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Optical Modelling of Low-Energy Antiproton Transport

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Summary

Low energy antiproton beams at the AD complex are guided using magnetic and electrostatic field. Dipole and quadrupole elements have large strengths and optical modelling on their effects rely on tracking to extract and use their linear transport matrices. This study explores two different method to extract linear transport matrix coefficient from tracking data and applies it to two straight elements: the drift and the quadrupole.

Contents

1 Introduction

The accelerator elements, such as dipole and quadrupole, are designed to transport, bend or focus charged particles. As these particles travel through the accelerator, they experience forces from electromagnetic fields, which guide their motion through the accelerator's structure. The simplest method to model this motion uses a mathematical tool called the linear transport matrix, which allows us to predict how a particle's position and angle will change as it moves through an accelerator element [2, 5].

For this project, a notebook was created, which computes the transport matrix of an element by tracking antiprotons from the entrance to the exit sides. This is done for two different elements: the drift space and the quadrupole. We can also determine the transport matrix of a succession of several different elements in a row by multiplication of the individual matrices.

To create the notebook, several python packages were used, including *SciPy*, *NumPy*, *pandas* and *matplotlib*, as well as a local python package *PyBT* [1].

2 Theoretical background

The force guiding a charged particle in the electromagnetic field is the Lorentz force:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \tag{1}$$

where $\vec{F}$ is the force exerted on the particle, q and $\vec{v}$ are the charge and the velocity of the particle respectively, and $\vec{E}$ and $\vec{B}$ are electric and magnetic fields respectively [5, 3]. In high energy machines, particle velocities approach the speed of light (c), and as a result, the influence of the magnetic field $\vec{B}$ becomes dominant over the electric field $\vec{E}$. It is important to note that only the component of the magnetic field transverse to the velocity creates a force, meaning that it can change the direction of motion but doesn't accelerate the particles [5, 2, 4].

In straight elements, the ensemble of particles travels along a straight path, which is here defined as the longitudinal direction z, with the transverse directions being in x and y axes. The reference particle defines the centre of the transport line.

In this notebook, the transport of an antiproton bunch is simulated. The antiproton bunch is a group of antiprotons around the reference particle, and every particle is characterised at any given step by spatial coordinates (x, y, z) and momentum coordinates (p_x, p_y, p_z). This space is known as the particle phase space. In addition to these coordinates, momenta angle xp and is used. This is defined by $xp = \frac{p_x}{p_z}$ [3].

The arbitrary solution to the equation of motion for any particle is

$$\begin{aligned} x_{final} &= x_{initial} \cdot C + xp_{initial} \cdot S && (2) \\ xp_{final} &= x_{initial} \cdot C' + xp_{initial} \cdot S' && (3) \end{aligned}$$

where C, C', S and S' are functions and x and xp represent the distance from the reference particle and the angle of momentum of the particle respectively.

We can rewrite Eq. 2 and 3 in a matrix form:

$$\begin{bmatrix} x_{final} \\ xp_{final} \end{bmatrix} = M \cdot \begin{bmatrix} x_{initial} \\ xp_{initial} \end{bmatrix}, \tag{4}$$

where

$$M = \begin{bmatrix} C & S \\ C' & S' \end{bmatrix}. \tag{5}$$

Using this we can establish a linear transport matrix M for any element in the ring[5].

2.1 Drift space

A drift space is an element with no electromagnetic fields. After entering a drift space, charged particles continue to move in a straight line. The final positions and momenta of a drift space are calculated by using:

$$xp_{final} = xp_{initial} \tag{6}$$

$$x_{final} = x_{initial} + L \cdot \tan(xp_{initial}) \tag{7}$$

where L is the length of the drift space element. Since the values of the initial angles are very small, we can use

$$\lim_{xp \to 0} L \cdot \tan(xp) = L \cdot xp \tag{8}$$

Thus the transport matrix for a drift space can be approximated to[5, 2]:

$$\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \tag{9}$$

2.2 Quadrupole

The quadrupolar magnetic field varies linearly with the position of the particle and is given by

$$\vec{B} = [g \cdot y, -g \cdot x, 0], \tag{10}$$

where g is the gradient of a quadrupole field, and x and y are the transverse coordinates of the particle. The resulting field focuses the particles in one plane while defocusing in another. Usually, a series of alternating quadrupoles ensure that the particle beam remains focused in both planes [5, 3]. This element may also use electric fields instead of magnetic ones but resulting in a very similar behaviour.

We define the focal length f as:

$$f = \frac{1}{k \cdot l} \tag{11}$$

With $k \cdot l = K$ the integrated normalised quadrupole strength. Using the reduced equation of motion for particle motion

$$x'' + Kx = 0 \tag{12}$$

we obtain the general focusing matrix:

$$\begin{bmatrix} x_{final} \\ xp_{final} \end{bmatrix} = \begin{bmatrix} \cos\sqrt{k}l & \frac{1}{\sqrt{k}}\sin\sqrt{k}l \\ -\sqrt{k}\sin\sqrt{k}l & \cos\sqrt{k}l \end{bmatrix} \begin{bmatrix} x_{initial} \\ xp_{initial} \end{bmatrix} \tag{13}$$

and the general form of the defocusing matrix [5]:

$$\begin{bmatrix} x_{final} \\ xp_{final} \end{bmatrix} = \begin{bmatrix} \cosh\sqrt{|k|}l & \frac{1}{\sqrt{|k|}}\sinh\sqrt{|k|}l \\ \sqrt{|k|}\sinh\sqrt{|k|}l & \cosh\sqrt{|k|}l \end{bmatrix} \begin{bmatrix} x_{initial} \\ xp_{initial} \end{bmatrix} \tag{14}$$

In a quadrupole one plane is focusing while the other is defocusing and we usually refer to the function of the quadrupole as its effect in the horizontal plane.

2.3 Thin lens approximation

When the length of the quadrupole is much smaller than the focal length, thin lens approximation can be applied for 13 and 14. The limit $l \rightarrow 0$ is applied while keeping $K = k \cdot l$ constant. Therefore, both the focusing and the defocusing matrices approach [5]:

$$M = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}. \tag{15}$$

By multiplying the transport matrices for individual elements in sequence, we can describe the particle's motion through a combination of those elements. This matrix approach provides a powerful yet simple tool for modeling particle accelerators and beam transport systems.

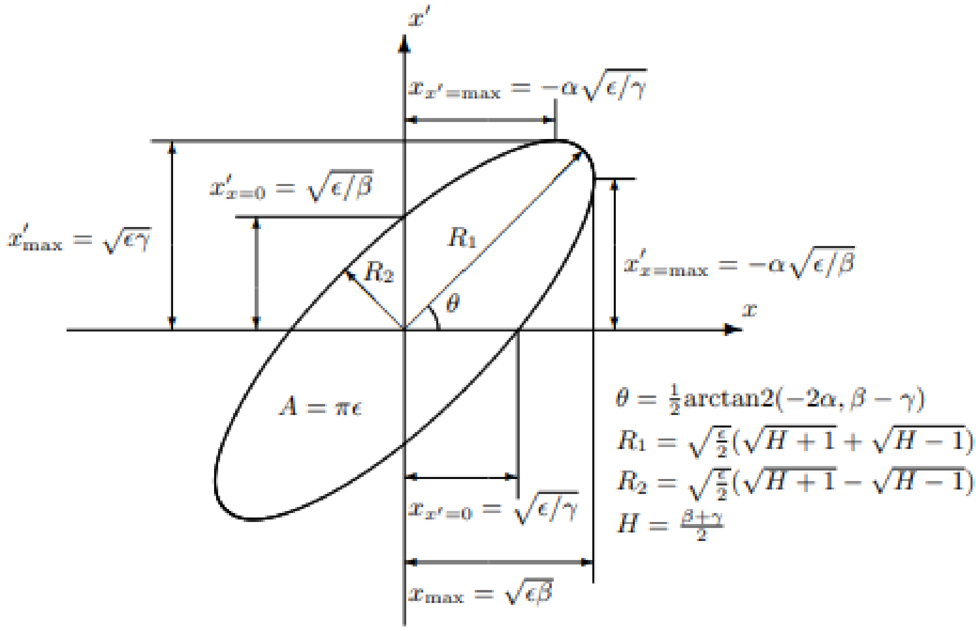

Figure 1: Enter Caption

3 Method

A beam distribution is created and visualized using *PyBT*. The distances from the reference particle and angles of the momenta are organized into a *dataframe* and then transformed into initial coordinates and momenta vectors. We choose the distances from the reference particle to be projected on the y axis, and the x and z coordinates to be set to zero. Subsequently, the tracker computes the particles' trajectory through multiple steps, recording the positions and momenta at each one. The particles' coordinates and momenta are converted back into angles and positions and the phase space is re-centered around the reference particle. Finally, the matrix elements are extracted from the processed data.

3.1 Beam generation and conversions

An elliptical particle distribution, which represents a particle's angle of the momentum vector with respect to the x and y axes and the distance from the reference particle on the transverse plane is generated using *PyBT*. Here, particles with positive distance represent particles generated above the y coordinate of the reference particle. The equation for an origin-centred ellipse is

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \epsilon. \tag{16}$$

Here ϵ is the two-dimensional transverse emittance, and α, β and γ are known as the Twiss parameters defining the ellipse orientation and aspect ratio [3]. In this notebook, the Twiss parameters were chosen to be $\alpha = 1.27976562$, $\beta = 4.636782612$. The chosen RMS emittance was $\epsilon = 10^{-6}$ m rad, and the standard deviation of the distribution $\sigma = 1$.

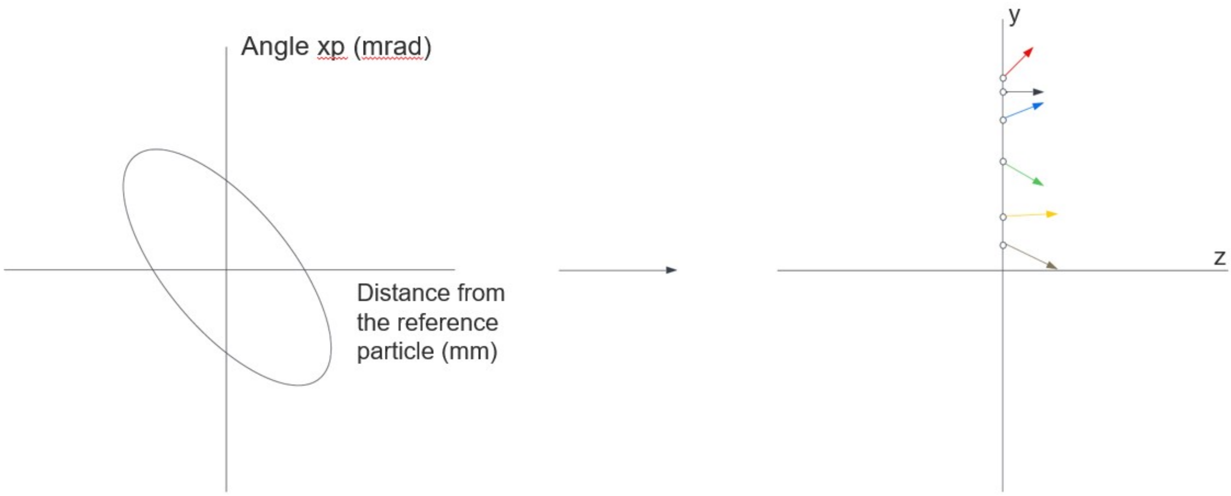

Figure 2: Translation from phase space to real phase. The positions on the y axis correspond to the distance between the tracked particle and the reference particle, and the arrows illustrate the initial momenta vectors of the tracked antiprotons.

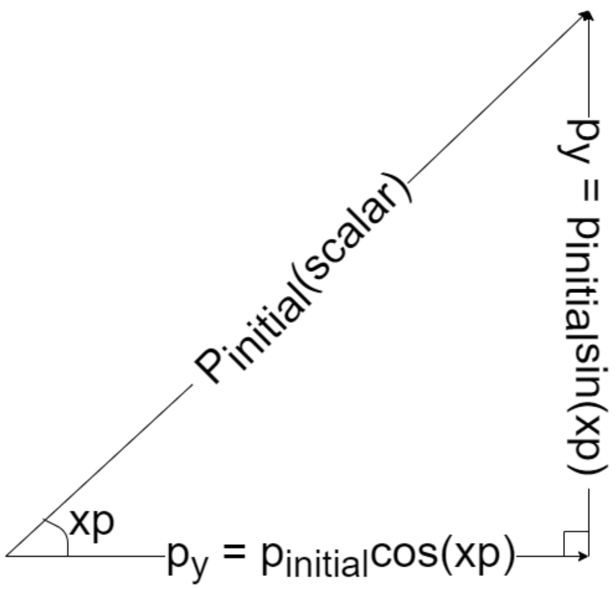

Figure 3: x and y components of the momentum

3.2 Tracking

Particles were tracked using two main techniques: direct computation and the tracker method. The direct computation method employed a formula to immediately obtain the final coordinates and momenta for the drift space. For the tracker method, a step-wise tracking function, based on the Boris algorithm, named `track_to_condition` was utilized for both the drift space and the quadrupole. `track_to_condition` accepts the initial Cartesian coordinates, momenta, mass, charge, initial time, time step, magnetic field, and the `track_is_inside` function, which sets the tracking termination criteria. In each tracking step, the particle's position and momentum are updated, moving the particle along its path. The tracking continues until the termination condition is met. For straight elements, the tracking stops when $z \geq L$, where z is the longitudinal coordinate and L is the magnet length.

3.2.1 Centering the final distribution

After tracking, the final coordinates and momenta are transformed into phase space angles and distances.

To calculate the phase space angles, the following formulae are used:

$$P = \sqrt{p_x^2 + p_y^2 + p_z^2} \;\rightarrow\; xp = \arctan(\frac{p_x}{p_z}),\; yp = \arcsin(\frac{py}{P}) \tag{17}$$

where P is the scalar momentum, and p_x, p_y and p_z represent the scalar momenta in their respective axes.

To determine the new distances from the reference particle, the following formulae are used:

$$r = (x^2 + y^2 + (L - z)^2) - \sqrt{(< x >)^2 + (< y >)^2 + (L - < z >)^2} \tag{18}$$

$$< x >= min(x) + \frac{max(x) - min(x)}{2} \tag{19}$$

$$< y >= min(y) + \frac{max(y) - min(y)}{2} \tag{20}$$

$$< z >= min(z) + \frac{max(z) - min(z)}{2} \tag{21}$$

The substraction of the average values provides the center of the new phase space and selects a new reference frame, in which the reference particle is at the origin. We take the sign of the new distance to be positive if the sign of the final y coordinate is positive, and negative otherwise.

Finally, the acquired data is organised into a *dataframe*. Initial and final distributions are plotted side-by-side for comparative analysis.

3.3 Computing the matrix elements

The matrix elements are extracted using two methods: the best fit method and the direct matrix element extraction.

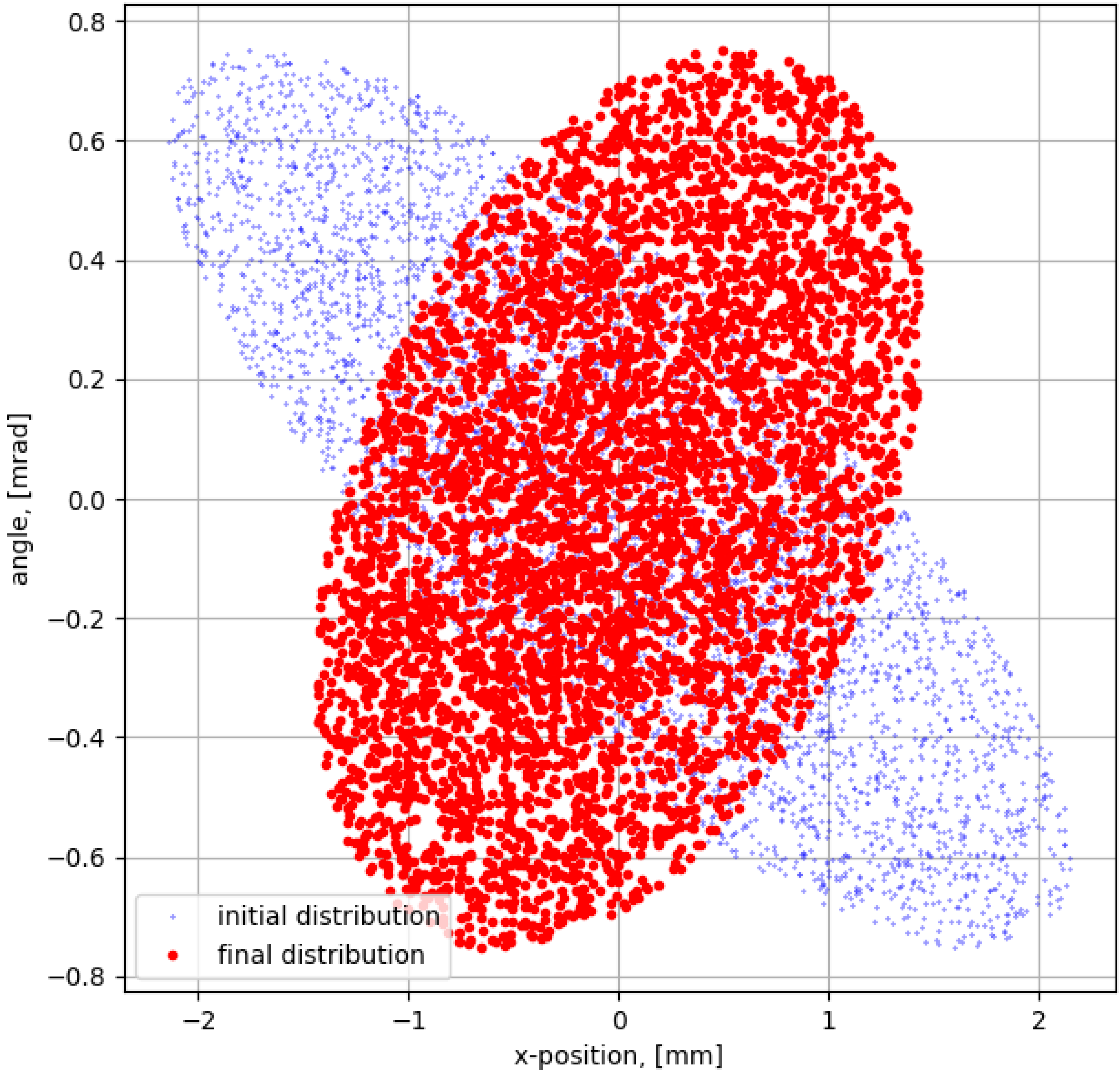

Figure 4: Initial and final distributions of the drift space of length 3.

3.3.1 Best fit method

We use a fitting procedure to find the value of the matrix coefficients which reproduce the transport:

$$\begin{bmatrix} x_f \\ xp_f \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \cdot \begin{bmatrix} x_i \\ xp_i \end{bmatrix} = \begin{bmatrix} R_{11}x_i + R_{12}xp_i \\ R_{21}x_i + R_{22}xp_i \end{bmatrix} \tag{22}$$

Given the variables x_f, xp_f, x_i and xp_i the equations are transformed to facilitate the use of the `curve_fit` function:

$$\begin{bmatrix} \frac{x_f}{xp_i} \\ \frac{xp_f}{xp_i} \end{bmatrix} = \begin{bmatrix} R_{11}\frac{x_i}{xp_i} + R_{12} \\ R_{21}\frac{x_i}{xp_i} + R_{22} \end{bmatrix} \tag{23}$$

We then define

$$x\_data = \frac{x_i}{xp_i} \tag{24}$$

$$y\_data1 = \frac{x_f}{xp_i} \tag{25}$$

$$y\_data2 = \frac{xp_f}{xp_i} \tag{26}$$

We rewrite the equations as

$$y\_data1 = x\_data \cdot R_{11} + R_{12} \tag{27}$$

$$y\_data2 = x\_data \cdot R_{21} + R_{22} \tag{28}$$

The *SciPy* `curve_fit` function, which utilizes non-linear least squares, is applied to determine the unknown matrix elements R_{11}, R_{12}, R_{21} and R_{22} by inputting the known y and x data. The computed matrix elements are then structured into a matrix.

3.3.2 Direct matrix element extraction

In this method, the matrix is perceived as a sum of its components. With this approach, we decompose the matrix multiplication:

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \cdot \begin{bmatrix} x_i \\ xp_i \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \cdot (\begin{bmatrix} x_i \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ xp_i \end{bmatrix}) \tag{29}$$

First, the beam is generated with the initial angles set to zero:

$$\begin{bmatrix} x_f \\ xp_f \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \cdot \begin{bmatrix} x_i \\ 0 \end{bmatrix} = \begin{bmatrix} R_{11}x_i \\ R_{21}x_i \end{bmatrix}. \tag{30}$$

The beam is tracked through a tracker as usual and the final phase space is obtained. We then plot x_i against x_f for R_{11} and x_i against xp_f for R_{21}.

For the second one, the initial distances and initial angles to y direction are set to zero:

$$\begin{bmatrix} x_f \\ xp_f \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ xp_i \end{bmatrix} = \begin{bmatrix} R_{12}xp_i \\ R_{22}xp_i \end{bmatrix}. \tag{31}$$

The methods follows:

1. A beam is generated with the initial x_positions set to zero. The final phase space is determined in the same way as described before. The parameters are then plotted to extract the slopes for the matrix elements R_{11} and R_{12}.

   For R_{11}, we plot the initial x positions on the x axis and the final x-positions on the y axis. For R_{21}, we plot the initial x positions on the x axis and the final angles on the y axis.

2. A beam is generated with the initial angles set to zero. The process from the first step is then repeated, but this time for the matrix elements R_{12} and R_{22}. For R_{12}, we plot the initial angles on the x axis and the final x positions on the y axis. For R_{22}, we plot the initial angles on the x axis and the final angles on the y axis.

4 Conclusions

We have explored two different methods to extract linear transport matrix from drift and quadrupole elements using a custom charged particles tracker. The beam characteristics and field strengths used here closely relate to the typical ones encountered by low energy antiprotons of the CERN AD complex in order to benchmark the method in those specific conditions.

This method ultimately aims at extracting linear transport matrices from curved elements such as dipoles and using computed field maps, which may be the topic of further studies.

References

[1] Y. Dutheil. *PyBT repository, https://gitlab.cern.ch/abt-optics-and-code-repository/simulation-codes/pybt/-/tree/master/pybt/trackers.* 2021, accessed on 2023-09-21.

[2] B. J. Holzer. *Transverse impedance of axially symmetric tapered structures.* 2013.

[3] T. Kalvas. *Beam Extraction and Transport.* PhD thesis, University of Jyvaskyla, Jyvaskyla, Finland, January 2014.

[4] S. Y. Lee. *Accelerator Physics.* World Scientific, 2012.

[5] H. Wiedemann. *Particle accelerator physics.* Springer Nature, 2015.