The Post-Newtonian Approximation in Massive Gravity

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Abstract

We briefly review the post-Newtonian expansion as a way to yield the approximate equations of motion from a metric theory of gravity in the slow-motion, weak-field limit. Then, the parameterized post-Newtonian (PPN) formalism is introduced, as it provides a method of characterizing metric theories of gravity (again, in the slow-motion, weak-field limit). Finally, the field equations for Lorentz-violating massive gravity are derived from the Lagrangian; these equations will allow for the future determination of the PPN parameters.

1 Introduction

Thus far, the predictions of general relativity (GR) have been in very good agreement with experimental observations. The problem with GR is that it does not seem compatible with a quantum theory of gravity. Therefore, many alternative theories have been proposed which may be more easily quantizable; in particular, we are interested in Lorentz-violating theories. We first introduce the post-Newtonian expansion, which yields the approximate equations of motion in the slow-motion, weak-field limit. Following this, we review the parameterized post-Newtonian formalism, which allows for the classification of a general metric theory of gravity based on the values of 10 parameters which appear in a post-Newtonian expansion of the metric. Since metric theories differ only in the way that matter (and possible other fields) generate the metric, these parameters allow one to characterize theories according to their physical predictions (such as Lorentz violation, momentum conservation violation, and non-linearity in the superposition principle). Finally, Lorentz-violating massive gravity is briefly explored. The Lagrangian for a specific theory is used to derive the field equations, which may later be utilized in the computation of the PPN parameters. We use the conventions of Blas and Sibiryakov [3].

2 The Post-Newtonian Expansion

NOTE: This section follows the discussion of Weinberg[1].

Suppose we have a system of slowly moving particles bound by gravitational forces. Taking the slow-motion, weak-field limit, we can expand in powers of the velocity and determine the equations of motion to one order higher than that given by Newtonian mechanics. This is the post-Newtonian expansion. The equations of motion are given by the geodesic equation:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0 \tag{1}$$

where τ denotes the proper time. This implies that we can compute the accelerations using

$$\frac{d^2x^i}{dt^2} = -\Gamma^i{}_{\nu\lambda}\frac{dx^\nu}{dt}\frac{dx^\lambda}{dt} + \Gamma^0{}_{\nu\lambda}\frac{dx^\nu}{dt}\frac{dx^\lambda}{dt}\frac{dx^i}{dt}. \tag{2}$$

We must compute the Christoffel symbols to various orders so that we can obtain our post-Newtonian approximation. We treat the metric tensor $g_{\mu\nu}$ as consisting of the background Minkowski metric $\eta_{\mu\nu}$ with the corrections being expandable in powers of the velocity. Thus we have

$$g_{00} = 1 + \overset{2}{g}_{00} + \overset{4}{g}_{00} + \ldots \tag{3}$$

$$g_{ij} = -\delta_{ij} + \overset{2}{g}_{ij} + \overset{4}{g}_{ij} + \ldots \tag{4}$$

$$g_{0i} = \overset{1}{g}_{0i} + \overset{3}{g}_{0i} + \ldots \tag{5}$$

where $\overset{N}{g}_{\mu\nu}$ denotes the term of order v^N in $g_{\mu\nu}$. We can then apply the formula

$$\Gamma^\mu{}_{\nu\lambda} = \frac{1}{2}g^{\mu\rho}\left(g_{\rho\nu,\lambda} + g_{\rho\lambda,\nu} - g_{\nu\lambda,\rho}\right) \tag{6}$$

to determine the Christoffel symbols. We see now that we must determine the metric in order to carry out this computation. The Ricci tensor is also expanded in a series and computed in terms of the Christoffel symbols, allowing us to relate it to the metric:

$$\overset{2}{R}_{00} = -\frac{1}{2}\nabla^2\overset{2}{g}_{00} \tag{7}$$

$$\overset{4}{R}_{00} = -\frac{1}{2}\nabla^2\overset{4}{g}_{00} + \frac{1}{2}\frac{\partial^2\overset{2}{g}_{00}}{\partial t^2} + \frac{1}{2}\overset{2}{g}_{ij}\frac{\partial^2\overset{2}{g}_{00}}{\partial x^i x^j} - \frac{1}{2}\left(\nabla^2\overset{2}{g}_{00}\right)^2 \tag{8}$$

$$\overset{3}{R}_{0i} = -\frac{1}{2}\nabla^2\overset{3}{g}_{0i} \tag{9}$$

$$\overset{2}{R}_{ij} = -\frac{1}{2}\nabla^2\overset{2}{g}_{ij}. \tag{10}$$

At this point, we make use of the Einstein equations to relate the metric components to the energy-momentum tensor (which is also expanded in a series). The resulting equations can be used to solve for the components of the metric and express it in terms of various potentials. The metric can then be used to derive the post-Newtonian equations of motion.

3 The Parameterized Post-Newtonian Formalism

The form of the metric up to post-Newtonian order in a general theory of gravity can be summarized by 10 parameters $(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma, \zeta_1, \zeta_2, \zeta_3, \zeta_4, \xi)$ when the metric is given in terms of specific potential functions. This formalism is discussed in more depth by Will[2]. The parameters appear as follows:

$$\begin{aligned} g_{00} = 1 &- 2U + 2\beta U^2 + 2\xi\Phi_W - (2\gamma + 2 + \alpha_3 + \zeta_1 - 2\xi)\Phi_1 \\ &- 2(3\gamma - 2\beta + 1 + \zeta_2 + \xi)\Phi_2 - 2(1+\zeta_3)\Phi_3 - 2(3\gamma + 3\zeta_4 - 2\xi)\Phi_4 + (\zeta_1 - 2\xi)A \end{aligned} \tag{11}$$

$$g_{ij} = -(1+2\gamma U)\delta_{ij} \tag{12}$$

$$g_{0i} = \frac{1}{2}(4\gamma+3+\alpha_1-\alpha_2+\zeta_1-2\xi)V_i + \frac{1}{2}(1+\alpha_2+\zeta_1+2\xi)W_i. \tag{13}$$

Physically, γ represents the amount of space curvature produced by a unit rest mass. β represents non-linearity in the superposition principle for gravity. ξ represents the presence of preferred-location effects. $\alpha_{1,2,3}$ represent the presence of preferred-frame effects, and $\zeta_{1,2,3,4}$ and α_3 represent the presence of violation of total momentum conservation. Note that α_3 thus contributes to two different physical effects. In general relativity, the only non-zero parameters are $\gamma = \beta = 1$. The potentials appearing in the metric all have the form

$$F(x) = G_N \int d^3y \frac{\rho(y) f}{|x-y|}. \tag{14}$$

The correspondences $F \leftrightarrow f$ are given by

$$\Phi_1 \leftrightarrow v_i v_i \qquad \Phi_3 \leftrightarrow \Pi \qquad U \leftrightarrow 1 \qquad V_i \leftrightarrow v^i \tag{15}$$

$$\Phi_2 \leftrightarrow U \qquad \Phi_4 \leftrightarrow \frac{p}{\rho} \qquad A \leftrightarrow \frac{(v_i(x-y)_i)^2}{|x-y|^2} \qquad W_i \leftrightarrow \frac{v_j(x_j-y_j)(x^i-y^i)}{|x-y|^2} \tag{16}$$

$$\Phi_W \leftrightarrow \int d^3z \rho(z) \frac{(x-y)_j}{|x-y|^2}\left(\frac{(y-z)_j}{|x-z|} - \frac{(x-z)_j}{|y-z|}\right) \tag{17}$$

Note that for U, $\Phi_{1,2,3,4}$, and V_i, we have

$$F_{,ii} = -4\pi G_N \rho f. \tag{18}$$

We also define the superpotential

$$\chi = -G_N \int d^3y \rho |x-y| \tag{19}$$

which satisfies

$$\chi_{,ii} = -2U \tag{20}$$

$$\chi_{,0i} = V_i - W_i. \tag{21}$$

4 Lorentz-Violating Massive Gravity

As their name suggests, massive gravity theories endow the graviton with mass. We examine a specific form of Lorentz-violating massive gravity as introduced by Blas and Sibiryakov[3]. Lorentz invariance is broken down to the subgroup of spatial rotations via the introduction of four scalar fields, ϕ^0, ϕ^a where $a = 1, 2, 3$. The Lagrangian (including Einstein-Hilbert and matter terms $\mathcal{L}_{EH}$ and $\mathcal{L}_m$, respectively) is given by

$$\mathcal{L} = \frac{1}{16\pi G}R + \mathcal{L}_m + \frac{1}{8\mu_0^4}\left((\partial_\mu \phi^0)^2 - \mu_0^4\right)^2$$

$$
\begin{aligned}
&- \frac{\kappa_0}{4\mu_0^4}\left((\partial_\mu \phi^0)^2 - \mu_0^4\right)\left(\sum_a P^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^a + 3\mu^4\right) \\
&- \frac{1}{8\mu^4}\sum_{a,b}\left(P^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^b + \mu^4\delta^{ab}\right)^2 \\
&+ \frac{\kappa}{8\mu^4}\left(\sum_a P^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^a + 3\mu^4\right)^2 . \qquad (22)
\end{aligned}
$$

We denote the massive gravity Lagrangian by $\mathcal{L}_S = \mathcal{L} - (\mathcal{L}_{EH} + \mathcal{L}_m)$ and define

$$P_{\mu\nu} = g_{\mu\nu} - \frac{\partial_\mu\phi^0\partial_\nu\phi^0}{g^{\lambda\rho}\partial_\lambda\phi^0\partial_\rho\phi^0}. \qquad (23)$$

We first vary the action $S = \int \mathcal{L}\sqrt{-g}d^4x$ with respect to the metric and find $\delta S = \int \left(\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}\mathcal{L}\right)\delta g^{\mu\nu}\sqrt{-g}d^4x$. Thus, the condition that the action be stationary ($\delta S = 0$) implies

$$\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}\mathcal{L} = 0. \qquad (24)$$

Computing, we find

$$
\begin{aligned}
\frac{\delta\mathcal{L}_S}{\delta g^{\mu\nu}} =& - \frac{\kappa_0}{4\mu_0^4}\left((\partial_\alpha\phi^0)^2 - \mu_0^4\right)\sum_a K^{aa}_{\mu\nu} - \frac{1}{4\mu^4}\sum_{a,b}\left(P^{\alpha\beta}\partial_\alpha\phi^a\partial_\nu\phi^b + \mu^4\delta^{ab}\right)K^{ab}_{\mu\nu} \\
&+ \frac{\kappa}{4\mu^4}\left(\sum_a P^{\alpha\beta}\partial_\alpha\phi^a\partial_\beta\phi^a + 3\mu^4\right)\left(\sum_b K^{bb}_{\mu\nu}\right). \qquad (25)
\end{aligned}
$$

We have defined

$$K^{ab}_{\mu\nu} \equiv \left(\frac{\delta g^{\sigma\rho}}{\delta g^{\mu\nu}} - \left(\frac{\delta g^{\sigma\alpha}}{\delta g^{\mu\nu}}u_\alpha u^\rho + \frac{\delta g^{\rho\alpha}}{\delta g^{\mu\nu}}u_\alpha u^\sigma\right) + u^\sigma u^\rho u_\alpha u_\beta\frac{\delta g^{\alpha\beta}}{\delta g^{\mu\nu}}\right)\partial_\sigma\phi^a\partial_\rho\phi^b \qquad (26)$$

where we have defined $u_\mu \equiv \frac{\partial_\mu\phi^0}{\sqrt{X}}$ with $X \equiv g^{\alpha\beta}\partial_\alpha\phi^0\partial_\beta\phi^0$. Note that a and b are not tensor indices, but merely labeling indices; thus, the summation convention does not apply to them. We can now define our energy-momentum tensor as

$$T^S_{\mu\nu} \equiv -2\left(\frac{\delta\mathcal{L}_S}{\delta g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}\mathcal{L}_S\right). \qquad (27)$$

Thus, the field equations ultimately take the form

$$G_{\mu\nu} = 8\pi G\left(T^m_{\mu\nu} + T^S_{\mu\nu}\right) \qquad (28)$$

where $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is the familiar Einstein tensor and $T^m_{\mu\nu}$ denotes the matter energy-momentum tensor.

Using the Euler-Lagrange equations $\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} - \frac{\partial\mathcal{L}}{\partial\phi} = 0$ and noting that the second term vanishes for all ϕ fields, we find for ϕ^0

$$\frac{1}{2\mu_0^4}\left((\partial_\alpha\phi^0)^2-\mu_0^4\right)\partial_\mu\delta^{\mu\beta}\partial_\beta\phi^0-\frac{\kappa_0}{2\mu_0^4}\partial_\mu\delta^{\mu\alpha}\partial_\alpha\phi^0\left(\sum_a P^{\beta\lambda}\partial_\beta\phi^a\partial_\lambda\phi^a+3\mu^4\right)$$
$$-\frac{\kappa_0}{4\mu_0^4}\left((\partial_\alpha\phi^0)^2-\mu_0^4\right)\left(\sum_a\frac{1}{\sqrt{X}}\partial_\mu(-g^{\mu\beta}u^\nu-g^{\mu\nu}u^\beta+2u^\mu u^\beta u^\nu)\partial_\beta\phi^a\partial_\nu\phi^b\right)$$
$$-\frac{1}{4\mu^4}\sum_{a,b}(P^{\alpha\beta}\partial_\alpha\phi^a\partial_\beta\phi^b+\mu^4\delta^{ab})\left(\frac{1}{\sqrt{X}}\partial_\mu(-g^{\mu\lambda}u^\nu-g^{\mu\nu}u^\lambda+2u^\mu u^\lambda u^\nu)\partial_\lambda\phi^a\partial_\nu\phi^b\right)$$
$$+\frac{\kappa}{4\mu^4}\left(\sum_a P^{\alpha\beta}\partial_\alpha\phi^a\partial_\beta\phi^a+3\mu^4\right)\left(\sum_b\frac{1}{\sqrt{X}}\partial_\mu(-g^{\mu\lambda}u^\nu-g^{\mu\nu}u^\lambda+2u^\mu u^\lambda u^\nu)\partial_\lambda\phi^b\partial_\nu\phi^b\right)=0. \quad (29)$$

For ϕ^i, we find

$$-\frac{\kappa_0}{2\mu_0^4}\left((\partial_\lambda\phi^0)^2-\mu_0^4\right)\left(\partial_\mu P^{\mu\nu}\partial_\nu\phi^i\right)$$
$$-\frac{1}{2\mu^4}\sum_{a,b}\Big((P^{\lambda\rho}\partial_\lambda\phi^a\partial_\rho\phi^b+\mu^4\delta^{ab})\big(\partial_\mu\delta^{ia}[P^{\mu\nu}\partial_\nu\phi^b+\delta^{ab}(P^{\mu\nu}\partial_\nu\phi^i)]$$
$$+\delta^{ib}[P^{\mu\nu}\partial_\nu\phi^a+\delta^{ab}(P^{\mu\nu}\partial_\nu\phi^i)]\big)\Big)$$
$$+\frac{\kappa}{2\mu^4}\left(\sum_a P^{\lambda\rho}\partial_\lambda\phi^a\partial_\rho\phi^a+3\mu^4\right)\left(\partial_\mu P^{\mu\nu}\partial_\nu\phi^i\right)=0. \quad (30)$$

5 Conclusion

We have found the field equations for Lorentz-violating massive gravity in the form investigated by Blas and Sibiryakov. In the future, these equations can be used to determine the PPN parameters for Lorentz-violating massive gravity. This would allow for a proper characterization of the theory within the PPN formalism and provide a step towards empirical solar system tests of the theory.

References

[1] S. Weinberg, ”Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity,” *John Wiley & Sons, Inc. (1972), 657 p.*

[2] C. M. Will, The Confrontation between General Relativity and Experiment, Living Rev. Rel. 4, 4 (2001) [arXiv:gr-qc/0103036].

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