On the descriptions of beam instabilities

Antoine Maillard*

Département de Physique, École Normale Supérieure, 24 rue Lhomond, 75005 Paris, France

(Dated: August 4, 2016)

We investigate two interesting features of beam instabilities in accelerators : First, we provide the equivalence between two models to describe transverse instabilities, the circulant matrix model (based on a longitudinal phase space discretization) and the Vlasov formalism. Secondly, we show how to derive dispersion integrals for transverse detuning effects in the Vlasov formalism, thus allowing for Landau damping mechanism.

* antoine.maillard@ens.fr

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I. INTRODUCTION

CERN, the European Organization for Nuclear Research, founded in 1954, is designed to gain an insight in the fundamental structure of the universe, particles. They are accelerated close to the speed of light, before they collide, and the colliding process gives informations about particle interaction, therefore about the most fundamental laws that build our universe.

CERN now counts 22 member states, collaborating to this great scientific adventure.

There are many physics or engineering departments at CERN, ranging from pure theoretical physics to very applied studies about the accelerator itself. I personally did my Summer Student project in the BE (Beam) department, in the ABP (Accelerator and Beam Physics) group, under the supervision of Benoit Salvant.

The purpose of the ABP group is to study the beam dynamics over all the CERN accelerator complex, from injectors to the LHC (the Large Hadron Collider, the biggest collider now at CERN, with a circumference of 27 km) itself. I had the chance to participate in two Machine Developments on the accelerators, one on the SPS (Super Proton Synchrotron) and one on the LHC, and I would like to thank again Benoit Salvant and all the team for giving me such an opportunity.

The close future of the LHC is its improvement into the HL-LHC (High Luminosity LHC), a project in which the group implication is capital. For instance, the study of collective beam instabilities is of critical importance, since instabilities in beam can pose serious limits on the luminosity of the machine. Understanding how such instabilities emerge, and designing both analytical and numerical tools to predict them is thus of fundamental importance for the design of future colliders, from the close HL-LHC, to the (maybe) future FCC (Future Circular Collider, a porject of a collider to replace and improve LHC).

Besides Benoit, I had the chance to work and discuss with many people in the area, and I would like to especially thank Elias Metral, Xavier Buffat, Nicolo Biancacci, David Amorim, Kevin Li, and Michael Schenk for all the discussions we had.

II. THE CIRCULANT MATRIX MODEL AND THE VLASOV FORMALISM

In this section, I present the first part of my work, about two different models used to study instabilities. I will not detail all calculations, and all the conventions are taken from the sources that will be cited.

A. Definitions of the models

1. Vlasov description

The Vlasov formalism is basically a reformulation of the Vlasov(or Boltzmann) equation that expresses the conservation of the phase space volume :

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = 0 \tag{1}$$

Where $\psi(s, J_y, \theta_y, z, \delta)$ is the phase space distribution density, as a function of $s = vt$, (J_y, θ_y) (the action-angle variables in the transverse y plane), the longitudinal position z and the longitudinal momentum deviation δ. In its expanded form, Eq.1 reads :

$$\frac{\partial \psi}{\partial s} + \frac{\partial J_y}{\partial s}\frac{\partial \psi}{\partial J_y} + \frac{\partial \theta_y}{\partial s}\frac{\partial \psi}{\partial \theta_y} + \frac{\partial z}{\partial s}\frac{\partial \psi}{\partial z} + \frac{\partial \delta}{\partial s}\frac{\partial \psi}{\partial \delta} = 0 \tag{2}$$

Assuming that a perturbation is developing with a (complex) frequency $\Omega_c = \omega_0 Q_c$, using the proper form for the Hamiltonian (see [1]), and after a few tricks, one can rewrite Eq.2 as

$$(Q_c - Q_{y,0} - \Delta Q_y(z,\delta) - jQ_s\partial_\phi)g_1(r,\phi) = -\frac{g_0(r)}{4\pi}\sum_{k=-\infty}^{+\infty} e^{-2\pi jQ_c k}\int \mathrm{d}\tilde{z}\mathrm{d}\tilde{\delta} W_X(z - \tilde{z} - 2\pi kR)g_1(\tilde{z},\tilde{\delta}) \tag{3}$$

In this equation, the physical important quantities are :

- $g_1(r,\phi)$ is the longitudinal perturbation distribution, with (r,ϕ) being the polar coordinates in the longitudinal (z,δ) phase space.
- $g_0(r)$ is the unperturbed longitudinal perturbation.
- Q_c : the frequency of the perturbation. If $\Im(Q_c) < 0$, an instability is growing.
- $Q_y = Q_{y,0} + \Delta Q_y$ is the transverse tune, ΔQ_y being the transverse detuning with the longitudinal coordinates.
- Q_s is the synchrotron tune.
- W_X is the wake function (the X index denotes a particular normalization convention), it is related to the impedance.
- The sum over $k \in \mathbb{Z}$ is the multiturn sum.

2. *The Circulant Matrix Model (CMM)*

The CMM (see [2] for a further description) is based on a discretization of the longitudinal phase space (r,ϕ) in N_s slices and N_r rings. Each cell (i,j) is weighted by the free distribution $g_0(r)$. The evolution of the system after one turn is described by the one-turn map M. Since we will investigate the properties of this matrix and relate them to the Vlasov formalism, we have to consider a single turn and not the multiturn sum in Eq.3.

For simplicity, we assume all weights of the cells equal to $\frac{1}{N_r N_s}$.In the case where there is no transverse detuning, $Q_y = Q_{y,0}$, then $M = M_0(\mathbb{I} + W)$, with

$$M_0 = \frac{1}{N_r N_s}\mathbb{I}_{N_r} \otimes P_{N_s}^{N_s Q_s} \otimes B_0(2\pi Q_{y,0})$$

W is a matrix that contains the impedance effects, $B_0(\theta)$ is a 2x2 rotation matrix of angle θ and

$$P_{N_s} \equiv \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

is a permutation matrix.

B. Equivalence of the two models

1. *Without transverse detuning effects*

In this case, and if there is no impedance effect ($W_X = 0$) one can actually (taking care of the fractional matrix power) completely diagonalize the one turn matrix M_0 of the CMM, to obtain its eigenmodes and eigenvalues.

The Vlasov equation in the same case reduces to a simple equation that can be solved analytically. However, in order to make the parallel with the CMM, we decide to proceed to the same discretization of phase space as the one that is conducted in the CMM. In concrete terms, we write

$$g_1(r,\phi) \equiv \sum_{i,j} \alpha_{i,j} \theta_{\Delta\phi}(\phi - \phi_j)\theta_{\Delta r_{i+1}}(r - r_i) \tag{4}$$

In which the $\theta_a(x)$ are window functions, non-zero only on $[0;a]$. Using this decomposition in the Vlasov equation yields a matrix equation for $\alpha_{i,j}$, that can be solved analytically. In the end, one finds for the two models the following eigenvalues :

$$Q_c = Q_{y,0} + lQ_s \quad \text{(CMM)} \tag{5}$$

$$Q_c = Q_{y,0} - \frac{1 - e^{\frac{2j\pi l}{N_s}}}{\frac{2j\pi}{N_s}} Q_s \quad \text{(Vlasov)} \tag{6}$$

So in the limit $N_s \to \infty$, one easily sees that the two models are indeed equivalent. Moreover, the corresponding eigenvectors are also equal in this limit.

Let us now investigate what happens when introducing a dipolar wake function. In the Vlasov formalism, it adds a right-hand-side term to the equation, while in the CMM, one multiplies the one-turn-map by a matrix $(I + W)$.

One can no longer analytically solve the equations or diagonalize the CMM one-turn matrix. We consequently proceeded by perturbation theory in the wake function. Going to first and to second order in the wake function, and applying standard perturbation theory methods, one can find the analytical expression of perturbation matrices, the spectrum of which gives the perturbation in Q_c induced by the wake function. Showing that these two matrices are the same in both models provide a way to prove the equivalences of the models at the corresponding order in the wake function. For instance, at second order, we find that, in the limit $N_s \gg 1$, for both models, one needs to diagonalize the *same* matrix :

$$\Lambda^{(2),l}_{r,r'} = \sum_{l' \neq l, r''} \frac{1}{l - l'} \sum_{b,b',c,c'} W_X(z_{(r,b)} - z_{(r'',c)}) W_X(z_{(r'',c')} - z_{(r',b')}) e^{-j\frac{2\pi l}{N_s}(b-b')} e^{-j\frac{2\pi l'}{N_s}(c'-c)} \tag{7}$$

In which $l \in \mathbb{Z}$ is an azimuthal mode, $r'' \in [|1, N_r|]$ and the $b, c \in [1, N_s]$ indices denote angular cells.

2. *Including transverse detuning effects*

I was also able to prove the equivalence of the two models with arbitrary transverse detuning effects. In the Vlasov equation, we already introduced such effects.

However, in the CMM, I was able to prove that the only consistent way to introduce transverse detuning effects is to multiply , not the one turn map M, but the permutation matrix P_{N_s} itself, by a well-defined diagonal matrix

$$\begin{pmatrix} e^{2\pi j \frac{\Delta Q_y(z_1,\delta_1)}{N_s Q_s}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{2\pi j \frac{\Delta Q_y(z_{N_s},\delta_{N_s})}{N_s Q_s}} \end{pmatrix}$$

And again, in the limit $N_s \to \infty$, and using first and second order perturbation theory in the wake function, one can prove the equivalence of the two models.

Finally, using the discretization of phase space (introduced by the circulant matrix model) in the Vlasov equation , we managed to show that the two models are indeed completely equivalent in the limit $N_s >> 1$, taking into account arbitrary transverse detuning effects and dipolar impedance effects.

III. TRANSVERSE DETUNING EFFECTS AND LANDAU DAMPING

We investigate here how I was able to derive dispersion integrals that can explain Landau damping from transverse detuning effects (as chromaticity effects). Once again, the calculations are lengthy and I will just point out the main results.

Introducing arbitrary transverse detuning effects in the Vlasov equation (Eq.3), and integrating out the radial dependency , I was able to derive a matrix equation, that can be written as :

$$\sigma_{l,p} = j\pi \frac{Q_s v}{\eta R^2} \sum_{p',l'} \sigma_{l',p'} Z\left[-\omega_0(Q_{y,0}+p')\right] \int \mathrm{d}r g_0(r) r \frac{H_l^p(r)\overline{H_l^{p'}}(r)}{Q_c - Q_{y,0} - lQ_s - \langle \Delta Q_y \rangle_\phi (r)} \tag{8}$$

The prefactor is of no importance in the Landau damping mechanism, and we shall forget about it. The H_l^p functions are generalizations of Bessel functions of the first kind, and $\langle \Delta Q_y \rangle_\phi (r)$ denotes the average over ϕ of the transverse detuning. If $\langle \Delta Q_y \rangle_\phi (r)$ is not constant, one can not reduce Eq.8 to an eigenvalue equation, but rather to an equation of the type $\det(I+M)=0$, that can be solved numerically.

From Eq.8, using complex analysis tricks, and taking particular values for the unperturbed distribution g_0, I was able to derive a set of equations that one can use to numerically find Q_c (and these equations could complete for instance the DELPHI code of [1]). I was also able to find that, in a weak wake limit, and in the limit where the frequencies of the impedance are very small compared to the other frequencies of the system, I recover the dispersion integral from Scott Berg and Ruggiero [3] :

$$\int \mathrm{d}r g_0(r) r \frac{r^{2|l|}}{Q_c - Q_{y,0} - lQ_s - \langle \Delta Q_y \rangle_\phi (r)} \tag{9}$$

This dispersion integral is widely used to compute Landau damping effects in accelerators. I was however able to derive a more general equation, involving more complicated functions, where one can consider any impedance, and take into account azimuthal mode coupling.

IV. CONCLUSION

Finally, I was able during my short stay at CERN to derive two important results :

- The complete proof of the equivalence of the CMM (used for instance in [2]) and the Vlasov description of collective beam instabilities.
- The derivation of Landau damping effects, due to transverse detuning, and the computation of the corresponding dispersion integral.

I would finally thank again all the team I had the chance to work with, and all the other summer students I met, and especially for the last week. If there was not much work done by me this week, we were able to organize a great fund-raiser for victims in Syria, and that was an amazing experience to meet you all.

[1] N. Mounet, (2014).
[2] X. Buffat, *Transverse beams stability studies at the Large Hadron Collider*, Ph.D. thesis, ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE (2014).
[3] J. S. Berg and F. Ruggiero, in *Particle Accelerator Conference, 1997. Proceedings of the 1997*, Vol. 2 (IEEE, 1997) pp. 1712–1714.