

Evaluation of different scaling laws for dynamic aperture variation over time and intensity evolution in hadron storage rings

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Abstract

Two proposed fit models for Dynamic Aperture (DA) evolution have been studied in detail and applied to different numerical models and experimental data. The performance and the characteristics of these fits are analyzed in the context of finding a fast estimate of DA and intensity loss for a charged particle beam.

1 Introduction

The stability domain of a particle beam might be greatly reduced by the nonlinearities in the accelerator's magnetic field. The estimation of such stability domain is related to the so-called Dynamic Aperture (DA) and it's carried out in Ref. [1], where original numerical methods of DA estimation are tested over different numerical simulations (Hénon map, LHC-like cell lattice, and SPS lattice). Understanding the behavior of such a stable region is fundamental for estimating long-term beam dynamics on existing machines and defining the phase space region where one can safely operate with the beam.

In Ref. [2], a scaling law is proposed for extrapolating long-term behavior of the DA of a modulated Hénon map and on LHC data with tune modulation. The scaling law proposed (1) is in accordance both with the Kolmogorov–Arnold–Moser (KAM) theory and with the Nekhoroshev estimate for beam dynamics [3], as long as its three parameters D_∞, b, κ are all positive. This proposed scaling law proved to be very effective in estimating DA dynamics and intensity losses in beam dynamics, unfortunately, there are some configurations where the parameters assume negative values during the fitting procedure. Some of these configurations go beyond the proposed picture on phase space partitioning [3].

In Ref. [3], the scaling law used for extrapolating DA dynamics is used for proposing an intensity evolution law for accelerators. This approach tries to link the value of DA and beam losses by making hypothesis on the initial beam distribution and then integrating it over the inverse logarithmic decay of the DA.

In this Summer Student Project, we aim to pursue the analysis conducted in [2, 3] on the scaling law (1) and to look for possible angular dependencies for DA decay by executing different angular partitions of the phase space. Moreover, a new scaling law (2) is proposed for fitting DA decay and its performance and characteristics are explored.

2 Fit Models

The two proposed fit models are:

$$D(N) = D_\infty + \frac{b}{\log(N)^\kappa} \quad (1)$$

$$D(N) = \frac{b}{\log(a \times N)^\kappa} \quad (2)$$

The first fit (FIT1) has been applied in Ref. [2] over a modulated Hénon map and it has been widely commented in Ref. [3]. As pointed out in both references, such description of DA evolution is compatible with the physical hypothesis that the phase space can be partitioned into two regions: a stable region $r < D_\infty$ where KAM surfaces confine the motion and where stable behavior take place (apart from a small measure where Arnold diffusion take place) and an outer part $r > D_\infty$ where chaotic motion occurs and the escape rate is given by a Nekhoroshev-like estimate. As pointed out in [3], the three parameters D_∞, b, κ can both be all positive values (and therefore indicate the presence of such stable region) or be particular combinations of negative and positive values that can indicate global chaoticity or even going beyond such KAM-Nekhoroshev partitioning.

The second fit (FIT2) is supported by the good results obtained in the Nekhoroshev part of the analysis made in Ref. [2] and hopes to solve the issues with the negative parameters of FIT1 by removing the KAM related part and focusing only on the Nekhoroshev-like dynamics.

2.1 Fitting procedure

Since both scaling laws present nonlinear parts, a scan on the nonlinear parameters is carried out and, for each scanned value, the optimal values of the linear parameters are carried out with a standard linear fit. The value of the scanned parameter with the smallest χ^2 value is then chosen. More precisely:

- **For FIT1**, we perform a scan over the κ parameter, as it is proposed in Ref. [2], but, in order to increase the precision over the κ parameter, we perform more scans in the interval near the firsts picked κ values with a smaller search step.
- **For FIT2**, we firstly rewrite the fit in the form

$$\log D(N) = B - \kappa \log(\log(a \times N))$$

this allows us to work in a form where the κ term is linear and therefore make a scan strictly necessary only on the a parameter. We then execute a scan over a values near the minimum value it can assume without having logarithms of non-positive values (i.e. we must always have $\log(a \times N) > 1$, therefore $a > 1/N_{\min}$ where $N_{\min}$ is the smallest N value we are fitting data over), if the minimum found in the χ^2 is a trivial one, we enlarge the value interval where we are performing the scan by an order of magnitude until we find a non-trivial minimum. If no non-trivial a values are found within $a < N_{\max}$, where $N_{\max}$ is the largest N value we are fitting data over, we then establish that FIT2 does not properly converge for this sample of data (see Subsection 4.1.1 for more insight on this topic) and we then manually set the highest value of κ for which we still manage to have non naive minimums for the a parameter.

3 Numerical models and experimental data

In this section we briefly present the numerical models and experimental data used in the project.

3.1 Modulated Hénon map

The 4D Hénon map contains most of the characteristics of a nonlinear lattice and it offers the possibility to include tune modulation. The implementation and the parameters used for the map are the same reported in Ref. [2]. The stability tracking was done for 10^7 turns for many different values of ϵ between 0 and 64. Both Cartesian and radial scan of the map stability have been performed. In particular, we performed a Cartesian scan with $dx = 0.01$ A U precision and an angular scan over 100 angles between

0 and $\pi/2$, with same dx . As reported in Ref. [2], the Hénon map with different modulation intensities (i.e. different ϵ values) is a good testing ground for scaling laws.

The DA of the map is computed by using the angular average indicated in Ref. [1] and the intensity loss over time is computed by considering an initial Gaussian distribution of the intensity with different sigma values as it is proposed in Ref. [3].

The modulated 4D Hénon map reads

$$\begin{pmatrix} x^{(n+1)} \\ p_x^{(n+1)} \\ y^{(n+1)} \\ p_y^{(n+1)} \end{pmatrix} = \mathbf{L} \begin{pmatrix} x^{(n)} \\ p_x^{(n)} + [x^{(n)}]^2 - [y^{(n)}]^2 \\ y^{(n)} \\ p_y^{(n)} - 2x^{(n)}y^{(n)} \end{pmatrix},$$

where (x, p_x, y, p_y) are the phase space coordinates and $\mathbf{L}$ is the direct product of two 2D rotations R ,

$$\mathbf{L} = \begin{pmatrix} R(\omega_x^{(n)}) & 0 \\ 0 & R(\omega_y^{(n)}) \end{pmatrix},$$

where the linear frequencies $\omega_x^{(n)}$, $\omega_y^{(n)}$ are slowly varying with the discrete time n and are described by

$$\begin{aligned} \omega_x^{(n)} &= \omega_{x0} \left(1 + \epsilon \sum_{k=1}^m \epsilon_k \cos(\Omega_k n) \right), \\ \omega_y^{(n)} &= \omega_{y0} \left(1 + \epsilon \sum_{k=1}^m \epsilon_k \cos(\Omega_k n) \right). \end{aligned}$$

As for the values for the parameters, we have considered $\omega_{x0} = 0.168$ and $\omega_{y0} = 0.201$ and for the Ω_k frequencies and ϵ_k amplitudes the parameters reported in Table 1.

A plot of the map stability is reported in Fig. 1.

k	Ω_k	$10^4 \epsilon_k$
1	$2\pi/868.12$	1.000
2	$2\Omega_1$	0.218
3	$3\Omega_1$	0.708
4	$6\Omega_1$	0.254
5	$7\Omega_1$	0.100
6	$10\Omega_1$	0.078
7	$12\Omega_1$	0.218

Table 1: Parameters of the modulated Hénon map.

3.2 LHC data

As experimental data for the DA evolution at the LHC we use the same dataset utilized in the analysis executed in Ref. [4]. The data is obtained by computing a second order symplectic-tracking using the SixTrack code [5]. The ring model utilized is the most accurate description of the LHC lattice including the measured field errors [6] and the details regarding the parameters taken into consideration for the computation can be found in Ref. [7].

The data consist of 60 different realizations of the LHC lattice for both particle beams, with and without IR correctors taken into account, and with DA observed over all angles, x -coordinate close angles and y -coordinate close angles. The polar grid of initial conditions is obtained by dividing the first quadrant of the phase space in 59 angles. For the angles considered during the fitting procedure, the following notation is used: ‘HV’ label indicates that all 59 angles were considered, the ‘H’ label means that only the 10 angles close to the x coordinate were considered and the ‘V’ label means that only the 10 angles close to the y coordinate.

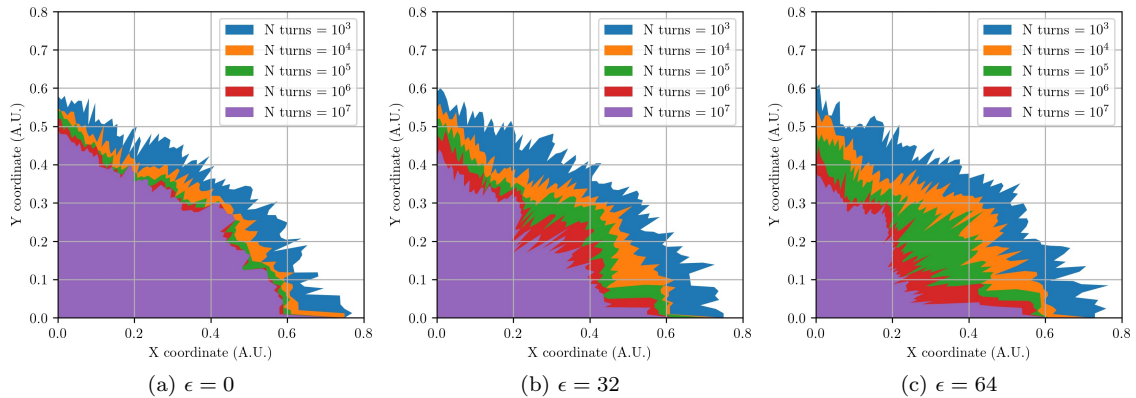


Figure 1: Stability scheme for modulated Hénon map for 3 different ϵ values.

4 Data Analysis

4.1 DA scaling laws

4.1.1 Hénon Map

We used Hénon map data as testing ground for analyzing the behavior of both fits.

FIT1. As it was observed in Ref. [2], FIT1 behaves properly for low values of ϵ but, as ϵ increases, we start to observe negative values for the fit parameters. For every scenario we observed a parabolic shape for the $\chi^2(\kappa)$ function (Fig. 2 is an example of such shape), this allows us to justify a more accurate scan over the κ value, once we have found a first minimum in such parabola with a larger step. FIT1 results are reported in Table 2 and three example plots are reported in Fig. 3.

ϵ	D_∞	b	κ
0	0.4551 ± 0.0002	7.73 ± 0.02	2.116 ± 0.001
1	0.3955 ± 0.0008	1.66 ± 0.01	1.146 ± 0.001
2	0.364 ± 0.001	1.44 ± 0.01	0.980 ± 0.001
4	0.278 ± 0.002	1.10 ± 0.01	0.673 ± 0.001
8	0.095 ± 0.003	1.093 ± 0.008	0.420 ± 0.001
12	-1.66 ± 0.01	2.63 ± 0.02	0.084 ± 0.001
16	-1.70 ± 0.01	2.70 ± 0.01	0.088 ± 0.001
32	1.472 ± 0.007	-0.573 ± 0.004	-0.231 ± 0.001
64	0.863 ± 0.003	-0.0621 ± 0.0005	-0.795 ± 0.001

Table 2: FIT1 parameters obtained for the modulated Hénon map.

FIT2. By observing the behavior of the $\chi^2(a)$ for increasing values of ϵ (Fig. 4), it can be seen how it switches from being a curve with a minimum near the lowest possible value of a to being a curve with no absolute minimum at all. By executing for one of these cases an analysis of the χ^2 function for different values of κ and a (Fig. 5), it can be observed how $\chi^2(a)$ stops having a minimum under the maximum explored value (i.e. 10^7 turns) for κ higher than a certain value. FIT2 results are reported in Table 3 and some representative plots are reported in Fig. 6.

One thing that was observed is how the slow change of concavity of the DA decay over time on logarithmic scale for larger values of ϵ (Fig. 7) seems to be directly correlated with the impossibility for FIT2 to converge properly. It can also be seen from Fig. 8 how FIT2 seems to have slightly better performance than FIT1 for large ϵ , since there are cases in which FIT1 does not assume positive parameters

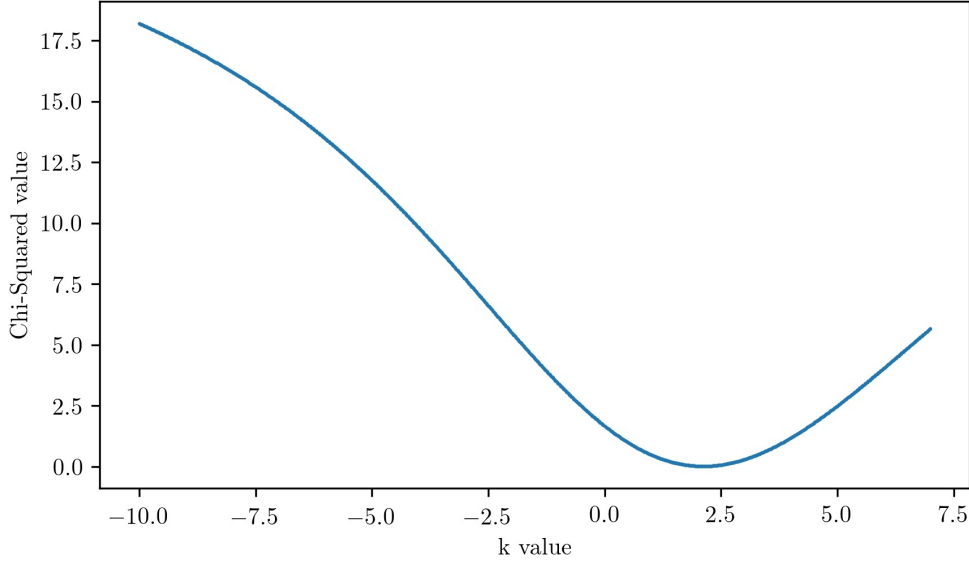


Figure 2: Example plot of the χ^2 function for FIT1. An analogous parabolic shape is maintained in every FIT1 performed in this report.

where instead FIT2 converges properly. Moreover, a strong correlation between the κ parameter and the $B = \log(b)$ parameter has been observed (Fig. 9).

ϵ	a	b	κ
0	$(2.5 \pm 0.5) \times 10^{-3}$	0.5838 ± 0.0002	0.09 ± 0.01
1	$(7.5 \pm 0.5) \times 10^{-3}$	0.6350 ± 0.0002	0.13 ± 0.01
2	$(7.0 \pm 0.5) \times 10^{-3}$	0.6441 ± 0.0004	0.14 ± 0.01
4	$(3.2 \pm 0.5) \times 10^{-2}$	0.7447 ± 0.0004	0.20 ± 0.01
8	$(3.0 \pm 0.5) \times 10^{-1}$	0.9811 ± 0.0006	0.30 ± 0.01
12	$(3.5 \pm 0.5) \times 10^1$	1.824 ± 0.001	0.49 ± 0.01
16	$(5.5 \pm 0.5) \times 10^1$	2.106 ± 0.001	0.54 ± 0.01
32	$(9 \pm 0.5) \times 10^6$	24.81 ± 0.02	1.20 ± 0.01
64	$(8.5 \pm 0.5) \times 10^6$	213.3 ± 0.6	1.88 ± 0.01

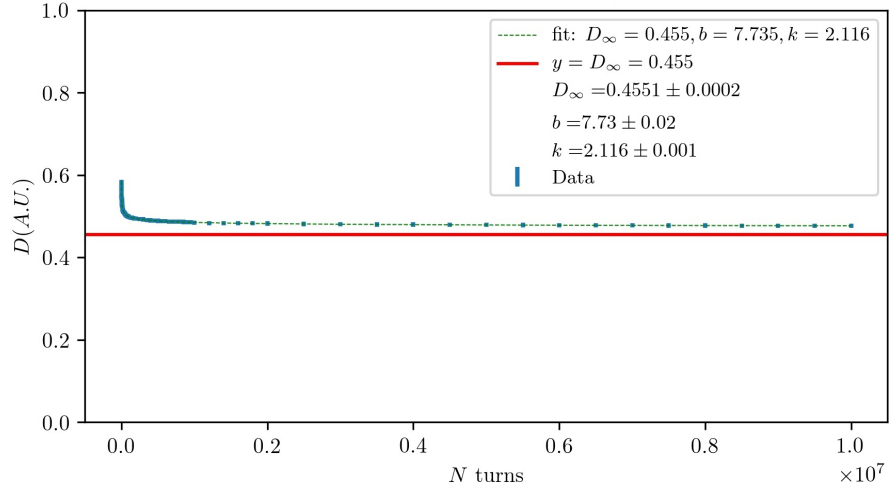
Table 3: FIT2 parameters obtained for the modulated Hénon map.

Fitting with different angular partitions. In order to observe the eventual dependency of the DA decay on the angle in the xy plane, we executed the fitting process over an increasing number of angular partitions of the same size (from a single partition including 100 scanned angles to 10 partitions including 10 scanned angles each). We observed both for FIT1 and FIT2 that for small ϵ values there is a small fluctuation in the fitting parameters while for large values of ϵ the parameters has so high fluctuation that they can start to interfere with the convergence itself of the fitting procedure. The plots of the fitting values for different angular partitions are reported in Fig. 10.

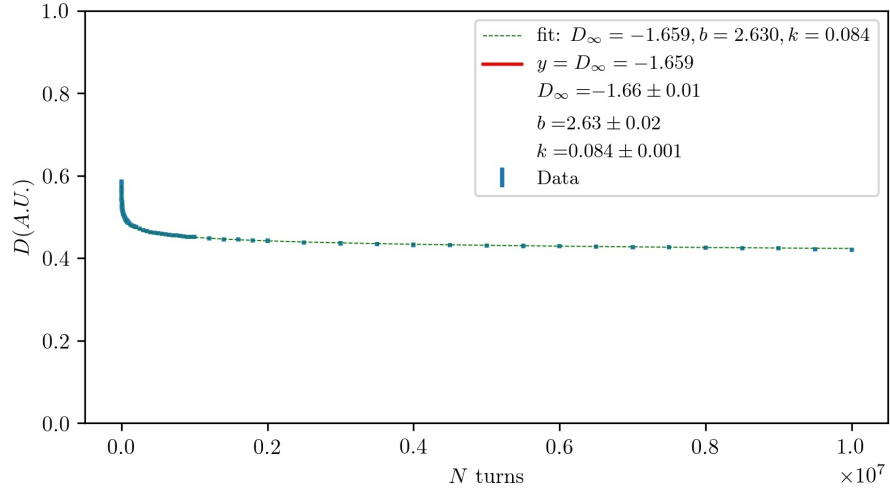
4.1.2 LHC data

We performed the fitting procedure over every realization of the LHC data sample and then we analyzed the distribution of the fit parameters for each analysis group. Moreover, we looked for possible correlations between the various distributions of the fit parameters.

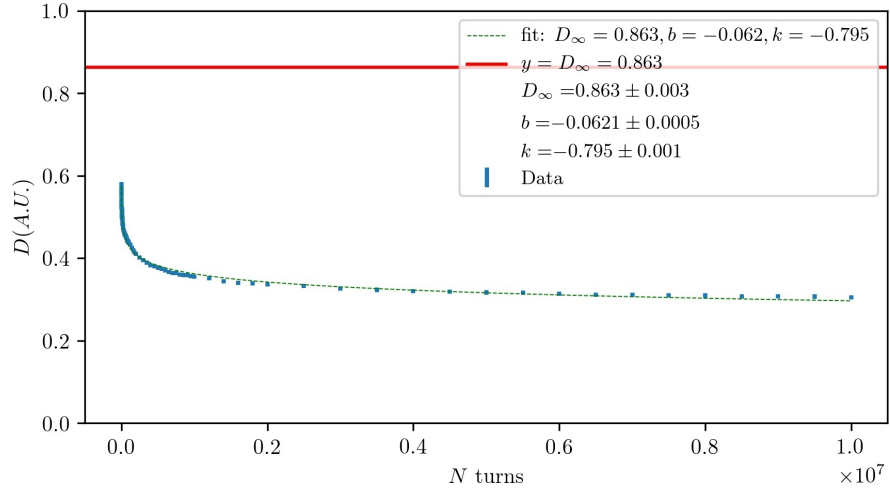
FIT1. In general, most of the data returns negative parameters for FIT1, we can observe that some rare realizations have positive parameters but not in an apparently consistent way. However, the correlation



(a) $\epsilon = 0$



(b) $\epsilon = 12$



(c) $\epsilon = 64$

Figure 3: 3 example plots of FIT1 over 3 different ϵ values. The three observed combinations of signs for the fit parameters (D_∞, b, κ) are represented in these examples.

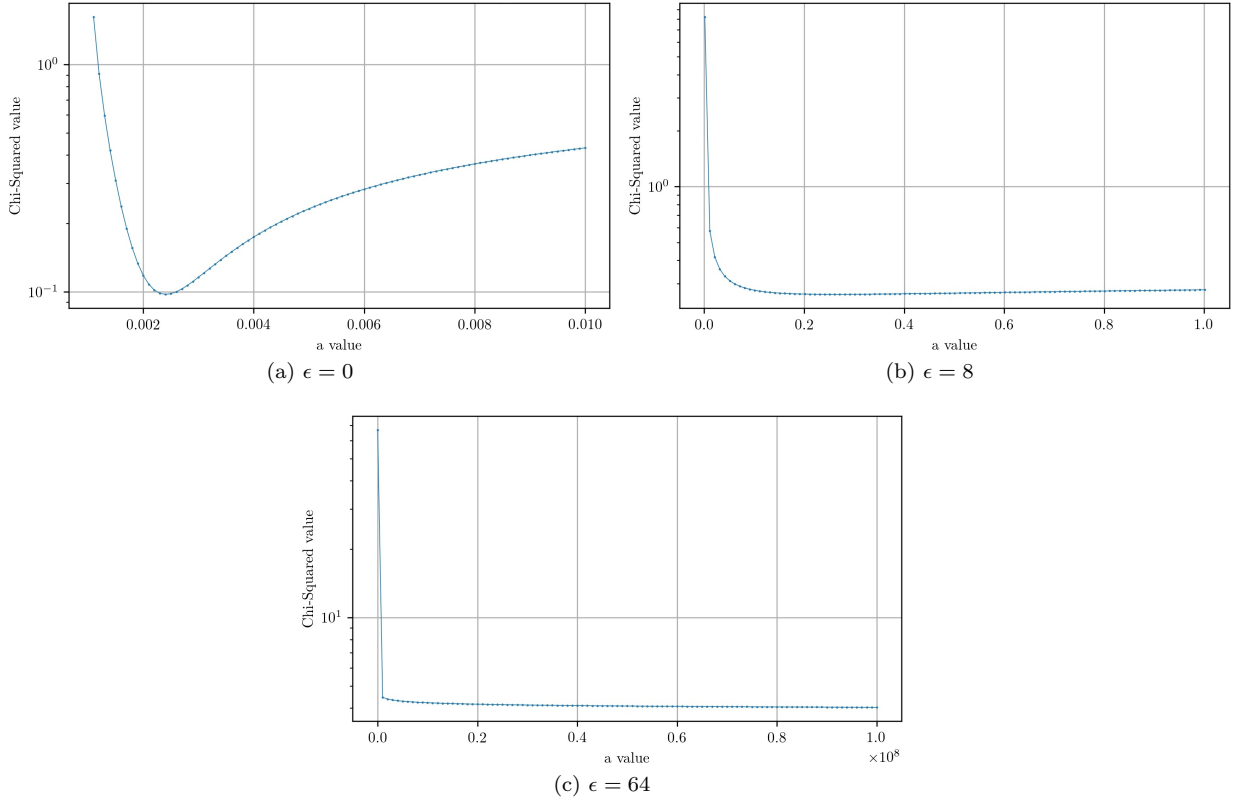


Figure 4: Plots of the χ^2 function of FIT2 over 3 different ϵ values. It can be seen how the function minimum starts to diverge for large values of ϵ .

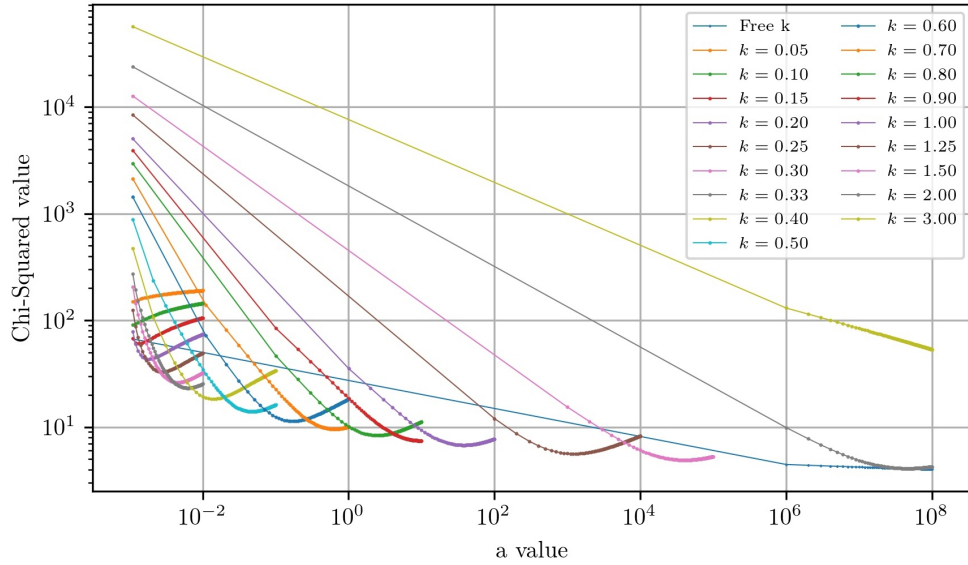
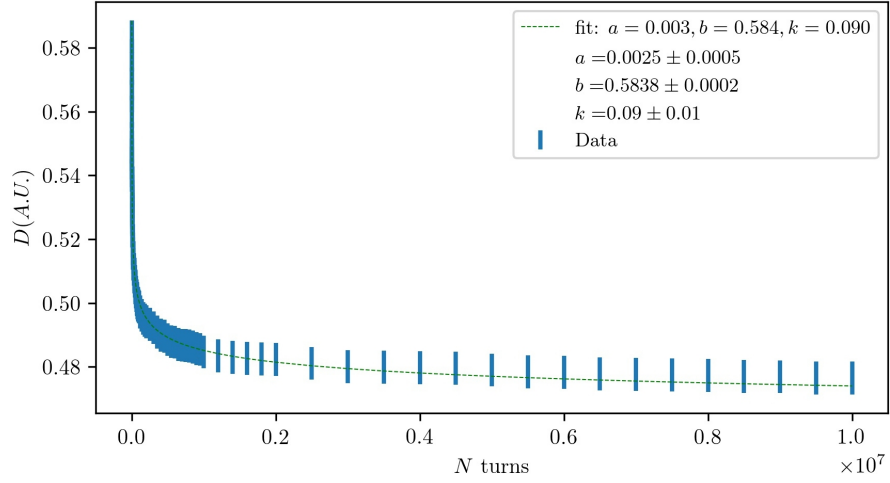
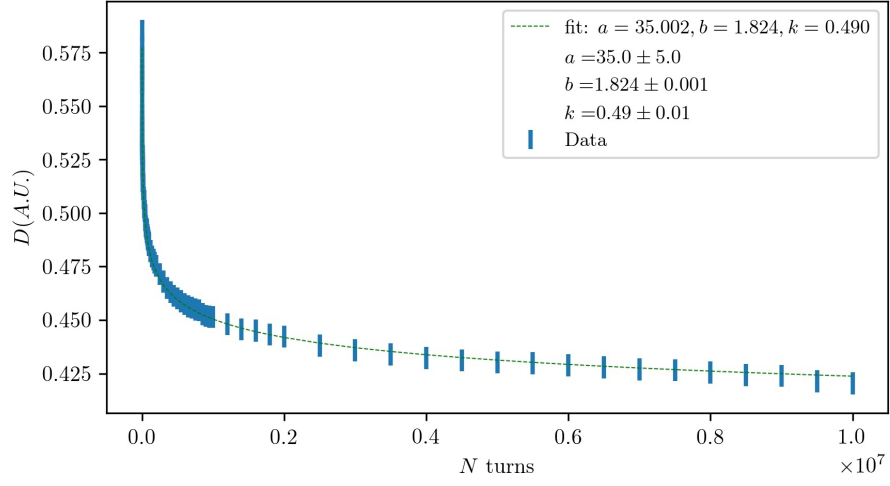


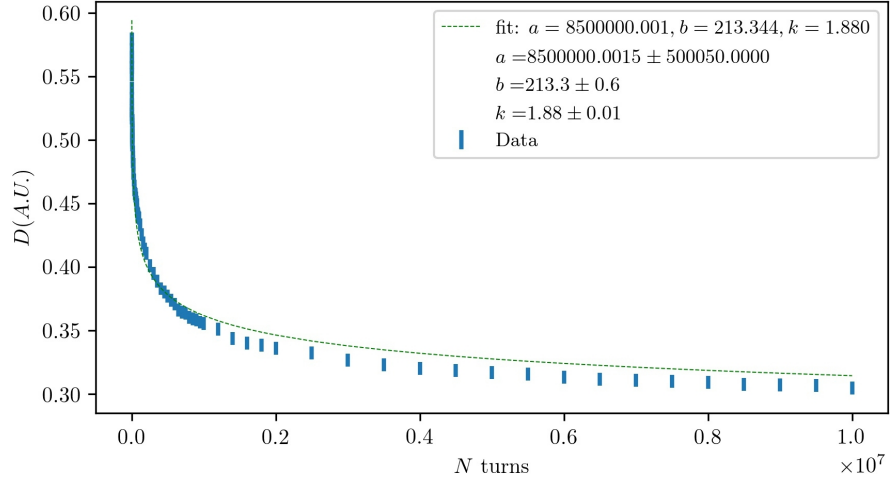
Figure 5: Evolution of the χ^2 function of FIT2 for different values of κ and a . It can be seen how in this situation (where a standard scan on a only with free κ diverges) we have that for κ over a certain value the χ^2 function does not have a minimum anymore.



(a) $\epsilon = 0$

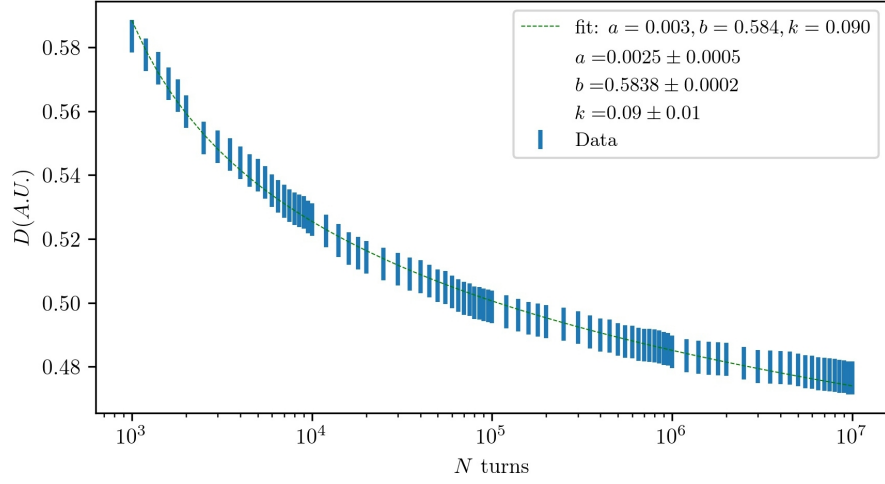


(b) $\epsilon = 12$

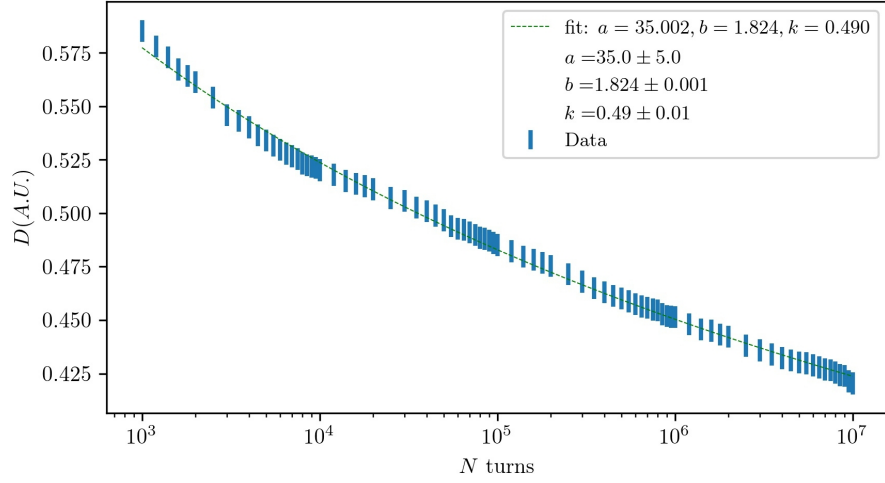


(c) $\epsilon = 64$

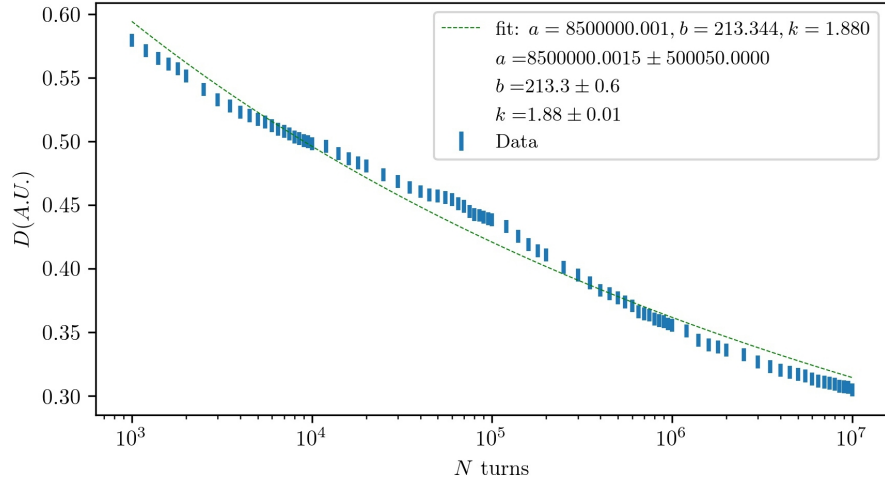
Figure 6: 3 example plots of FIT2 over 3 different ϵ values. It can be seen how the quality degrades for large ϵ values.



(a) $\epsilon = 0$



(b) $\epsilon = 12$



(c) $\epsilon = 64$

Figure 7: The 3 plots in Fig. 6 in logarithmic scale for the number of turns. It can be seen how there is a slight change of concavity on the data distribution for larger values of ϵ and how this affects the performance of FIT2.

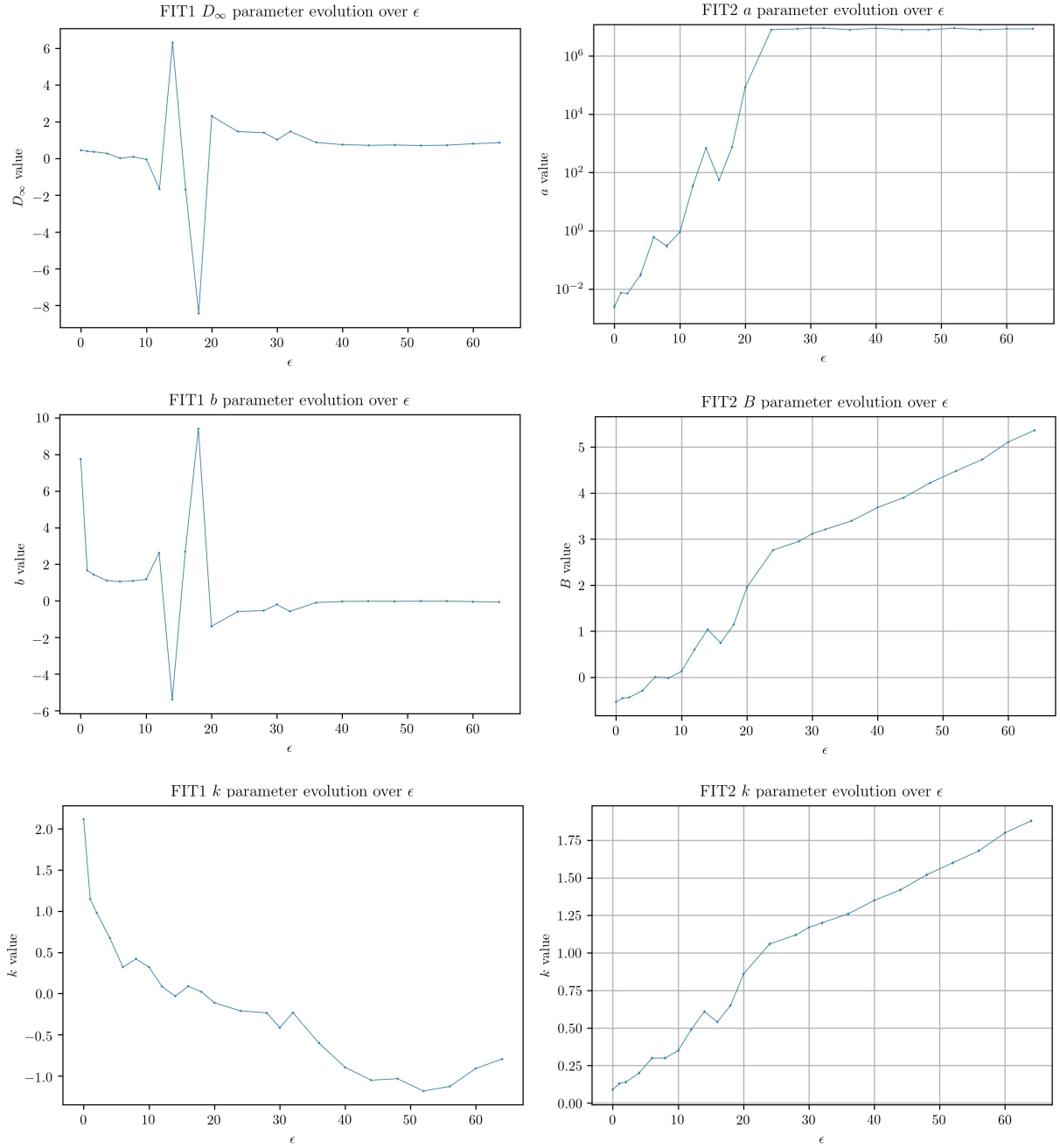


Figure 8: Comparison of FIT1 and FIT2 parameters over every analyzed ϵ value. It can be observed how there are some cases in which FIT2 does not cap the a value while FIT1 does not have all positive values for its three parameters, meaning that FIT2 had a better performance in describing the Hénon map progression. A strong correlation between B and κ values from FIT2 can also be observed.

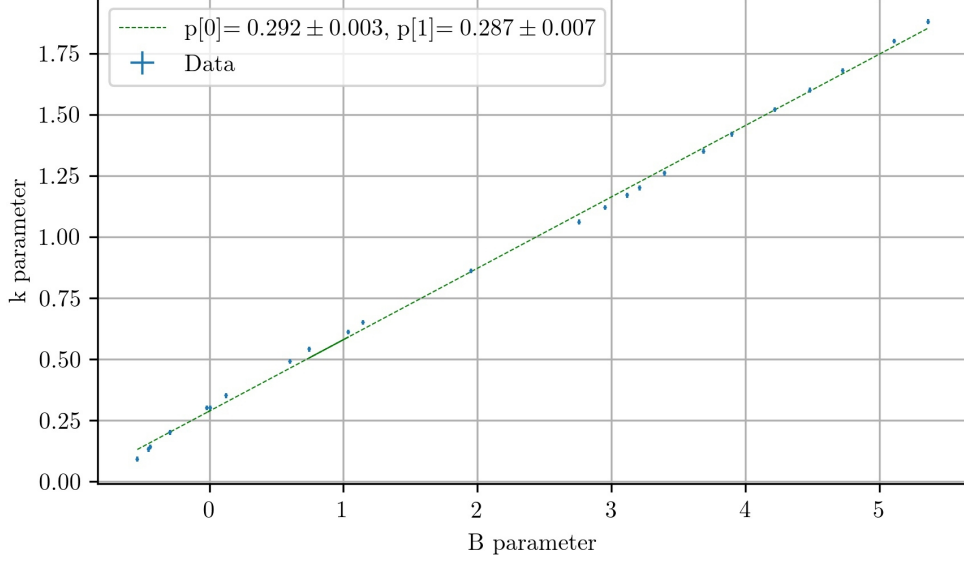


Figure 9: Correlation of B and κ for every Hénon map analyzed (ϵ values between 0 and 64). Fit formula is $y = p[0]x + p[1]$.

plots seem to show some consistent logarithmic curves that will need further investigations. All the results produced are reported in Fig. 11 and Fig. 12. Since all the correlation and distribution plots shows mostly the same behavior, we report just two of them in Fig. 13 and Fig. 14.

FIT2. As it can be predicted by the concavity of the data displayed considering the logarithm of the number of turns, most of the FIT2 instances do not converge with a scan only on the a parameter and require a double scan over κ and a . Still, more ‘successful’ fits were observed: in certain realizations where FIT1 did not return positive parameters, FIT2 converged with a single scan over the a parameter. This suggests that FIT2 actually might have better performances over FIT1 in describing the DA decay. As for the correlation plots, some of the insights seen in the Hénon map analysis can be also found here, but still, it needs more investigations. All the results produced are reported in Fig. 15, Fig. 16. Since all the correlation and distribution plots shows mostly the same behavior, we report just two of them in Fig. 17, Fig. 18.

4.2 Intensity losses

As stated in Ref. [3], a possible approach for describing the intensity loss inside an accelerator is by going through the definition of DA and defining an initial beam distribution.

We start assuming for instance an initial 2D Gaussian distribution for the beam

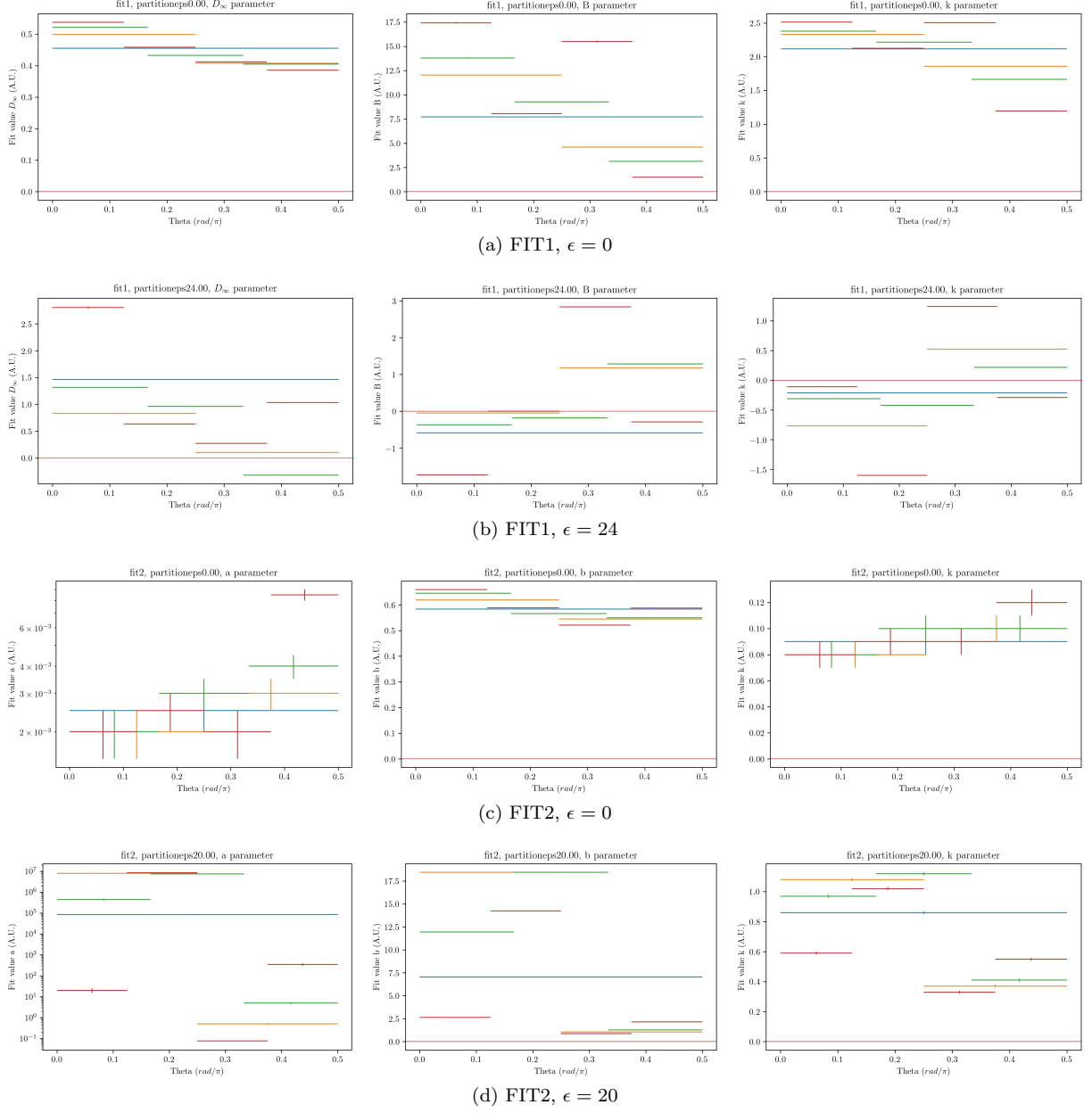
$$\rho_G(x, y) = \frac{1}{2\pi\sigma_x\sigma_y} \exp \left[- \left(\frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2} \right) \right],$$

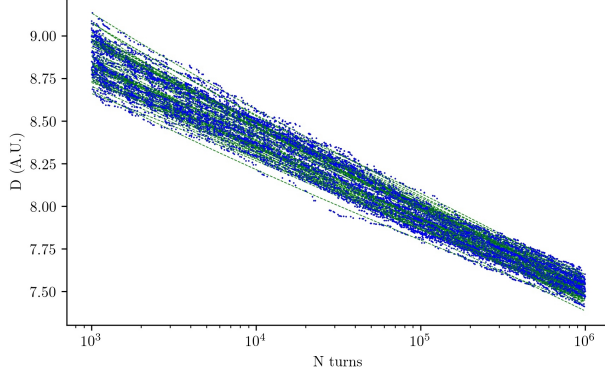
then, we state that, after a discrete number of turns N , every particle with amplitude beyond $D(N)$ is lost. Therefore, if we switch to polar coordinates and express $D(N)$ itself in units of sigma, we have that the evolution of the beam intensity $I(N)$ can be written as

$$\frac{I(N)}{I_1} = 1 - \int_{D(N)}^{+\infty} e^{-(r^2/2)} r dr = 1 - e^{-D^2(N)/2}$$

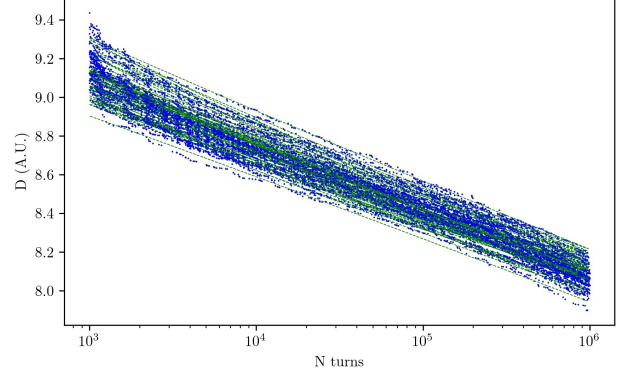
where $I_1 = I(1)$ and it is assumed that $\lim_{N \rightarrow 1} D(N) = +\infty$.

In order to evaluate how our DA fits are capable to estimate the intensity loss, we computed the actual intensity loss of the Hénon map for the various analyzed ϵ values. We considered several different Gaussian distributions with $\sigma_x = \sigma_y = \sigma$ and σ values between 0.20 (A.U.) and 1.00 (A.U.). We then plotted both the actual intensity loss and the estimated loss from the various DA fitted scaling functions.

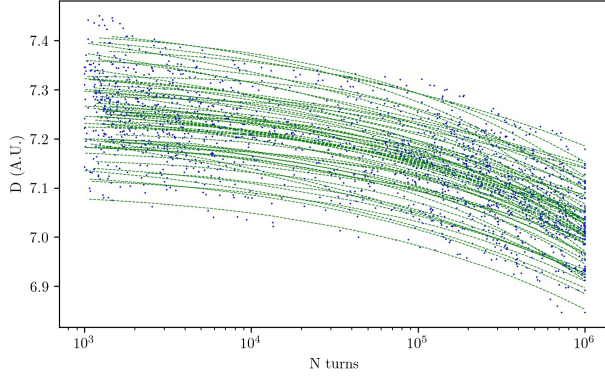




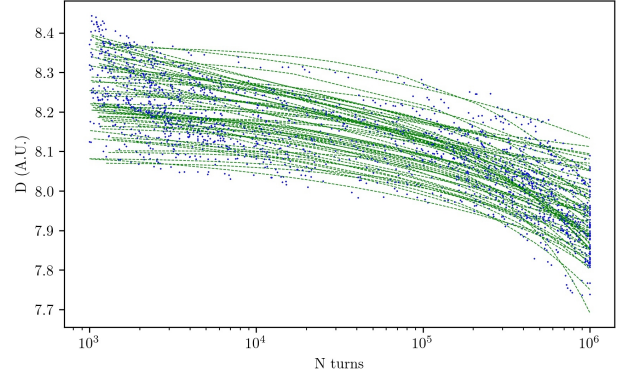
(a) All angles analyzed, no IR correction.



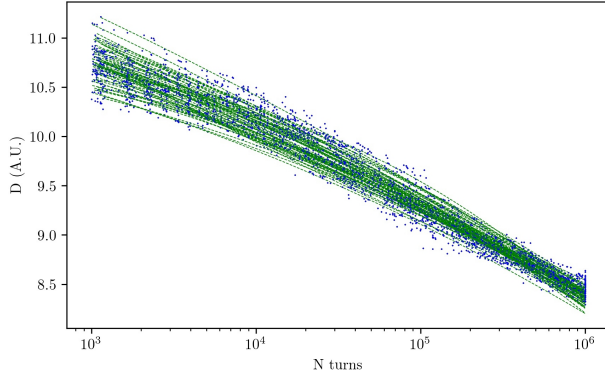
(b) All angles analyzed, IR correction present.



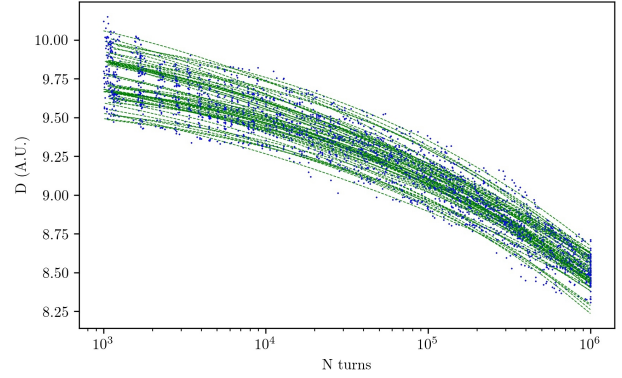
(c) Only x near angles analyzed, no IR correction.



(d) Only x near angles analyzed, IR correction present.



(e) Only y near angles analyzed, no IR correction.



(f) Only y near angles analyzed, IR correction present.

Figure 11: FIT1 execution over LHC Beam 1 data. Blue dots are the data, green lines are the various fits.

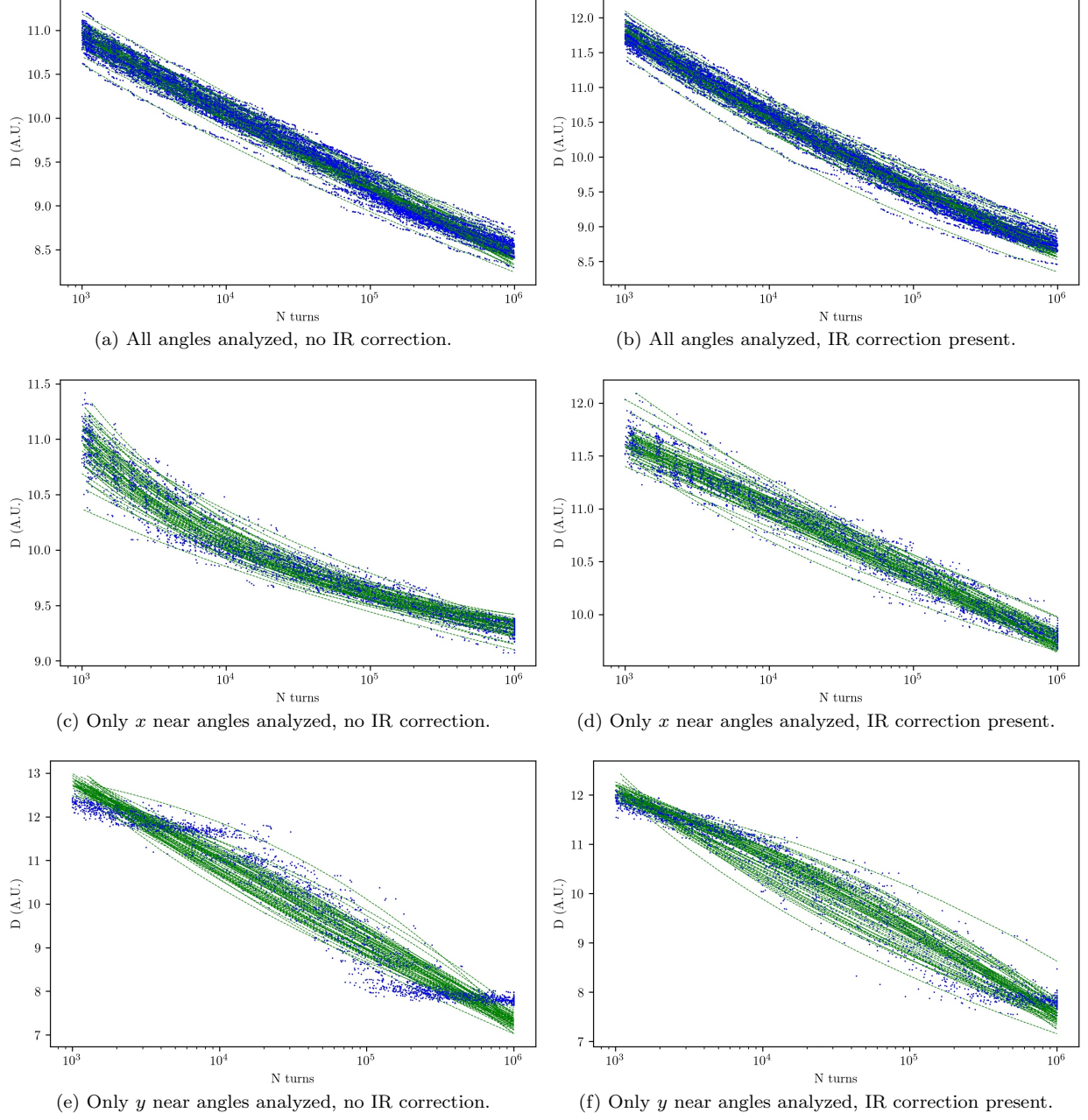


Figure 12: FIT1 execution over LHC Beam 2 data. Blue dots are the data, green lines are the various fits.

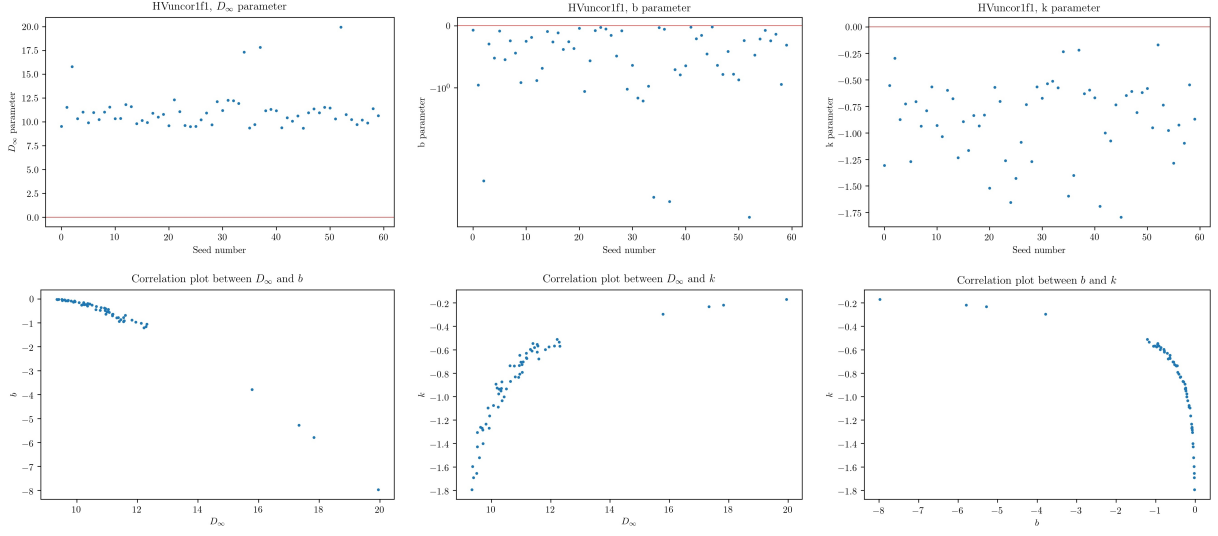


Figure 13: FIT1 parameter analysis for the realizations of LHC Beam 1 without IR correction, all angles considered.

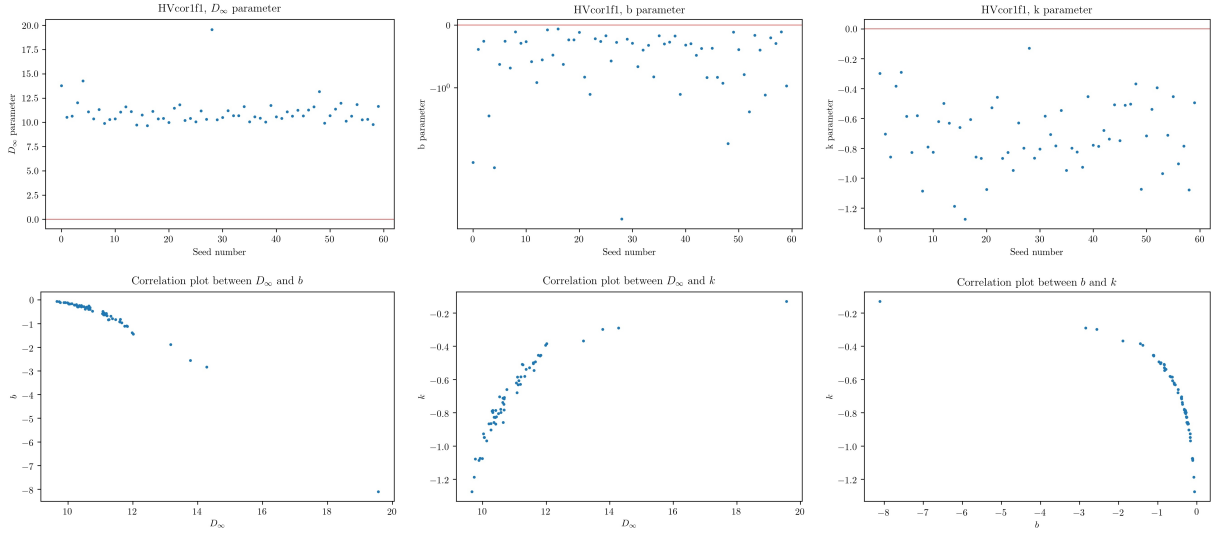
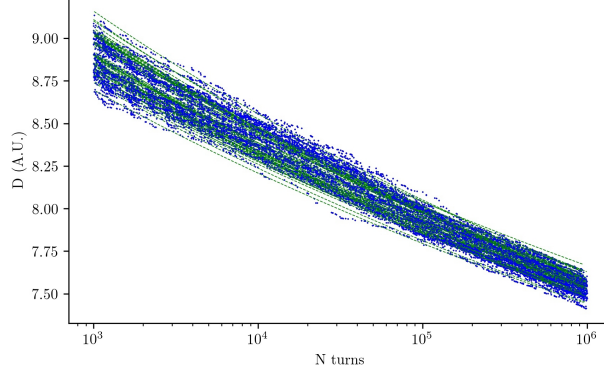
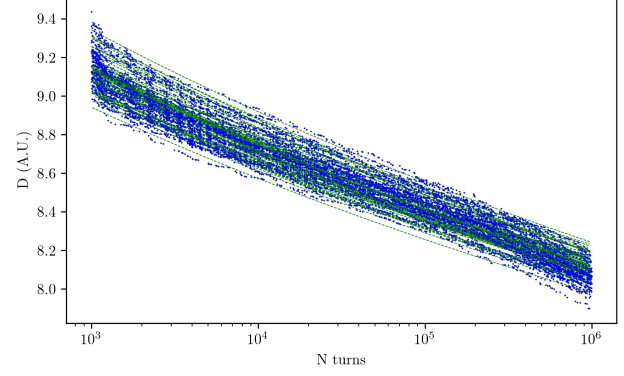


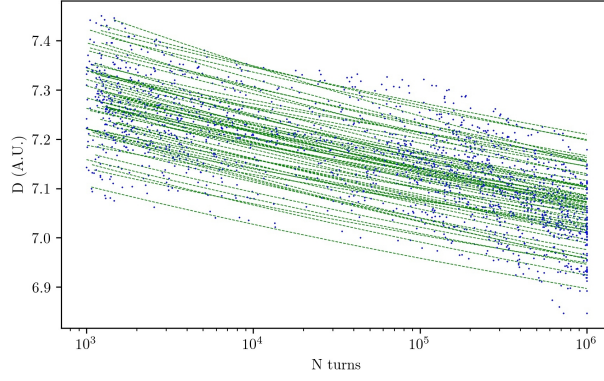
Figure 14: FIT1 parameter analysis for the realizations of LHC Beam 1 with IR correction, all angles considered.



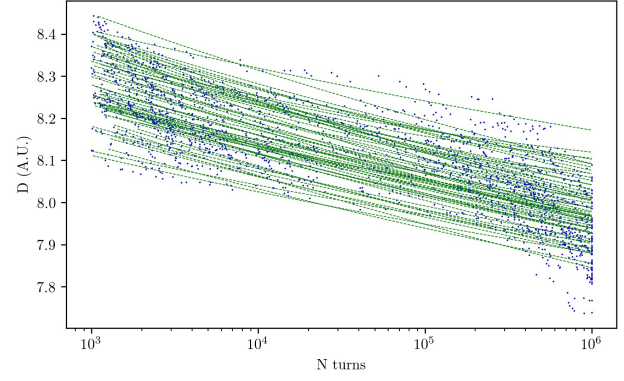
(a) All angles analyzed, no IR correction.



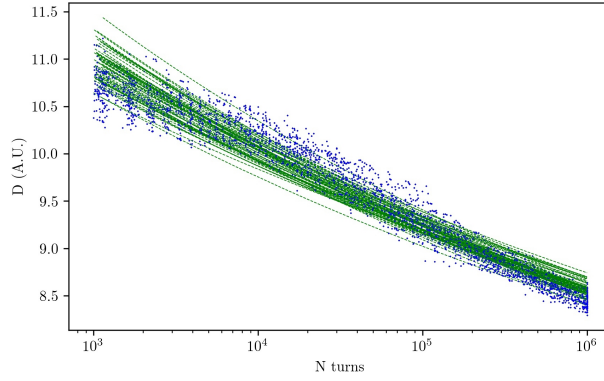
(b) All angles analyzed, IR correction present.



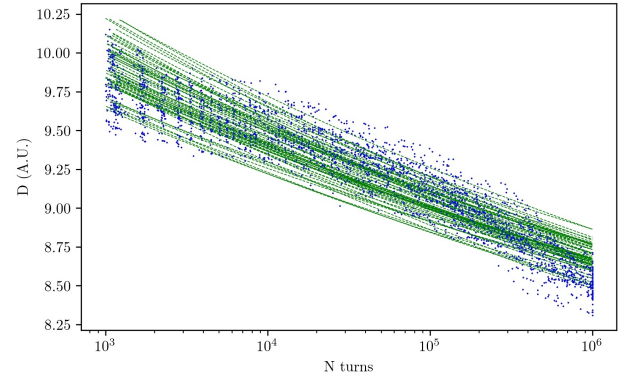
(c) Only x near angles analyzed, no IR correction.



(d) Only x near angles analyzed, IR correction present.

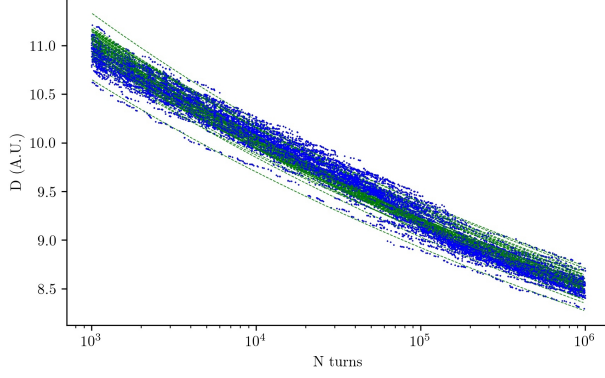


(e) Only y near angles analyzed, no IR correction.

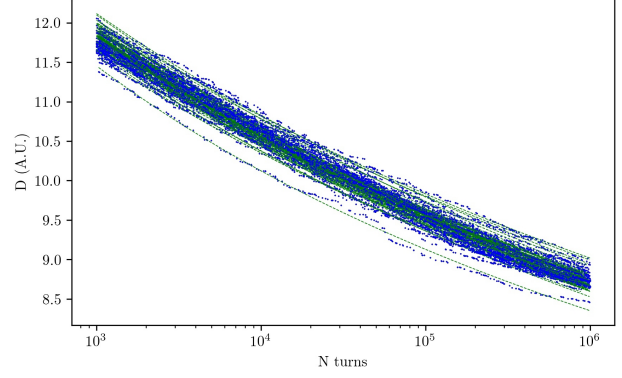


(f) Only y near angles analyzed, IR correction present.

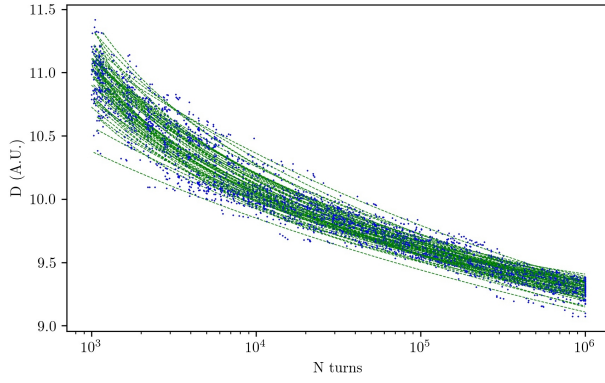
Figure 15: FIT2 execution over LHC Beam 1 data. Blue dots are the data, green lines are the various fits.



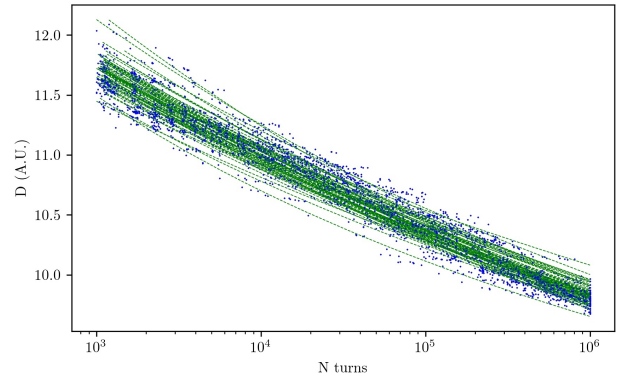
(a) All angles analyzed, no IR correction.



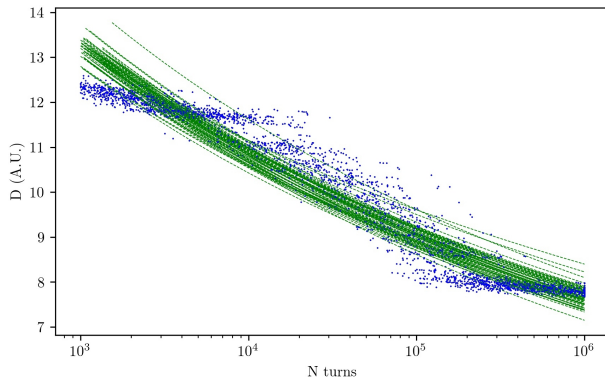
(b) All angles analyzed, IR correction present.



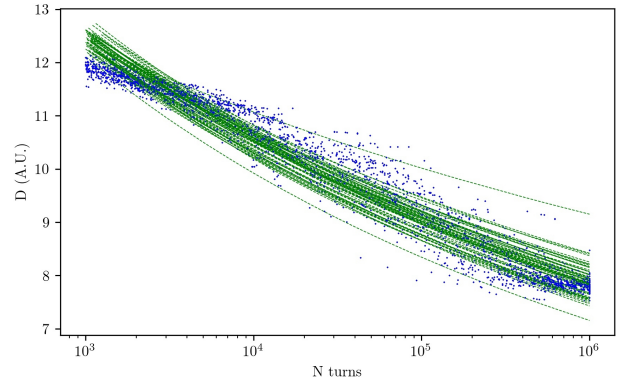
(c) Only x near angles analyzed, no IR correction.



(d) Only x near angles analyzed, IR correction present.



(e) Only y near angles analyzed, no IR correction.



(f) Only y near angles analyzed, IR correction present.

Figure 16: FIT2 execution over LHC Beam 2 data. Blue dots are the data, green lines are the various fits.

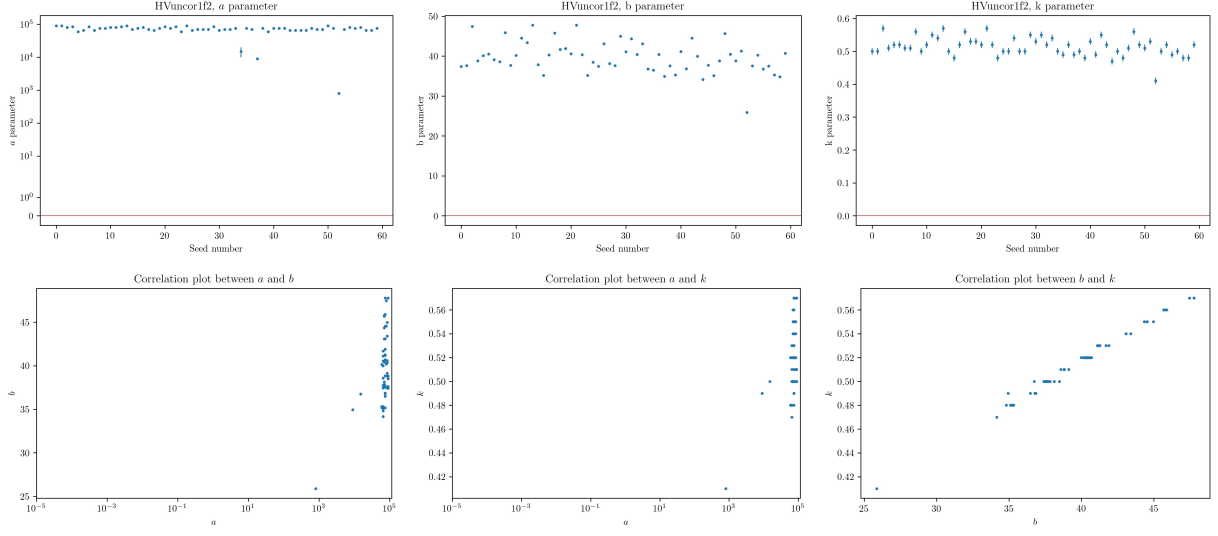


Figure 17: FIT2 parameter analysis for the realizations of LHC Beam 1 without IR correction, all angles considered.

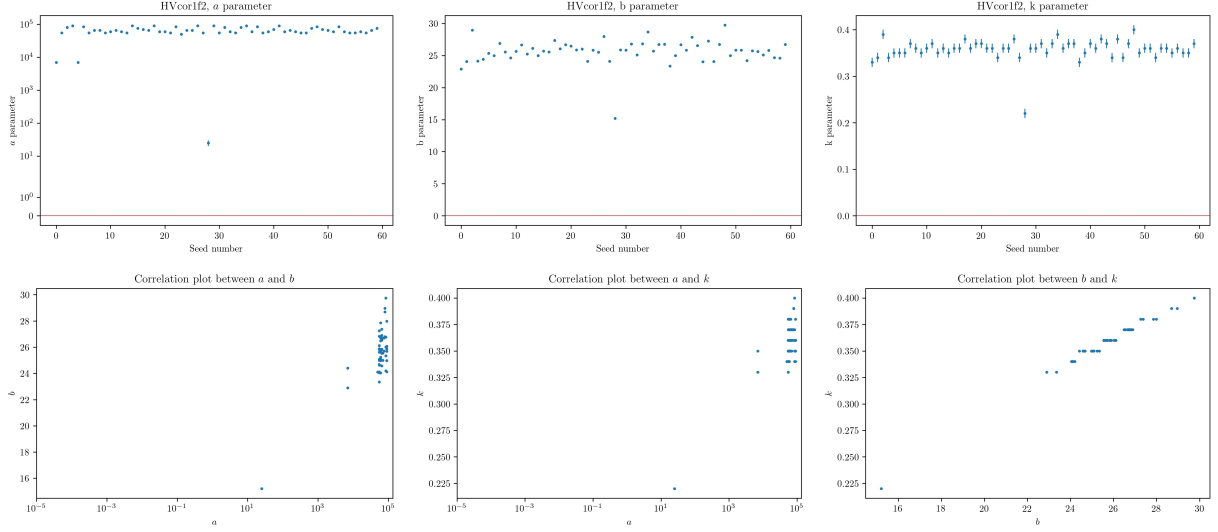


Figure 18: FIT2 parameter analysis for the realizations of LHC Beam 1 with IR correction, all angles considered.

By executing then the fitting procedure on more than one angular partition, we tried to observe if describing the DA evolution on smaller angle intervals could lead to better intensity loss estimate. We indeed observed improvements by increasing the number of partitions for small values of σ but, as σ increases, this improvement appears to be mostly lost (Fig. 19 for FIT1, Fig. 20 for FIT2). This can be observed better from the plots of the L^2 -norm evolution for increasing number of partitions (Fig. 22 for FIT1, Fig. 22 for FIT2).

As for the error measure of such estimates, we have two sources of error: the uncertainty on the computed fitting parameters and the uncertainty over the computed simulation data. Further research is required for evaluating how these two contributions actually adds up to each other. Therefore, the presented plots have no error bars for this reason.

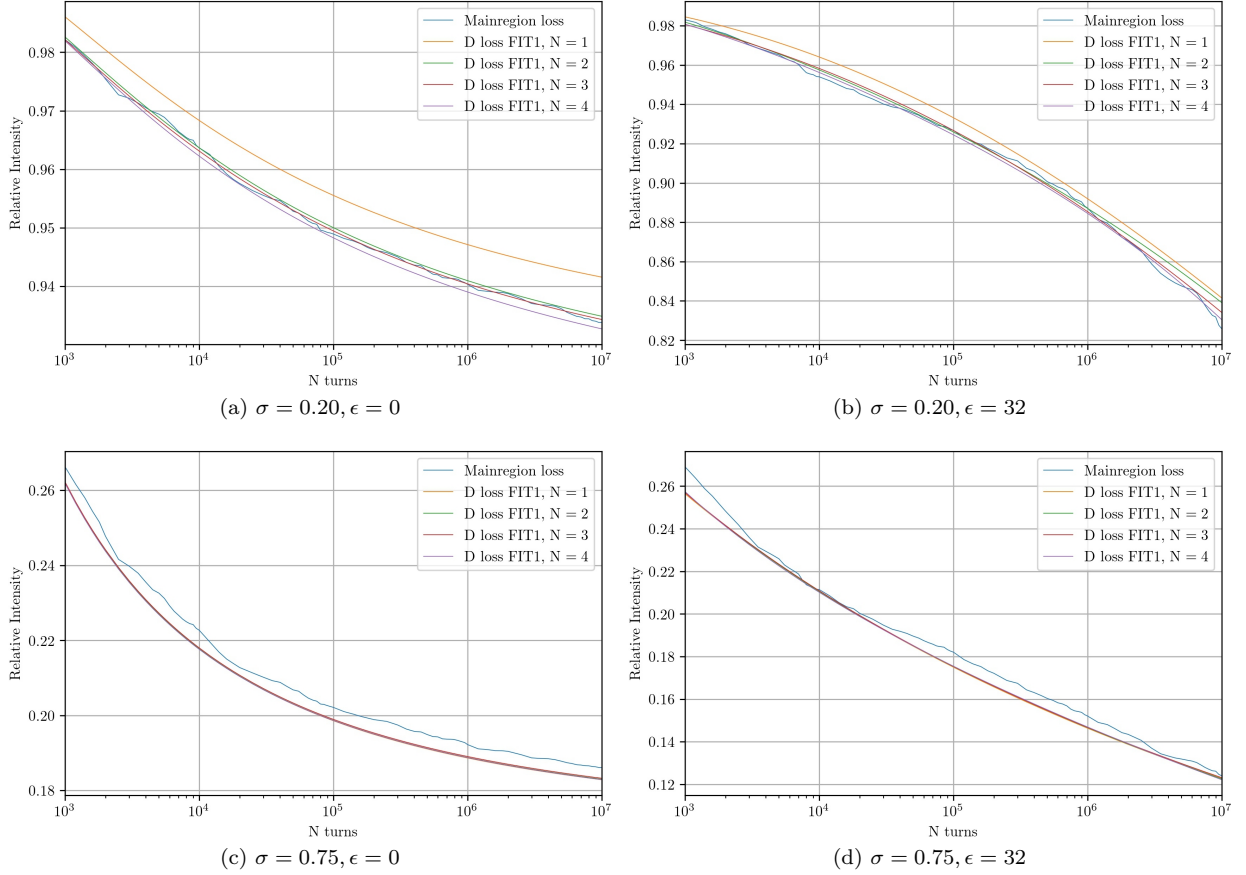
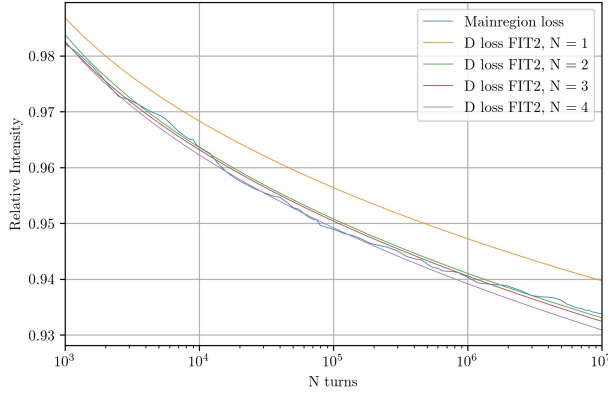
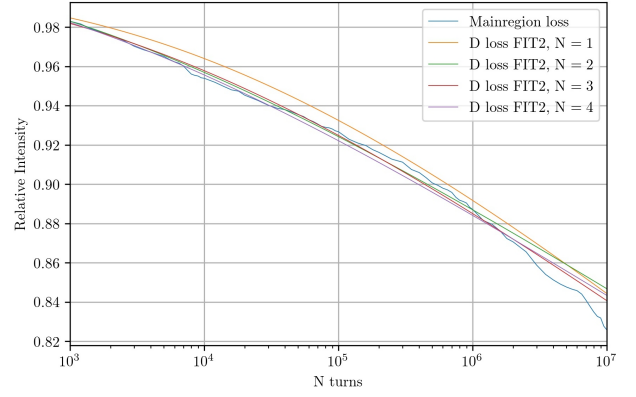


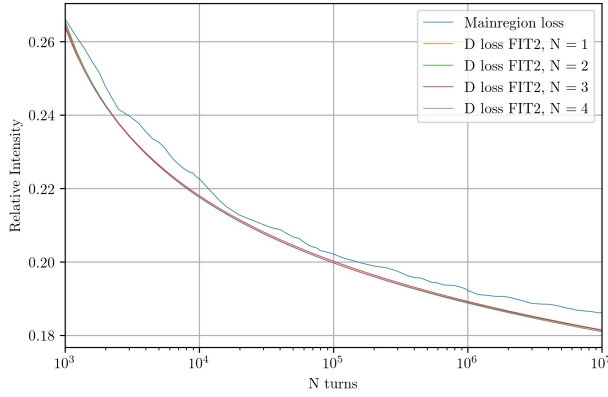
Figure 19: Comparison of FIT1 relative intensity loss estimation with actual Hénon map computed intensity loss.



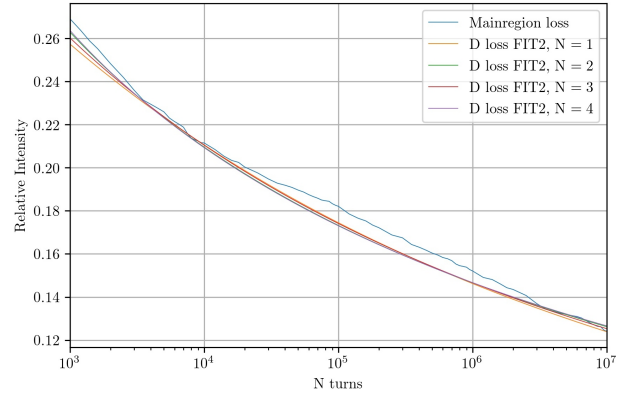
(a) $\sigma = 0.20, \epsilon = 0$



(b) $\sigma = 0.20, \epsilon = 32$

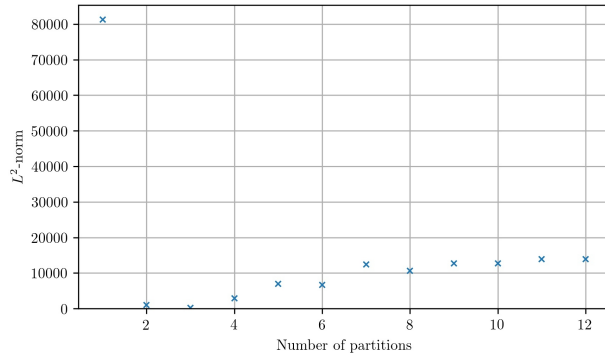


(c) $\sigma = 0.75, \epsilon = 0$

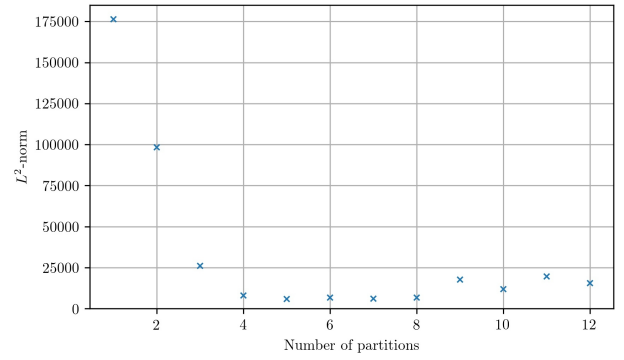


(d) $\sigma = 0.75, \epsilon = 32$

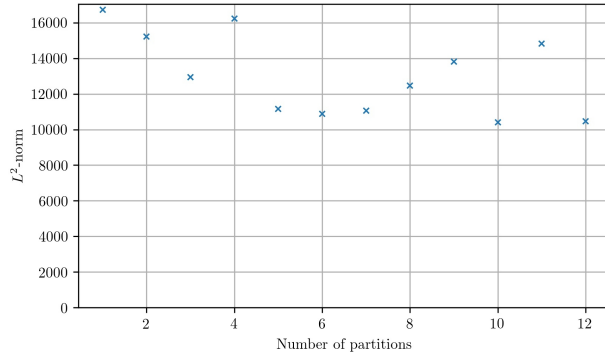
Figure 20: Comparison of FIT2 relative intensity loss estimation with actual Hénon map computed intensity loss.



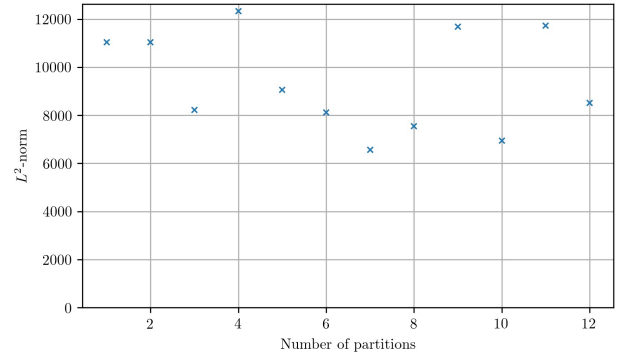
(a) $\sigma = 0.20, \epsilon = 0$



(b) $\sigma = 0.20, \epsilon = 32$

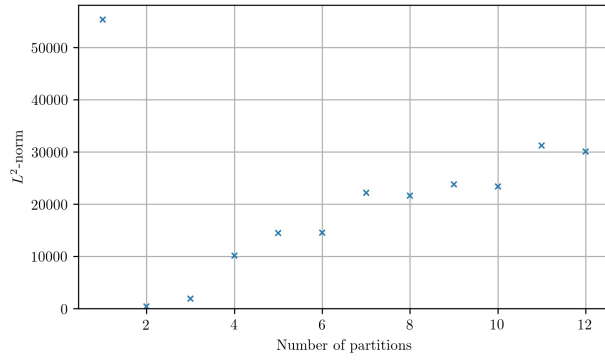


(c) $\sigma = 0.75, \epsilon = 0$

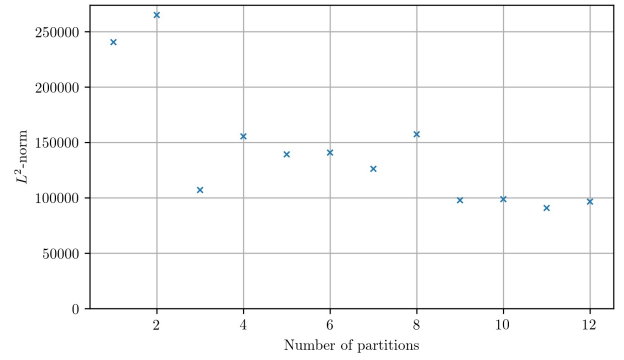


(d) $\sigma = 0.75, \epsilon = 32$

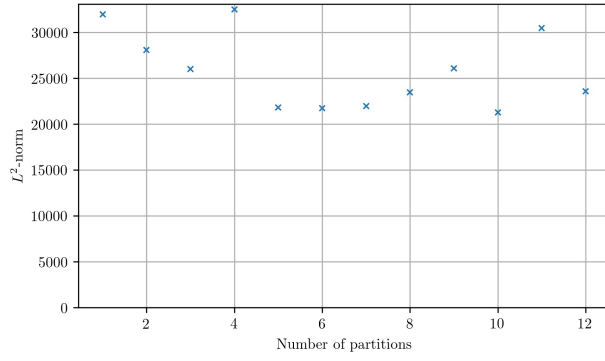
Figure 21: L^2 -norm evolution in intensity loss estimation for increasing number of angular partitions used for DA computation and fitting procedure with FIT1.



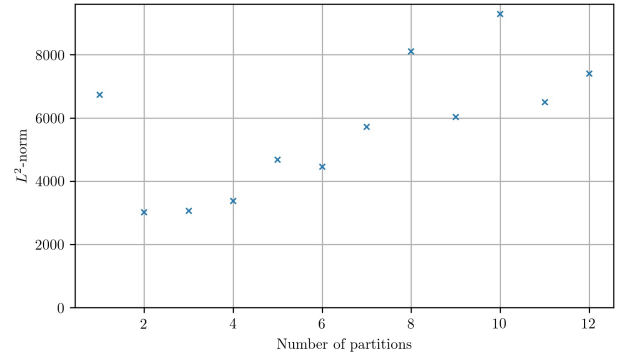
(a) $\sigma = 0.20, \epsilon = 0$



(b) $\sigma = 0.20, \epsilon = 32$



(c) $\sigma = 0.75, \epsilon = 0$



(d) $\sigma = 0.75, \epsilon = 32$

Figure 22: L^2 -norm evolution in intensity loss estimation for increasing number of angular partitions used for DA computation and fitting procedure with FIT2.

5 Conclusions and outlook

In this study we have presented a first comparison between two scaling laws for describing DA variation over time and, consequently, intensity variation over time for an hadron storage ring. Most of the results presented in [2] about FIT1 have been reproduced and used to evaluate the performance of FIT2, that on first sight seems more promising, as more data samples could be properly described than FIT1.

Moreover, analysis of DA evolution over different angular partition shows indeed strong differences in the resulting fit parameter, further studies in that direction are therefore needed. The promising results for the intensity loss estimation encourage more analysis on this direction.

Further study is also to be performed on analyzing the correlations between the fit parameters, in order also to understand what exactly breaks the convergence on the a parameter for FIT2. Understanding that could lead to more proper fitting procedures for the LHC data.

6 Acknowledgments

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