The Hilbert transform takes a signal to the Fourier space, shifts the phase of its negative frequency components by +90 degrees and the phase of the positive frequency components by -90 degrees, and then returns to the real (time) space. As an example, the Hilbert transform of  $\cos(2t)$  and  $\sin(-t)$  are

$$\mathcal{H}(\cos(2\tau))(t) = \cos\left(2t - \frac{\pi}{2}\right) = \sin(2t) \quad (47)$$

$$\mathcal{H}(\sin(-\tau))(t) = \sin\left(-t + \frac{\pi}{2}\right) = \cos(t) \quad (48)$$

We can see that for periodic signals this understanding is superior to the convolution integral.

### C.2.2 Important properties

An important result is that **the Hilbert transform doesn't depend on the offset of the signal on the  $y$  axis**. For  $c \in \mathbb{R}$

$$\mathcal{H}(u + c) = \mathcal{H}(u) + \mathcal{H}(c) = \mathcal{H}(u) + c * \frac{1}{\pi t} = \mathcal{H}(u) + \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{c}{t - \tau} d\tau = \mathcal{H}(u) \quad (49)$$

Let us now analyse the extrema of the Hilbert transform. For simplicity we assume that the signal is periodic in  $2\pi$  and of zero mean<sup>5</sup>. The two sided Fourier series of the signal, its Hilbert transform and corresponding derivatives can then be written for  $m \neq 0$  as:

$$u = \sum_{m=-\infty}^{\infty} A_m e^{imt} \quad u' = i \sum_{m=-\infty}^{\infty} m A_m e^{imt} \quad (50)$$

$$\mathcal{H}(u) = -i \sum_{m=-\infty}^{\infty} \text{sgn}(m) A_m e^{imt} \quad \mathcal{H}(u)' = \sum_{m=-\infty}^{\infty} |m| A_m e^{imt} \quad (51)$$

Finally if we also require that the signal is even (or odd) we can get a cos (sin) series to simplify the above.

$$u = \sum_{m=1}^{\infty} a_m \cos(mt) \quad u' = - \sum_{m=1}^{\infty} m a_m \sin(mt) \quad (52)$$

$$\mathcal{H}(u) = \sum_{m=1}^{\infty} a_m \sin(mt) \quad \mathcal{H}(u)' = \sum_{m=1}^{\infty} m a_m \cos(mt) \quad (53)$$

We immediately notice that for  $t = 0$  both  $u'$  and  $\mathcal{H}(u)$  become zero. Also for  $t = \pi/2$  both  $\mathcal{H}(u)'$  and  $u$  become 0. This is the principle of the method. In general, **For a symmetric signal  $u$ , there exist extrema of  $u$  that coincide with zeros of  $\mathcal{H}(u)$ .**

### C.3 Phase algorithm for symmetric signals

Analytic signal is defined as

$$s_a = u + i\hat{u} = r e^{i\theta} \quad (54)$$

where  $r$  is called the analytic envelope and  $\theta$  is the instantaneous phase of  $u$ , with  $r = \sqrt{u^2 + \hat{u}^2}$  and  $\theta = \tan^{-1} \left( \frac{\hat{u}}{u} \right)$ . We immediately see that

$$u = \mathcal{R} \{s_a\} = \mathcal{R} \{r e^{i\theta}\} = r \cos(\theta) \quad (55)$$

What follows is that  $u = 0$  when  $\theta = \pm \frac{\pi}{2}$ . Moreover the solution to  $\theta = 0$  is the same as the solution to  $\hat{u} = 0$ :

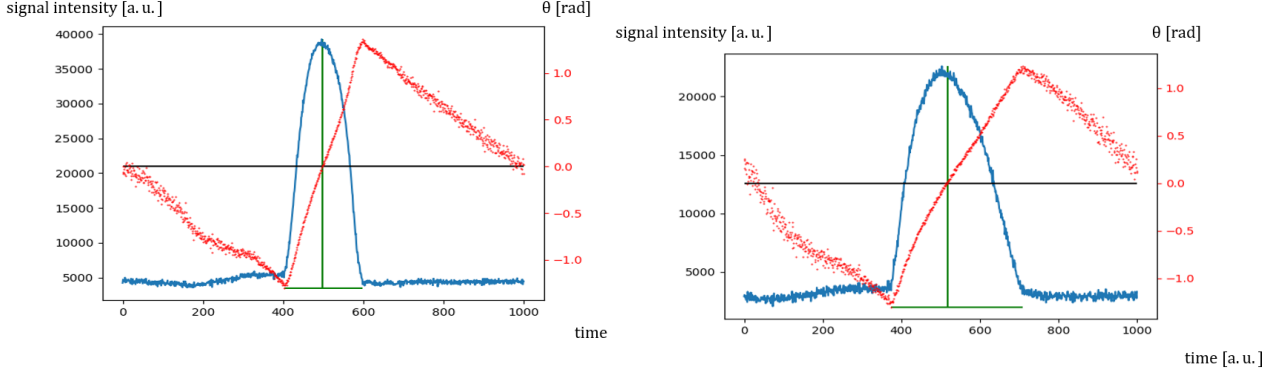
$$\theta = \tan^{-1} \left( \frac{\hat{u}}{u} \right) = 0 \quad (56)$$

$$\frac{\hat{u}}{u} = 0 \quad (57)$$

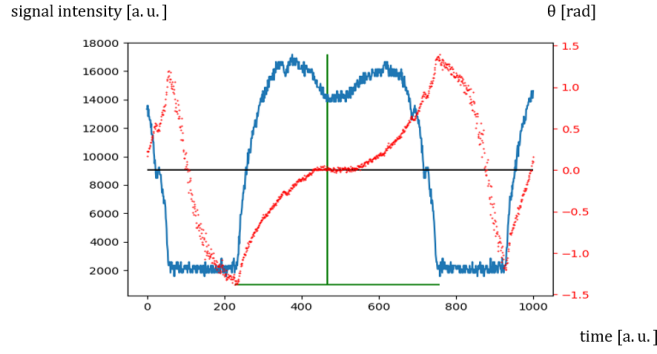
$$\hat{u} = 0 \quad (58)$$

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<sup>5</sup>The result is similar in the general case.

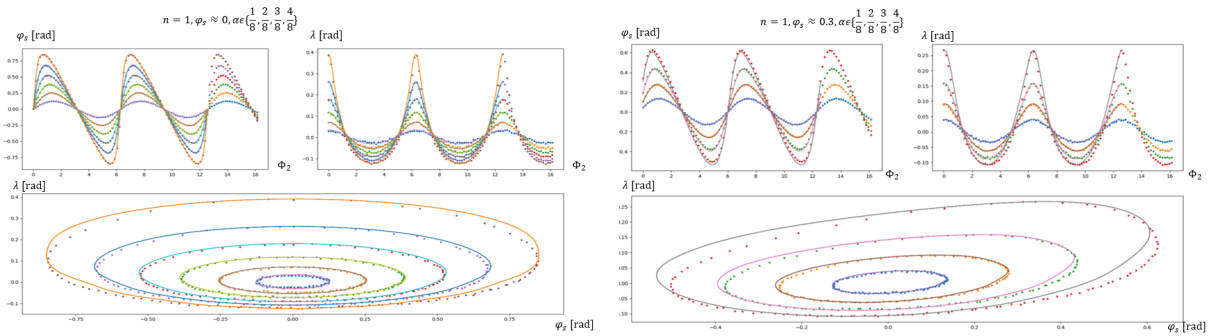


(a) The measurement of a symmetric signal,  $\phi_{s0} = 0$ . We can see how the zero of  $\tilde{\theta}$  corresponds to the maximum of  $u$ ; and how the extrema of  $\tilde{\theta}$  align with the endpoints of the profile. (b) In the case of an asymmetric signal the  $\phi_s$  measurement is not reliable. However, the measurement of  $\lambda$  is still accurate.



(c) For a system with  $\phi_s$  multivalued, its measurement fails. The measurement of  $\lambda$  is still reliable

Figure 7: Measurement of  $\phi_s$  and  $\lambda$ . In blue - the measured profile. In red - its analytic phase. In green - the measured  $\phi_s$  and  $\lambda$ .



(a) Good fit at the flat-top, where  $\phi_{s0} = 0$  (b) Good fit outside of flat-top given small  $\alpha$ .

Figure 8: The model  $\lambda \sim \mathcal{H}(\phi_s)$ . Dotted -  $\phi_s$  and  $\lambda$  from a simulated data of a beam profile. Bold - plot of the Fourier series of  $\phi_s$  and its Hilbert transform scaled and shifted to model  $\lambda$ .