

CERN Summer Student Report: Testing Lorentz invariance emergence in Ising Model using lattice Monte Carlo simulations

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1 Theoretical Introduction

All measurements performed so far at the observable energy scales show no violation of Lorentz invariance. However, it is yet impossible to check experimentally whether this symmetry holds at high energies such as the Planck scale. Recently, theories of gravitation with Lorentz violation, known as Horava-Lifshitz gravity [1, 2] have gained significant attention by treating Lorentz symmetry as an emergent phenomenon. A Lifshitz type theory assumes an anisotropic scaling between space and time weighted by some critical exponent. In order for these theories to be viable candidates for quantum gravity description of the nature, Lorentz symmetry needs to be recovered at low energies.

A simple model, which allows for a non-perturbative study of a simplified version of the above described scenario, is a system of two interacting 2+1D Ising models, where one of them features an anisotropic behaviour between spatial and temporal directions. Monte Carlo simulations allow us to study this model non-perturbatively.

The Ising Model is a model of ferromagnetism in statistical physics. [3] It consists of a lattice with spins at the lattice points, with the Hamiltonian of an individual lattice site i : $H_i = \sum_{j_{nn}} S_i S_j$, where the sum runs over the nearest neighbors of S_i , with values of S_i restricted to a single axis (either parallel or antiparallel to it) and values $+1$ or -1 . In this project, we are considering the spin system consisting of two 3D Ising models and their corresponding interaction. Different coefficients are associated with the interaction in corresponding spatial/temporal direction within the individual Ising system: $c_k, d_k; k = 0, 1, 2$, and h is the coefficient of interaction between spins on the corresponding systems. The action of this interacting system then reads:

$$S = \sum_{k=0}^2 c_k \sum_{\langle i,j \rangle} S_i^k S_j^k + \sum_{k=0}^2 d_k \sum_{\langle i,j \rangle} S_i'^k S_j'^k + h \sum_i S_i S_i', \quad (1)$$

Here, S_i and S_i' represent the values of spin on the position i on the first and the second individual Ising

system, respectively, $\langle i, j \rangle$ represents summation over the values S_i in the first lattice (or S'_i in the second lattice) and their nearest neighbours S_j (or S'_j) in the direction k and sum in the last (interaction) term goes over all the spin positions i on both 3D lattices. The Hamiltonian H_i of a specific lattice site reads:

$$H_i = - \sum_{k=0}^2 c_k \sum_{j_{nn}} S_i^k S_j^k - \sum_{k=0}^2 d_k \sum_{j_{nn}} S_i'^k S_j'^k + h S_i S'_i \quad (2)$$

j_{nn} denotes the nearest neighbours of S_i (or S'_i) in the k direction. Lorentz invariance breaking is achieved in one of the interacting Ising systems by introducing anisotropy in one (or more) spacetime directions. The main goal of the project was to determine how the interaction between separate Ising systems and anisotropy in one of them (i.e. different values of coefficients h and d_3) affect the point of second-order phase transition in the described model.

2 Relevant Physical Quantities

The observables calculated using the Monte Carlo simulation ($\langle E \rangle$, $\langle E^2 \rangle$, $\langle M \rangle$, $\langle |M| \rangle$ and $\langle M^2 \rangle$) have been defined in analogy with the 2D Ising Model[3]. Mean magnetisation per lattice site is calculated in the following way:

$$\langle M \rangle = \frac{1}{N} \sum_{\alpha}^N M(\alpha) \quad (3)$$

where α runs over all sites of the double lattice (i.e. magnetisation is proportional to the 'excess' number of spins pointing up or down). $\langle |M| \rangle$ and $\langle M^2 \rangle$ are calculated in the similar way.

By making use of the Hamiltonian (2), it is possible to obtain the energy per lattice site:

$$\langle E \rangle = \frac{1}{2N} \langle \sum_i H_i \rangle \quad (4)$$

and in a similar way $\langle E^2 \rangle$.

Heat capacity (C) and susceptibility (χ) have also been evaluated, since they are expected to show a marked fluctuation at the Curie temperature (the temperature at which the phase transition from ferromagnetic to paramagnetic phase occurs).

$$C = \frac{\partial E}{\partial T} = \frac{\langle E^2 \rangle - \langle E \rangle^2}{k_b T^2} \quad (5)$$

$$\chi = \frac{\partial M}{\partial T} = \frac{\langle M^2 \rangle - \langle M \rangle^2}{k_b T^2} \quad (6)$$

The Binder cumulant was also calculated. It was introduced by Karl Binder in the context of finite size scaling[3], and it is used to determine the critical exponent and critical temperature.

$$U_L = 1 - \frac{\langle M^4 \rangle_L}{3 \langle M^2 \rangle_L^2} \quad (7)$$

In the thermodynamic limit, when $L \rightarrow \infty$, the Binder cumulant behaves as follows:

$$\begin{cases} U_L \rightarrow 0, & \text{if } T > T_c \\ U_L \rightarrow \frac{2}{3}, & \text{if } T < T_c \end{cases}$$

i.e. the function has a discontinuity at T_C . The point of intersection of Binder cumulants for different lattice sizes depends rather weakly on those sizes, allowing for an estimate of the critical temperature.

3 The Simulation - Monte Carlo Method

The statistical properties of the system are governed by the partition function: $Z = \sum_C e^{-\beta E(C)}$, where the sum runs over all possible configurations of the system (with the energy of each being $E(C)$). Any attempt to obtain the partition function by direct summation would prove to be futile, since the number of terms would amount to 2^N , where N represents the number of spins (L^2 for the 2D Ising Model, L^3 for the 3D Ising Model, and $2L^3$ for the target Double Ising Model). Therefore, one uses *importance sampling* techniques - i.e. sampling of the phase space made out of a set of configurations that contribute the most to the averages, avoiding the need to generate all configurations. *Monte Carlo simulations* [4] are one of the possible ways to achieve this.

The basic idea of a Monte Carlo simulation is to advance the system from an initial configuration C_i to a new trial configuration, C_{i+1}^t , and then decide if the new trial configuration is accepted ($C_{i+1} = C_{i+1}^t$) or declined ($C_{i+1} = C_i$). The resulting sequence of configurations ($i = 1, 2, \dots, M$) forms a *Markov chain* (i.e. a chain characterized by 'memorylessness' - the probability distribution of the next state depends only on the current one, and not on the events that preceded it). If $P(A)$ denotes the probability for the system to be in state A at time t , and $W(A \rightarrow B)$ the rate of transition from A to B , this Markov process can be described by the master equation:

$$\frac{dP(A)}{dt} = \sum_{A \neq B} [P(B)W(B \rightarrow A) - P(A)W(A \rightarrow B)] \quad (8)$$

$$W(A \rightarrow B) \neq 0, \sum_B W(A \rightarrow B) = 1, \forall A, B \quad (9)$$

Furthermore, one can decompose the transition probability into trial proposition probability and acceptance probability, so that $W(A \rightarrow B) = T(A \rightarrow B) \cdot A(A \rightarrow B)$.

One of the two natural requirements to impose on this Markov Chain is the property of ergodicity - the time average of a sequence of processes is equal to the ensemble average, i.e., any configuration with a non-zero Boltzmann weight should be reachable in a finite number of steps from an initial configuration C_0 with a non-zero Boltzmann weight.

The other requirement, a strong, but not necessary, is the one of *detailed balance*: for every pair of states A, B , the following condition must be satisfied: $P_A \cdot T(A \rightarrow B) \cdot A(A \rightarrow B) = P_B \cdot T(B \rightarrow A) \cdot A(B \rightarrow A)$. The meaning of this requirement is apparent upon closer inspection of (8), the condition $dP(A)/dt = 0$ (which prevents the Markov Chain from being trapped inside a cycle) is valid if every two terms in the summation on the right hand side cancel.

The trial proposition and acceptance probabilities need to be chosen so that the probability of sampling of a configuration A is equal to its Boltzmann weight: $P_A = \frac{e^{-\beta E_A}}{Z}$. Detailed balance condition then reads:

$$\frac{T(A \rightarrow B) \cdot A(A \rightarrow B)}{T(B \rightarrow A) \cdot A(B \rightarrow A)} = \frac{P_A}{P_B} = e^{-\beta(E_A - E_B)}$$

Local MC algorithm: Metropolis. One of the ways to realise detailed balance condition is to propose a small change in configuration A , resulting in a new configuration B , so that the reverse process is equally likely - so that the condition $T(A \rightarrow B) = T(B \rightarrow A)$ holds for all pairs A, B . As for the acceptance probabilities, simulations are fastest if one of them (either $A(A \rightarrow B)$ or $A(B \rightarrow A)$) is equal to 1. These conditions are fulfilled when using *Metropolis Algorithm*, in which the acceptance probability is set to:

$$A(A \rightarrow B) = \min[1, P_B/P_A] = \min[1, e^{-\beta(E_B - E_A)}].$$

Change of configuration (from A to B) that does not increase the energy is always accepted, but a proposed move which results in higher energy is accepted with a probability that decreases exponentially as the energy difference increases. In practical terms, while simulating, for example, the Ising Model, after the initialisation of the lattice, one randomly selects a spin site, and proposes the new configuration by flipping the spin (B , with the energy E_B). A random number (**rand**) is also generated, and the configuration B is accepted if either $\Delta E < 0$ or **rand** $< e^{-\Delta E}$ is true.

4 Correlation Functions

Two-point correlation functions are to be of essential importance for characterizing the violation of Lorentz invariance in our target model. Two-point correlation functions were calculated for the ordinary 2D and 3D Ising Models as a preparatory step, so that the analogous procedure could be implemented for our target Double Ising Model.

For the (ordinary) Ising Model, the two-point correlation function is computed in the following way:

$$C(r) = \langle S(0)S(r) \rangle - \langle S(0) \rangle \langle S(r) \rangle \quad (10)$$

In the 2D case, $S(r)$ represent the values of spins at the positions $y = r$, and the angle brackets denote averaging with respect to the x -axis.

In more detail:

$$C(r) = \frac{1}{N} \sum_{x=0}^{N-1} S(x, y=0)S(x, y=r) - \left(\frac{1}{N} \sum_{x=0}^{N-1} S(x, y=0) \right) \cdot \left(\frac{1}{N} \sum_{x=0}^{N-1} S(x, y=r) \right) \quad (11)$$

In the 3D case, $S(r)$ in the expression (10) represent the values of spins at the positions $z = r$, and the angle brackets denote averaging with respect to both x and y axes.

In more detail:

$$C(r) = \frac{1}{N^2} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} S(x, y, z=0)S(x, y, z=r) - \left(\frac{1}{N^2} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} S(x, y, z=0) \right) \cdot \left(\frac{1}{N^2} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} S(x, y, z=r) \right) \quad (12)$$

As for the calculation of the two-point correlation functions for our target Double Ising Model, objects such as $\langle S_i S'_j \rangle$ will be of interest. Here, S_i represents the spin value at the position i on the first lattice, and S'_j represents the spin value at the position j on the second lattice of the Double Ising system. Near the point of criticality, the correlators in the Lorentz invariant phase are expected to decay slower than the ones in the phase with Lorentz violation.

5 Testing the Validity of the Simulation

The program used for simulation of target model was written in C++ by expanding the code for the 2D Ising Model [3]. It employs initialization of two different lattices, one of which is chosen randomly at the beginning of every Metropolis step, and updated accordingly. Results obtained by the simulation of the target model have been compared to results obtained by the ordinary 3D Ising model simulation to test the validity of the written code. By setting the values of the parameters to $h = 0$, $c_k = 1$, $d_k = 0$ for $k = 0, 1, 2$ (or $h = 0$, $c_k = 0$, $d_k = 1$), Hamiltonian (2) reduces to the ordinary 3D Ising Model Hamiltonian (for a ferromagnet) in the absence of an external field:

$$H_i = - \sum_{j \text{ n n}} S_i S_j \quad (13)$$

where the sum runs over the nearest neighbours of S_i . Provided plots demonstrate that the two codes give the same results for the measured observables $\frac{C}{N}$ and $\frac{\langle E \rangle}{N}$. Both simulations have been run for $L = 2$ and 1000000 Monte Carlo steps.

The results for magnetisation and observables directly dependent on magnetisation show some discrepancy between results obtained by the target code and ordinary 3D Ising simulation, although both plots follow the same trend. This is expected, since the simulation for our target model has twice as many degrees of freedom than the simulation for ordinary 3D Ising Model - two lattice systems which are updated randomly at every MC step.

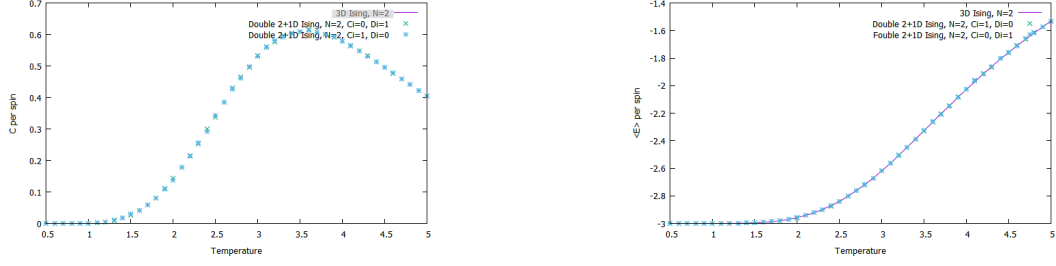


Figure 1: Plots which show the comparison of observables obtained with the new simulation for the target model, and the ordinary 3D Ising Model simulation. Plot on the left side represents heat capacity per spin vs temperature, and the plot on the right represents energy per spin vs temperature

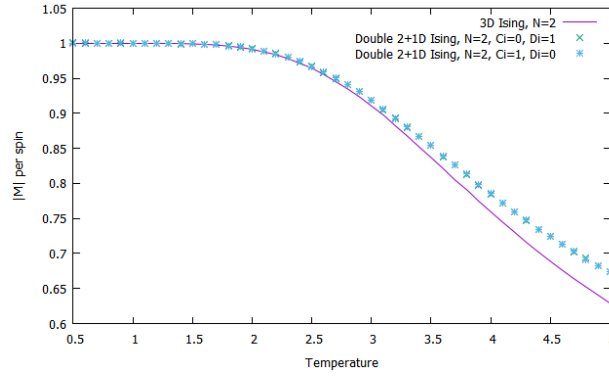


Figure 2: Comparison of absolute magnetisation obtained by our target model simulation, and the simulation for ordinary 3D Ising Model

6 Results for nonzero interaction between two 3D Ising Models

The effect of different interaction constants h on the critical temperature was examined, with values of other coefficients fixed to $d_1 = d_2 = d_3 = 1$, i.e. the case where both 3D Ising Models respect Lorentz symmetry. Theoretically, the point of second-order phase transition is the point where heat capacity and magnetic susceptibility are expected to diverge. Due to the finite size of examined lattice ($N = 8$), the points of transition are manifested as global maxima of heat capacity (magnetic susceptibility) vs temperature plots. The simulations were run with following parameters: 1000000 Monte Carlo steps, temperature resolution of 0.1.

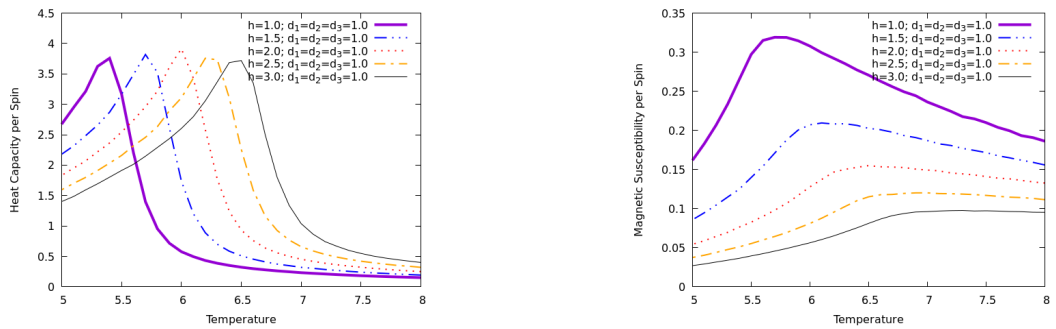


Figure 3: Plots which show the behaviour of critical point for various h : plot on the left shows heat capacities per spin vs temperature, and one on the right shows magnetic susceptibilities vs temperature

It is clear from Figure 3 that the critical points shift towards higher temperature values with the increasing

h . Figure 4 depicts values of critical temperatures as a function of the value of the interaction constant h .

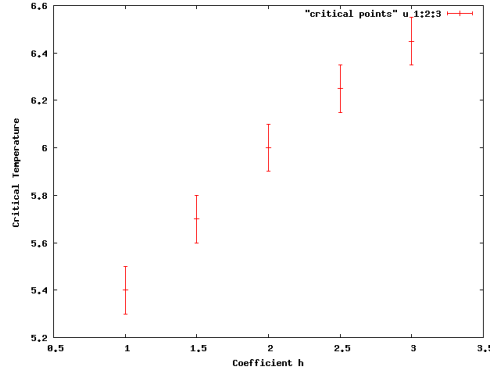


Figure 4: Values of critical temperatures vs h

7 Anisotropic Double Ising Model

In analogy with the previous case, the effect of different anisotropies in one of the Ising systems, manifested through different values of d_3 , was examined. Examined lattice size was $N = 8$. The values of $h = d_1 = d_2 = 1$ were fixed. The following parameters have been used to run the simulations: 1000000 Monte Carlo steps, temperature resolution of 0.1.

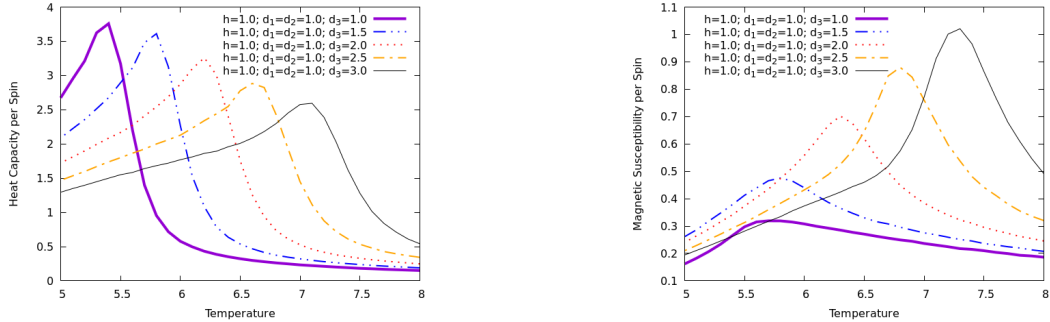


Figure 5: Plots which show the behaviour of critical point for various d_3 . Plot on the left shows heat capacities per spin vs temperature, and the one on the right shows magnetic susceptibilities vs temperature

From Figures 3 and 4 it is clear that the critical temperature shifts to the higher values with increasing anisotropy (i.e. d_3). The precision with which the critical temperature can be determined depends on the resolution used for temperature steps - i.e. in order to achieve greater precision, temperature steps need to be smaller than 0.1 used here. Also, Finite Size Scaling[3] procedure should be employed in order to determine the true critical temperature, which gives rise to the need to implement one of the faster algorithms for the simulation, which is discussed in the Conclusion and Outlook section.

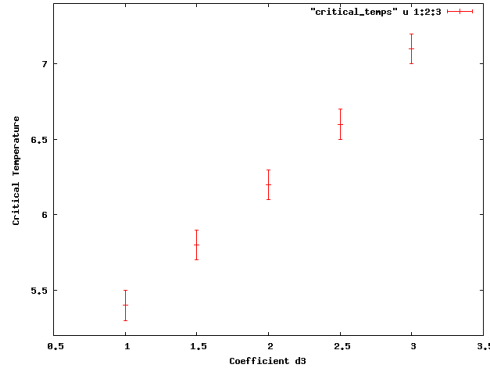


Figure 6: Values of critical temperatures vs d_3

8 Conclusion and Outlook

The results obtained during the Summer School, presented in this report, will serve as a first step towards characterization of the described 3D interacting Ising Model with Lorentz symmetry violation, and provide with the necessary framework for further investigation.

The most relevant physical implications are to be drawn from correlation functions in the region of criticality. Therefore, the next step should comprise of simulations of our described model for larger lattices (up to $N = 256$ or more), so that enough data would be available for proper utilization of Finite Size Scaling procedure in order to precisely determine the critical temperature. Due to the limitations of the local MC algorithm, studying lattices with lattice extents of $N = 256$ and higher using the presented algorithm would not prove feasible in terms of computing times. Therefore, one of the faster algorithms and/or parallelization techniques should be implemented. Code parallelization[5] assumes converting the code to a multi-threaded form, which can then be run on multiple processors. The other considered option would consist of implementing one of the cluster-based algorithms[6, 7] that update large cluster of spins and bring a significant speed-up of the simulation at and near the critical point.

After having managed to simulate larger lattices and apply Finite Size Scaling procedure, we would be able to study the relevant correlation functions in the region of criticality. This would allow us to extract the effective masses from the decay of the correlation functions in a particular direction, which could then be associated to the renormalised couplings[8]. By studying the running of these couplings from high- to low-energy scales, we can check for which choice of interaction and anisotropy parameters would the Lorentz symmetry be emergent at low energies.

9 Acknowledgements

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