Betatron Tune Determination: Interpolation Formulas

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Abstract

In order to obtain accurate estimations of the betatron tune, interpolation formulas have been derived (see [1] and [2]) that give estimates that approaches the real tune as $1/N^2$ and $1/N^4$ for signals of constant amplitude. In this document interpolation formulas for signals with exponential decaying amplitudes are derived, and its errors are analysed as a function of the input signal size and the decay constant. We obtain the same scaling law of $1/N^2$ for the case of constant amplitude, and an improvement over the previous methods in the case of decaying amplitude. Lower boundaries for the errors were observed, and methods for surpass this were analysed.

I. Introduction

In circular accelerators the transverse position of a beam of particles changes from turn to turn, oscillating around a stable trajectory. The main frequency of this oscillations is known as *betatron tune*, or just *tune*. It's determination is of great importance to keep the beam far from resonances, and several methods can be used. The basic Discrete Fourier Transform can be applied on the signal, which provides an estimate that approaches the tune with an error that is proportional to the inverse of the size of the signal, $1/N$. This can be improved if interpolation formulas are used, and for harmonic signals of constant amplitude this work has been developed in [1] and [2]. This methods have errors that scale like $1/N^2$, or $1/N^4$ when using windowing.

The goal of this report is to find an interpolation formula for harmonic signals with exponential decaying amplitudes, analyse its behaviour and compare with the methods described in [1] and [2].

II. Tune Determination

Let's assume that the signal ($z(n)$) is harmonic with exponential decay,

$$z(n) = a\, e^{2\pi i n\nu - \lambda n}, \tag{1}$$

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then the coefficients of its Discrete Fourier Transform (DFT) are

$$\phi(\nu_j) = \frac{1}{N}\sum_{n=1}^{N} z(n)e^{-2\pi i n\nu_j} = \frac{a}{N}\sum_{n=1}^{N} e^{2\pi i(\nu-\nu_j)n-\lambda n} \tag{2}$$

$$= \frac{a}{N}e^{2\pi i(\nu-\nu_j)-\lambda}\frac{1-e^{2\pi i(\nu-\nu_j)N-\lambda N}}{1-e^{2\pi i(\nu-\nu_j)-\lambda}} \tag{3}$$

$$= \frac{a}{N}e^{[\pi i(\nu-\nu_j)-\lambda/2](N+1)}\frac{\sin[\pi(\nu-\nu_j)N+i\lambda N/2]}{\sin[\pi(\nu-\nu_j)+i\lambda/2]} \tag{4}$$

$$= \frac{a}{N}e^{[\pi i(\nu-\nu_j)-\lambda/2](N+1)}\frac{\sin[\pi(\nu-\nu_j)N]\cosh(\lambda N/2)+i\cos[\pi(\nu-\nu_j)N]\sinh(\lambda N/2)}{\sin[\pi(\nu-\nu_j)]\cosh(\lambda/2)+i\cos[\pi(\nu-\nu_j)]\sinh(\lambda/2))}, \tag{5}$$

where we used the property

$$\sum_{n=1}^{N} e^{\xi n} = e^{\xi}\sum_{n=0}^{N-1} e^{\xi n} = e^{\xi}\frac{1-e^{\xi N}}{1-e^{\xi}}. \tag{6}$$

and

$$\sin(a+ib) = \sin(a)\cosh(b) + i\cos(a)\sinh(b). \tag{7}$$

This way we get for the amplitude of the DFT coefficients the following expression

$$\left|\phi(\nu_j)\right|^2 = \frac{a^2}{N^2}e^{-\lambda(N+1)}\frac{\sin^2[\pi N(\nu-\nu_j)]\cosh^2(\lambda N/2)+\cos^2[\pi N(\nu-\nu_j)]\sinh^2(\lambda N/2)}{\sin^2[\pi(\nu-\nu_j)]\cosh^2(\lambda/2)+\cos^2[\pi(\nu-\nu_j)]\sinh^2(\lambda/2)} \tag{8}$$

$$= \frac{a^2}{N^2}e^{-\lambda(N+1)}\frac{\sin^2[\pi N(\nu-\nu_j)]+\sinh^2(\lambda N/2)}{\sin^2[\pi(\nu-\nu_j)]+\sinh^2(\lambda/2)}. \tag{9}$$

Now let us make an observation: When we change j by ± 1, i.e. $\nu_j \to \nu_j \pm 1/N$, the numerator of this expression doesn't change, due to the fact that

$$\sin[\pi N(\nu-\nu_j)] \to \sin([\pi N(\nu-\nu_j)\mp\pi] = -\sin([\pi N(\nu-\nu_j)].$$

This means that we can factorise the trigonometrical expressions that have N on the argument, and arrive to a set of equations that are only function of $\tan(\pi N(\nu-\nu_j))$.

So, we have that

$$\left|\phi(\nu_{j\pm1})\right|^2 = \frac{a^2}{N^2}e^{-\lambda(N+1)}\frac{\sin^2[\pi N(\nu-\nu_j)\mp\pi]+\sinh^2(\lambda N/2)}{\sin^2[\pi(\nu-\nu_j)\mp\pi/N)]+\sinh^2(\lambda/2)} \tag{10}$$

$$= \frac{a^2}{N^2}e^{-\lambda(N+1)}\frac{\sin^2[\pi N(\nu-\nu_j)]+\sinh^2(\lambda N/2)}{\sin^2[\pi(\nu-\nu_j)\mp\pi/N]+\sinh^2(\lambda/2)}, \tag{11}$$

so we can define coefficients $\chi_\pm$ such that

$$\chi_\pm = \frac{|\phi(\nu_j)|^2}{|\phi(\nu_{j\pm1})|^2} = \frac{\sin^2[\pi(\nu-\nu_j)\mp\pi/N]+\sinh^2(\lambda/2))}{\sin^2[\pi(\nu-\nu_j)]+\sinh^2(\lambda/2)} \tag{12}$$

and so

$$\sinh^2(\lambda/2) = \frac{\sin^2[\pi(\nu-\nu_j)\mp\pi/N]-\sin^2[\pi(\nu-\nu_j)]\chi_\pm}{\chi_\pm-1} \tag{13}$$

$$= \frac{\left\{\sin[\pi(\nu-\nu_j)]\cos(\pi/N)\mp\sin(\pi/N)\cos[\pi(\nu-\nu_j)]\right\}^2-\sin^2[\pi(\nu-\nu_j)]\chi_\pm}{\chi_\pm-1}, \tag{14}$$

which is an expression independent of the sign in $\pm$, which implies that we can equal the expressions with both signs and get

$$\eta := \frac{\chi_+ - 1}{\chi_- - 1} = \frac{\left\{\cos(\pi/N)\tan[\pi(\nu-\nu_j)] - \sin(\pi/N)\right\}^2 - \tan^2[\pi(\nu-\nu_j)]\,\chi_+}{\left\{\cos(\pi/N)\tan[\pi(\nu-\nu_j)] + \sin(\pi/N)\right\}^2 - \tan^2[\pi(\nu-\nu_j)]\,\chi_-}, \tag{15}$$

and so we get our solution as

$$\tan[\pi(\nu-\nu_j)] = \frac{1}{\tan(\pi/N)}\left[\frac{\eta+1}{\eta-1} \pm \sqrt{\left(\frac{\eta+1}{\eta-1}\right)^2 + \tan^2(\pi/N)}\right]. \tag{16}$$

To solve for the sign, we can choose ν_j to be the position of the maximum of the DFT coefficients. So, if $|\phi(\nu_{j+1})| > |\phi(\nu_{j-1})|$ we know that the correction to j/N must be positive, and so we must take the minus sign (because in that case $\eta < 1$). The other sign corresponds to the other case and is equivalent to perform a reflection on the frequency spectrum, correct the value with the same recipe as before, and reflect back.

So, by inverting Eq. (16) it is possible to determine the signal frequency with an improved accuracy with respect to a plain FFT analysis. Moreover, the damping parameter λ can be obtained from Eq. (14) as

$$\lambda = 2\,\mathrm{arcsinh}\sqrt{\frac{|\phi(\nu_{j\pm1})|^2\sin^2[\pi(\nu-\nu_j)\mp\pi/N] - \sin^2[\pi(\nu-\nu_j)]|\phi(\nu_j)|^2}{|\phi(\nu_j)|^2 - |\phi(\nu_{j\pm1})|^2}}, \tag{17}$$

where one of the two signs has to be selected.

III. Numerical Results

We now consider as an input to our method, a harmonic signal in the presence of three harmonics of smaller amplitude, with exponential decay

$$z(n) = e^{-\lambda n}\left(e^{2\pi i\nu n} + e^{(2\pi i\nu n-1)2} + e^{(2\pi i\nu n-1)3} + e^{(2\pi i\nu n-1)4}\right).$$

The presence of harmonics modifies the Fourier coefficients near the main peak with a factor of order $1/N$, which is then propagated to the expression (**??**) as an error of order $1/N^2$.

We tested this, and the error is shown as a function of the signal size N in Figure 1 together with the errors obtained by applying a a simple Fast Fourier Transform, and the ones obtained by using the interpolation formulas described in [1] with and without data windowing.

In the case without damping ($\lambda = 0$), the scaling laws are clearly seen. When a damping is added, we can see that the behaviour gets worse, and the method developed here shows an improvement over the other methods. Nevertheless, due to the fact that after a certain value for n the exponential decay of the amplitude erases the signal, there's no further improvement in growing the size of the signal and we see that the error saturates for large N.

A similar behaviour can be observed in the error of the estimation of the damping constant λ as a function of N. In Figure 2 we can observe that it also scales as $1/N^2$ and presents a saturation for large values of N when the $\lambda > 0$. It is curious that for small non-zero values of λ, there seems to be a value of N that minimises the error.

To test if the saturation effect is due to the fact that the exponential erases the signal, we can do the following reasoning: The moment from which the signal stops being significant must be

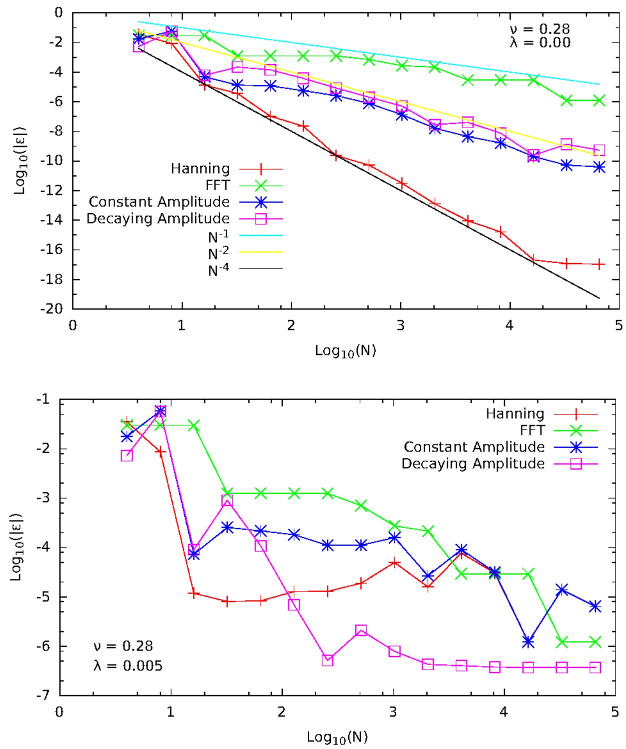


Figure 1: *The comparison between the method developed here (Decaying Amplitude) and in [1] (Constant Amplitude and Hanning) is shown for the cases without damping (left) and with a damping factor of $\lambda = 0.005$ (right).*

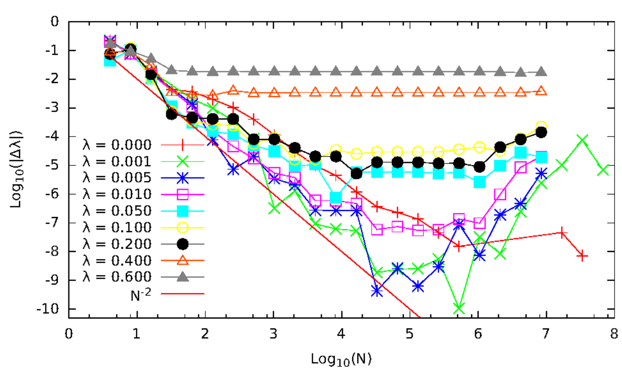


Figure 2: *Error in the estimation of the damping constant λ as a function of the size of the signal N, plotted for different values of λ.*

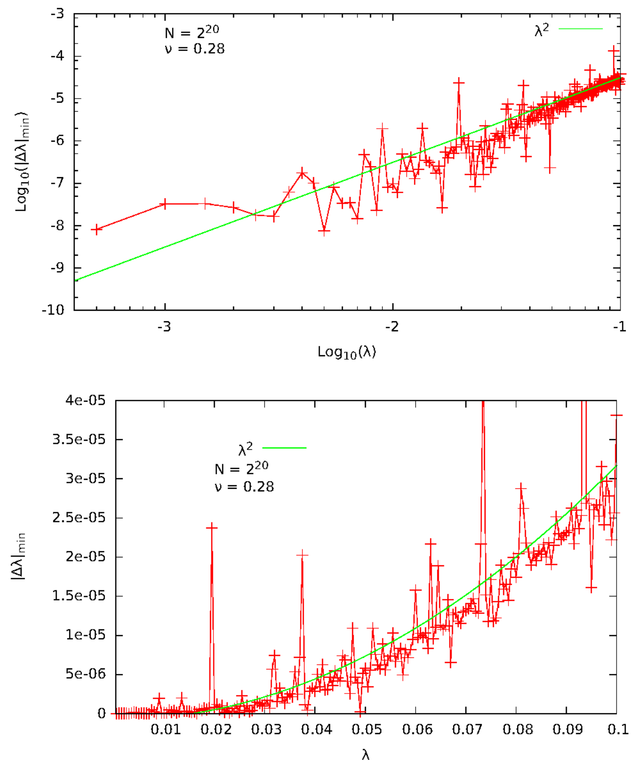


Figure 3: *Error in the estimation of the damping constant λ as a function of λ. The data is plotted in logarithmic (left) and linear (right) scales.*

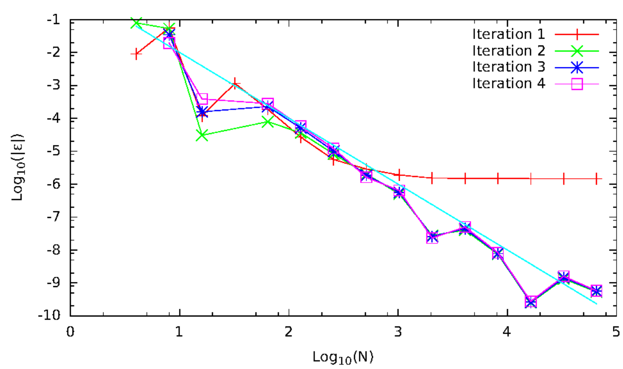
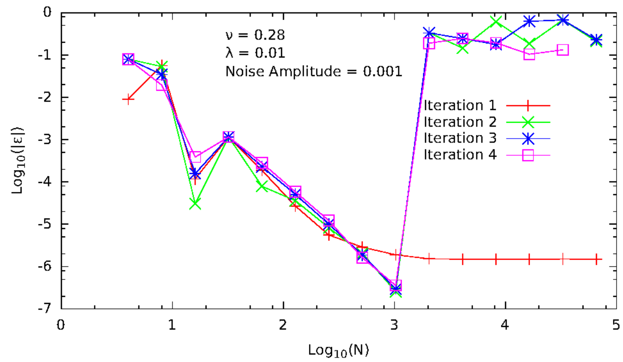


Figure 4: *The errors in frequency determination are showed as a function of N when iterating on the method. It is clear that the first iteration improves significantly the estimation.*



captionThe results of Figure 4 are reproduced for the case of a signal with a withe noise of constant amplitude.

proportional to the decay time $N_{max} = 1/\lambda$, and as the error scales as $1/N^2$, the error in this point should be of order $1/N_{max}^2 \propto \lambda^2$. We tested this, using as an estimation of the saturation value for the error in λ, its value at $N = 2^{20}$. The result of this test is shown in Figure 3.

We can also improve the analysis and avoid this saturation of the error, if we amplify the signal as it decays. We can do this multiplying the signal for the increasing exponential function e^{λ_n} using the estimation of λ that the method provides. Even more, we could then iterate this procedure to reduce even more the effects of decay. For the input signal used here we can see in Figure 4 that doing this we remove almost completely the exponential decay with the first iteration, which causes that there's no significant improvement with more iterations.

We can then test if this iteration method is robust to signal with noise. To see this we generated an input signal just as the one used so far, but we added a white noise of small amplitude. The result is presented in Figure **??** and it shows that after certain N, the method stop working. This is due to the fact that in our attempt to amplify the signal by a factor equal to the damping factor, we are also amplifying the noise, and after a certain value of n, it becomes much grater than the signal and the interpolation formulas stop giving good estimations for the tune.

IV. Discussion

The work done in [1] and [2] to find interpolation formulas that provide more accurate estimations of the betatron tune was extended to interpolate with a harmonic signal whose amplitude decays exponentially. Analytic formulas for the estimation of the main frequency, as well as the decay constant, were found. Through numerical methods, the error for this estimations were analysed as a function of the size of the input signal, and of the decay constant. It was shown that these errors scale as $1/N^2$, and for signal with decaying amplitudes provide a better estimate than the previous methods available in the literature. Nevertheless, it was shown that the error is bounded from below due to the fact that the signal eventually vanishes in the presence of damping. It was shown that iteration of the method improves the accuracy, except in the case of a signal with noise of non-vanishing amplitude where the method fails. Further analysis on this matter is required, as also on different functional forms for input signals in order to test the robustness of the method.

References

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[2] R. Bartolini, M. Giovannozzi, W. Scandale, A. Bazzani and E. Todesco (1996). PRECISE MEASUREMENT OF THE BETATRON TUNE *Particle Accelerators*, Vol 55, pp. 247-256.