CERN / Computing and Networks Division
CN/95/18
December 1995

The Polygamma Function $\psi^{(k)}(x)$ for $x = \frac{1}{4}$ and $x = \frac{3}{4}$

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ABSTRACT. Expressions for the polygamma function $\psi^{(k)}(x)$ for the arguments $x = \frac{1}{4}$ and $x = \frac{3}{4}$ are given in terms of Bernoulli numbers, Euler numbers, the Riemann zeta function for odd integer arguments, and the related series of reciprocal powers of integers $\beta(m)$.

1. Introduction

The polygamma function

$$\psi^{(k)}(x) \;=\; \frac{d^k}{dx^k}\psi(x) \;=\; \frac{d^{k+1}}{dx^{k+1}}\ln\Gamma(x)$$

appears in a number of theoretical and practical applications, and a number of its properties are listed in the relevant handbooks. In particular, it is well-known that for $k \geq 1$ [1, No. 6.4.4]

$$\begin{aligned}\psi^{(k)}(1) &= (-1)^k k!\,\zeta(k+1)\\ \psi^{(k)}(\tfrac{1}{2}) &= (-1)^{k+1} k!\,(2^{k+1}-1)\,\zeta(k+1),\end{aligned}$$

where $\zeta(n)$ is the Riemann zeta function for integer arguments. Together with the recursion formula [1, No. 6.4.6]

$$\psi^{(k)}(1+x) - \psi^{(k)}(x) \;=\; (-1)^k k!\,x^{-k-1} \tag{1}$$

and the reflection formula [1, No. 6.4.7]

$$\psi^{(x)}(1-z) + (-1)^{k+1}\psi^{(k)}(x) \;=\; (-1)^k\,\pi\,\frac{d^k}{dx^k}\cot\pi x, \tag{2}$$

it is easy to find expressions for $\psi^{(k)}(n)$ and $\psi^{(k)}(\frac{1}{2}\pm n)$, where $n \in \mathbb{N}$.

It seems surprising that the values of $\psi^{(k)}(x)$ for $x=\frac{1}{4}$ and $x=\frac{3}{4}$ and hence for $x=\frac{1}{4}\pm n$ and $x=\frac{3}{4}\pm n$ have received less attention; at least these values are seldom found in the handbooks. Sometimes, e.g. in [2], one finds a relation like $\psi'(\frac{1}{4})-\psi'(\frac{3}{4})=16G$, where $G = 0.91596\ldots$ is the Catalan constant.

It is the purpose of this note to present expressions for $\psi^{(k)}(\frac{1}{4})$ and $\psi^{(k)}(\frac{3}{4})$ in terms of the Bernoulli numbers, the Euler numbers, the Riemann zeta function for odd integer arguments, and the related series of reciprocal powers of integers $\beta(m)$.

2. Expressions for $\psi^{(k)}(\frac{1}{4})$ and $\psi^{(k)}(\frac{3}{4})$

For rational arguments $x = p/q$ the polygamma function $\psi^{(k)}(x)$ can be witten as [3, No. 8.363 8]

$$\begin{aligned}\psi^{(k)}\left(\frac{p}{q}\right) &= (-1)^{k+1}k!\,\zeta\left(k+1,\frac{p}{q}\right) \qquad (3)\\ &= (-1)^{k+1}k!\,q^{k+1}\sum_{n=0}^{\infty}\frac{1}{(p+qn)^{k+1}} \qquad (k\geq 1),\end{aligned}$$

where $\zeta(k+1,x)$ is the generalized zeta function. In particular we obtain

$$\begin{aligned}&\psi^{(k)}(\tfrac{1}{4}) + (-1)^k\psi^{(k)}(\tfrac{3}{4}) \qquad (4)\\ &= (-1)^{k+1}k!\,2^{2k+2}\left\{\sum_{n=0}^{\infty}\frac{1}{(4n+1)^{k+1}} + (-1)^k\sum_{n=0}^{\infty}\frac{1}{(4n+3)^{k+1}}\right\}.\end{aligned}$$

From the reflection formula (2) we find for $x = \frac{1}{4}$

$$\psi^{(k)}(\tfrac{1}{4}) - (-1)^k \psi^{(k)}(\tfrac{3}{4}) = -\pi \frac{d^k}{dx^k} \cot \pi x \bigg|_{x=1/4} . \tag{5}$$

In order to calculate the higher derivatives of $\cot \pi x$ at $x = \frac{1}{4}$, we make use of the trigonometric relation [4, No. 34:5:17]

$$\tan \pi(x + \tfrac{1}{4}) = \sec 2\pi x + \tan 2\pi x.$$

It is then not difficult to obtain the derivatives of the cotangent function at $x = \frac{1}{4}\pi$ from the power series expansions for the tangent and secant functions [1, No. 4.3.67, 69], namely

$$\tan z = \sum_{n=0}^{\infty} \frac{2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!} z^{2n-1} \qquad (|z| < \tfrac{1}{2}\pi)$$

and

$$\sec z = \sum_{n=0}^{\infty} \frac{|E_{2n}|}{(2n)!} z^{2n} \qquad (|z| < \tfrac{1}{2}\pi),$$

by noting that

$$\frac{d^k}{dx^k} \cot \pi x \bigg|_{x=1/4} = (-1)^k \frac{d^k}{dx^k} \tan \pi x \bigg|_{x=1/4} ,$$

where B_{2n} and E_{2n} are the Bernoulli and Euler numbers, respectively. This leads to

$$\frac{d^{2k-1}}{dx^{2k-1}} \cot \pi x \bigg|_{x=1/4} = -(2\pi)^{2k-1}\, 2^{2k}\, (2^{2k}-1) \frac{|B_{2k}|}{2k}.$$

and

$$\frac{d^{2k}}{dx^{2k}} \cot \pi x \bigg|_{x=1/4} = (2\pi)^{2k}\, |E_{2k}|.$$

By using the two series [1, No. 23.2.20, 21]

$$\sum_{j=0}^{\infty} \frac{1}{(2j+1)^m} = (1-2^{-m})\,\zeta(m) \qquad \text{and} \qquad \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^m} = \beta(m)$$

we obtain from (4) and (5), for odd $k = 2n - 1$,

$$\psi^{(2n-1)}(\tfrac{1}{4}) + \psi^{(2n-1)}(\tfrac{3}{4}) = \pi^{2n}\, 2^{4n-1}\, (2^{2n}-1) \frac{|B_{2n}|}{2n},$$

$$\psi^{(2n-1)}(\tfrac{1}{4}) - \psi^{(2n-1)}(\tfrac{3}{4}) = (2n-1)!\, 2^{4n}\, \beta(2n),$$

and for even $k = 2n$

$$\psi^{(2n)}(\tfrac{1}{4}) - \psi^{(2n)}(\tfrac{3}{4}) = -\pi(2\pi)^{2n}\, |E_{2n}|,$$

$$\psi^{(2n)}(\tfrac{1}{4}) + \psi^{(2n)}(\tfrac{3}{4}) = -(2n)!\, 2^{2n+1}\, (2^{2n+1}-1)\, \zeta(2n+1).$$

Hence for $n \in \mathbb{N}$,

$$\left.\begin{array}{l}\psi^{(2n-1)}(\frac{1}{4})\\ \psi^{(2n-1)}(\frac{3}{4})\end{array}\right\} = \frac{4^{2n-1}}{2n}\left\{\pi^{2n}(2^{2n}-1)\,|B_{2n}| \pm 2(2n)!\,\beta(2n)\right\}$$

and

$$\left.\begin{array}{l}\psi^{(2n)}(\frac{1}{4})\\ \psi^{(2n)}(\frac{3}{4})\end{array}\right\} = \mp 2^{2n-1}\left\{\pi^{2n+1}\,|E_{2n}| \pm 2(2n)!\,(2^{2n+1}-1)\,\zeta(2n+1)\right\}.$$

With the exception of $\beta(2) = G$, which is merely a definition, no expressions for $\zeta(2n+1)$ or $\beta(2n)$ in terms of other well-known constants are known.

The following list gives a few examples for $\psi^{(k)}(\frac{1}{4})$ and $\psi^{(k)}(\frac{3}{4})$:

$$\begin{array}{rclrcl}
\psi'(\frac{1}{4}) &=& \pi^2 + 8G, & \psi'(\frac{3}{4}) &=& \pi^2 - 8G,\\
\psi''(\frac{1}{4}) &=& -2\,[\,\pi^3 + 28\zeta(3)\,], & \psi''(\frac{3}{4}) &=& 2\,[\,\pi^3 - 28\zeta(3)\,],\\
\psi'''(\frac{1}{4}) &=& 8\,[\,\pi^4 + 96\beta(4)\,], & \psi'''(\frac{3}{4}) &=& 8\,[\,\pi^4 - 96\beta(4)\,],\\
\psi^{(4)}(\frac{1}{4}) &=& -8\,[\,5\pi^5 + 1488\zeta(5)\,], & \psi^{(4)}(\frac{3}{4}) &=& 8\,[\,5\pi^5 - 1488\zeta(5)\,],\\
\psi^{(5)}(\frac{1}{4}) &=& 256\,[\,\pi^6 + 960\beta(6)\,], & \psi^{(5)}(\frac{3}{4}) &=& 256\,[\,\pi^6 - 960\beta(6)\,],\\
\psi^{(6)}(\frac{1}{4}) &=& -32\,[\,61\pi^7 + 182880\zeta(7)\,], & \psi^{(6)}(\frac{3}{4}) &=& 32\,[\,61\pi^7 - 182880\zeta(7)\,].
\end{array}$$

References

[1] M. Abramowitz and I.A. Stegun, eds., *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 9th printing with corrections (Dover, New York, 1972).

[2] P.J. de Doelder, *On the Clausen integral* $\mathrm{Cl}_2(\theta)$ *and a related integral*, J. Comput. Appl. Math. **11** (1984), 325–330.

[3] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products,* 5th edition, A. Jeffrey, ed., (Academic Press, New York, 1994).

[4] J. Spanier and K.B. Oldham, *An Atlas of Functions,* (Hemisphere Publ., Washington DC and Springer-Verlag, Berlin, 1987).