Cross section of the induced neutrinoless double β decay

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Abstract
The neutrinoless double β decay is a process, that would prove the Majorana nature of neutrinos. But the process has not been measured yet. Therefore, the cross section and the expected number of events for the neutrinoless double β decay induced with energetic neutrinos was calculated in this report. The evaluation of the expected number of events in $10\,\text{yr}$ is smaller than 10^{-15} for energies of the inducing neutrinos up to $100\,\text{MeV}$. However, the cross section grows more than linearly with the energy, so the interaction could be measurable at higher energies. Inducing the decay with a scalar dark matter particle also does not lead to a measurable rate, as the expected number of events in $10\,\text{yr}$ is smaller than 10^{-43} for the dark matter mass $m_{\text{DM}} = 1\,\text{MeV}$.

Keywords
CERN Summer Student report; Neutrinoless double beta decay; Cross section; Majorana neutrinos.

1 Introduction

Neutrinos are elementary fermions with very small masses, which cannot be explained by the standard model of particle physics (SM). Their masses could be explained with the Seesaw mechanisms if neutrinos are Majorana particles.
The Majorana nature of neutrinos would be proven, if a neutrinoless double β decay could be measured. However, the lifetime of this decay is $T^{0\nu\beta\beta} > 10^{25}\,\text{yr}$ (see [1]) and is not measured yet.

This report therefore explores the possibility of an induced neutrinoless double β decay. In this interaction the decay is induced with energetic neutrinos of an external source interacting with the nucleus. The cross section of this decay should scale according to an yet to be determined power law and could therefore become big enough to be measurable for high energy neutrinos.
Therefore, the cross section of this interaction at different energy scales will be examined to observe the scaling and to possibly achieve a measurable rate.

The report is structured in the following way:
Firstly, the Seesaw mechanism and the singlet Majoron model are explained, as well as the neutrinoless double β decay and the idea behind the induced decay. Secondly, the calculation of the cross section for inducing the decay with low energy neutrinos is detailed and the result of the numerical evaluations is explained. After that, the cross section and the rate of the induced decay with low energy scalar dark matter instead of neutrinos is also evaluated. Lastly the calculation of the cross section at higher energies for quasi-elastic neutrino nucleus scattering and deep inelastic scattering is explained. The machinery to calculate these processes has been setup, but the full numerical evaluation will be pursued elsewhere.

2 Theory behind the neutrinoless double β decay

In the SM all massive fermions are Dirac fermions and neutrinos are assumed to be massless. Through oscillation measurements it was shown, that neutrinos do have very small masses (see [2]). One of the theories explaining the small masses of the neutrinos is the theory, that neutrinos are Majorana particles. If neutrinos are Majorana particles, their small masses could be explained with the seesaw mechanism, which is the leading mechanism for Majorana mass generation.

The neutrinoless double β decay is an experiment to investigate, whether neutrinos are Majorana particles, since the observation of such a decay would prove that neutrinos are Majorana particles.

2.1 Dirac and Majorana Fermions

This overview follows the explanations in [2] and [3]. The time evolution of fermions in the standard model is described by the Dirac Lagrangian

$$\mathcal{L} = i\bar{\Psi}\gamma^{\mu}\partial_{\mu}\Psi - m_D\bar{\Psi}\Psi. \tag{1}$$

The field Ψ can be further separated into independent left- and right-handed chiral fields with the projection operators $P_{R/L} = (1 \pm \gamma^5)/2$, so that $\Psi = \Psi_L + \Psi_R$. In terms of the left and right chiral states the Lagrangian is

$$\mathcal{L} = i\bar{\Psi}_R\gamma^{\mu}\partial_{\mu}\Psi_R + i\bar{\Psi}_L\gamma^{\mu}\partial_{\mu}\Psi_L - m_D\bar{\Psi}_L\Psi_R - m_D\bar{\Psi}_R\Psi_L. \tag{2}$$

The left and right chiral fields are only coupled over the mass term.

Majorana fermions also fulfil the equation of motion of Dirac fermions, but additionally they also fulfil the Majorana condition $\Psi = \Psi^c$ with the charge conjugated field $\Psi^c = -C\bar{\Psi}^T = -i\gamma^2\gamma^0\bar{\Psi}^T$ [4]. Majorana particles are therefore their own antiparticles. Only neutral fields can be Majorana fields, for charged fields charge conservation would be violated. Majorana fermions also have only 2 degrees of freedom, as the charge conjugation operator transforms right-handed fields into left-handed fields $\Psi_L = (\Psi_R)^c$.
This leads to a new possible mass term for Majorana fermions

$$\mathcal{L}_{\text{Majorana}} = -\frac{1}{2}m\overline{\Psi^c}\Psi + h.c. = -\frac{1}{2}m\overline{\Psi^c_L}\Psi_L + h.c.. \tag{3}$$

2.2 Seesaw mechanism

The neutrino masses are not part of the SM. If they would be Dirac masses like the other fermions, it is also not explained, why they would be several orders of magnitude smaller than other masses. One mechanism to explain the origin of the neutrino mass is the Seesaw mechanism. The explanation of the Seesaw mechanism follows [5] and [2].
In the Seesaw mechanism it is assumed, that neutrinos don't have only a Dirac Mass term, but also a right-handed Majorana mass term

$$\mathcal{L} \supset -m_D\overline{\nu}_L\nu_R - \frac{1}{2}m_M\overline{(\nu^c_R)}\nu_R + h.c.. \tag{4}$$

The Majorana mass term can be added to the SM, since the right-handed neutrinos form a singlet in the SM. A left-handed term cannot be added, as the left-handed neutrinos form a doublet with left-handed electrons under the weak interaction, which would lead to a Majorana

mass term also for electrons that violates charge conservation.
Diagonalizing the mass matrix

$$M = \begin{pmatrix} 0 & m_D \\ m_D & m_M \end{pmatrix}, \tag{5}$$

leads to two Majorana fermion fields χ_1 and χ_2 with masses m_1, m_2. For the case $m_D \ll m_M$ the fields are

$$\chi_1 = (\nu_L + \nu_L^c) - \frac{m_D}{m_M}(\nu_R + \nu_R^c), \tag{6}$$

$$\chi_2 = (\nu_R + \nu_R^c) + \frac{m_D}{m_M}(\nu_L + \nu_L^c), \tag{7}$$

with masses $m_1 \approx \frac{m_D^2}{m_M}$ and $m_2 \approx m_M$. The Dirac mass, m_D, stems from the Higgs VEV 246 GeV with a natural value for the Yukawa coupling of $\mathcal{O}(1)$ leading to $m_D \approx \mathcal{O}(100\,\mathrm{GeV})$. With $m_M \approx 10^{14}\,\mathrm{GeV}$ the mass of the first Majorana field is $m_1 \approx \mathcal{O}(0.1\,\mathrm{eV})$. Since the right-handed part is very small next to the left-handed part, it can be neglected for χ_1. Therefore, χ_1 is the left-handed very light neutrino given by the SM.
The second Majorana field χ_2 is effectively a very heavy right-handed neutrino, which decouples and does not influence the interactions.
Since there are 3 neutrinos in the SM, the Majorana fields χ_1 and χ_2 have 3 generations.
As seen above, the Seesaw mechanism can explain the small neutrino masses in the standard model, but the Majorana mass m_M is chosen arbitrarily.

2.2.1 Singlet Majoron Model

The Seesaw mechanism also leads to lepton number violation with $L = 2$. Therefore, the global $U(1)_{B-L}$ symmetry of Lepton and Baryon number conservation is broken.
If it is assumed, that this symmetry is spontaneously broken at the Seesaw scale f, the spontaneous symmetry breaking leads to a massless Goldstone Boson, the Majoron, which interacts with the neutrinos. The following explanations follow [6].

To describe this effect in field theory, a scalar $\sigma = (f + \sigma_0 + iJ)/\sqrt{2}$ with a very heavy decoupled scalar σ_0 and the Majoron, J, is added to the Lagrangian

$$\mathcal{L} \supset -y\bar{L}N_R H - \frac{1}{2}\lambda\overline{N_R^c}\sigma N_R. \tag{8}$$

N_R is a right-handed neutrino, L is the SM lepton doublet, H is the Higgs field and y and λ are Yukawa couplings of $\mathcal{O}(1)$. Spontaneous breaking of the $U(1)_{B-L}$ symmetry at the seesaw scale leads to the Majorana mass term $\frac{f}{2}\lambda\overline{N_R^c}N_R$ with the Majorana mass $m_M = (f\lambda)/2$. Spontaneous symmetry breaking at the Higgs scale leads to the Dirac mass term $(yv)/\sqrt{2}\bar{L}N_R$ with $m_D = (yv)/\sqrt{2}$.
For $f \gg v$ the situation is again as for the basic Seesaw mechanism, that $m_D \ll m_M$, which leads to very heavy decoupled right-handed neutrinos and to the light SM neutrinos.
However, in addition to the neutrinos, there also is the additional Goldstone Boson, the Majoron coupling to the neutrinos and the Majorana mass is not chosen arbitrarily anymore but given by the Seesaw breaking scale.
In the following it is also assumed, that the Majoron is not massless, but that it has a mass bigger than the neutrino mass. The parameter space for possible masses and neutrino-Majoron couplings can be found in [7].

2.3 Neutrinoless double β decay

Since the Seesaw mechanism would explain the neutrino masses, it is tested, if neutrinos are Majorana fermions. One process, that would prove the Majorana nature of neutrinos is the neutrinoless double β decay ($0\nu\beta\beta$), as explained in [1].

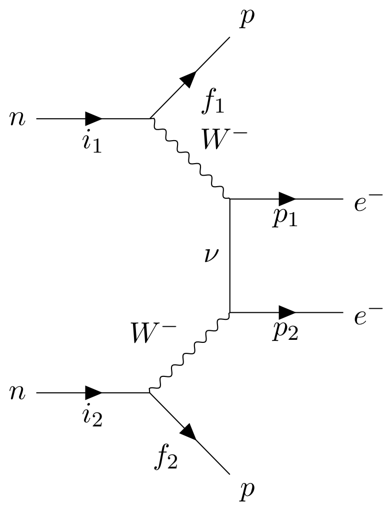


Figure 1: Feynman Diagram of $0\nu\beta\beta$. The initial neutrons are bound in a nucleus, that is capable of double β decay and the protons are bound in the daughter nucleus.

Nuclei, that undergo double β decay can also undergo neutrinoless double β decay, if the emitted neutrinos annihilate with each other. The decay rate for this process is

$$\Gamma^{0\nu} = |m_{\beta\beta}|^2 |M^{0\nu}|^2 G^{0\nu}(Q, Z) \tag{9}$$

where $m_{\beta\beta}$ is the effective Majorana mass of the neutrinos, $M^{0\nu}$ is the nuclear matrix element and $G^{0\nu}(Q, Z) \propto G_F^4$ is the phase space factor of the decay. Q is the total kinetic energy released in the decay and Z is the atomic number of the decaying nucleus.
The decay however is very suppressed with G_F^4 and also $m_{\beta\beta}$ which is expected to be very small. Measurements of this decay lead to a lower limit of

$$T^{0\nu\beta\beta} = \frac{\ln 2}{\Gamma^{0\nu\beta\beta}} > 10^{25}\,\mathrm{yr}. \tag{10}$$

Because of the low decay rate it is very difficult to measure the decay. Therefore, the induced neutrinoless double β decay is explored in the following sections to examine the possibility of a higher decay rate for high enough energies of the inducing particle.

2.4 Idea behind the neutrinoless double β decay

In the induced neutrinoless double β decay the decay is induced with a neutrino beam with incoming neutrino Energy E_ν.

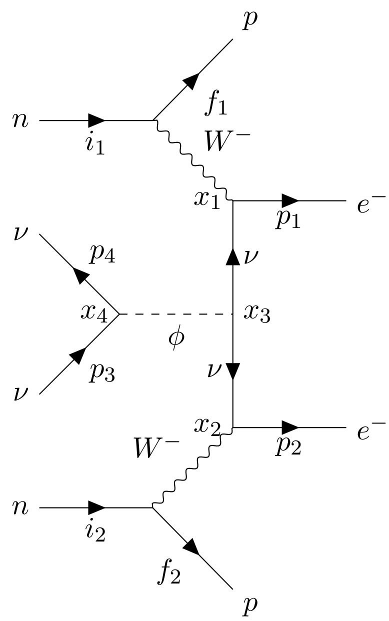


Figure 2: Feynman Diagram of the induced $0\nu\beta\beta$. The initial neutrons and final state protons are bound in a nucleus and can be seen as static.

The neutrino scatters off the nucleus and transfers its momentum trough a virtual Majoron ϕ with mass m_ϕ. This induces the decay in the nucleus, leading to the typical $0\nu\beta\beta$ decay of the nucleus with the 2 outgoing electrons. For low energies the nucleus can be seen as static, the outgoing protons in the diagram are bound in the daughter nucleus.
Following a dimensional analysis of the cross section, it is expected to scale with energy with a power law dependence. This could lead to a detectable number of events for high enough energies.
Therefore, the cross section of this process at different energies is calculated in the following sections.

3 Induced neutrinoless double β decay at low energies

In this section the cross section and expected number of events of the induced neutrinoless double β decay are calculated.
For that, first the amplitude of the decay induced with neutrinos is calculated from the Feynman diagram and with the amplitude, the cross section and the expected rates are numerically evaluated.
The possibility of inducing the decay with dark matter is also evaluated.

3.1 Calculation of the cross section

3.1.1 Evaluation of the scattering amplitude

The matrix element of the Feynman diagram in figure 2 for the induced $0\nu\beta\beta$ is given in position space by

$$\langle f| S^2 |i\rangle = \frac{1}{2!}\left(\frac{ig\cos(\theta_C)}{2\sqrt{2}}\right)^2 \left(\frac{ig}{\sqrt{2}}\right)^2 \left(\frac{i}{M_W}\right)^2 N_{p_1} N_{p_2} N_{p_4} \int \mathrm{d}^4x_1 \mathrm{d}^4x_2 \mathrm{d}^4x_3 \mathrm{d}^4x_4 \langle N_f| T\{J^\alpha(x_1) J^\beta(x_2)\} |N_i\rangle \tag{11}$$

$$
\begin{aligned}
&\bar{u_L}(p_1)e^{ix_1p_1}\gamma_\alpha \langle 0| T\{v_{e,L}(x_1)v_k^T(x_3)\} |0\rangle \hat{C} \\
&\left[\langle 0| T\{\phi(x_3)\phi^\dagger(x_4)\} |0\rangle \sum_i U_{ei}\bar{v}_i(p_4)e^{ip_4x_4}(-ig_\phi\delta_{ij})v_j^C(p_3)e^{-ip_3x_4}\right] \\
&(-ig_\phi\delta_{kl}) \langle 0| T\{v_l(x3)v_{e,L}^T(x_2)\} |0\rangle \gamma_\beta^T \bar{u_L}^T(p_2)e^{ix_2p_2} - (p_1 \rightleftharpoons p_2).
\end{aligned}
$$

The definitions of the nuclear and leptonic current as well as the nuclear matrix element follow the definitions made in section 4 of [1]. The calculations for the low-energy case, adapted for the inducing neutrino, are inspired by these calculations.
Here $N_{p_i} = \frac{1}{(2\pi)^{3/2}\sqrt{2E_i}}$ is the normalisation for outgoing relativistic particles, e.g the electrons and the neutrinos. The nucleus is static in the interaction, and therefore does not have a relativistic normalisation factor for the initial and final state of the nuclei.
U_{ei} is the PMNS matrix and the coupling of the neutrinos to the Majoron, a BSM scalar with mass m_ϕ, is assumed to couple to the mass eigenstates of the neutrinos with $-i\delta_{ij}g_\phi$.
The propagators of the Majorana neutrino are

$$
\langle 0| T\{v_{e,L}(x1)v_k^T(x_3)\} |0\rangle = -\frac{i}{(2\pi)^4}\sum_k U_{ek}\int \mathrm{d}^4q_1 e^{-iq_1(x_1-x_3)}\hat{P}_L\frac{\not{q}_1 + m_k}{q_1^2 - m_k^2}\hat{C}, \tag{12}
$$

$$
\langle 0| T\{v_l(x_3)v_{e,L}^T(x_2)\} |0\rangle = -\frac{i}{(2\pi)^4}\sum_l U_{el}\int \mathrm{d}^4q_2 e^{-iq_2(x_3-x_2)}\frac{\not{q}_2 + m_l}{q_2^2 - m_l^2}\hat{P}_L\hat{C}. \tag{13}
$$

And the Majoron propagator is

$$
\langle 0| T\{\phi(x_3)\phi^\dagger(x_4)\} |0\rangle = \frac{i}{(2\pi)^4}\int \mathrm{d}^4k\frac{e^{-ik(x_3-x_4)}}{k^2 - m_\phi^2}. \tag{14}
$$

Since the dependence of x_3 and x_4 in the matrix element lies only in exponentials, the integration over these variables can be made immediately and lead to 2 δ-functions

$$
\int \mathrm{d}^4x_4 e^{ip_4x_4 - ip_3x_4 + ikx_4} = (2\pi)^4\delta^{(4)}(p_4 - p_3 + k) \tag{15}
$$

$$
\int \mathrm{d}^4x_3 e^{iq_1x_3 - iq_2x_3 - ikx_3} = (2\pi)^4\delta^{(4)}(q_1 - q_2 - k). \tag{16}
$$

With the 2 δ-functions, the integrals over k and q_2 can be eliminated. Defining $k = p_3 - p_4$ and $q = q_1$ the matrix element becomes

$$
\begin{aligned}
\langle f| S^2 |i\rangle =& \frac{1}{2!}\left(\frac{ig\cos(\theta_C)}{2\sqrt{2}}\right)^2\left(\frac{ig}{\sqrt{2}}\right)^2\left(\frac{i}{M_W}\right)^2 N_{p_1}N_{p_2}N_{p_4} \\
&\int \mathrm{d}^4x_1\mathrm{d}^4x_2 \langle N_f| T\{J^\alpha(x_1)J^\beta(x_2)\} |N_i\rangle \\
&\left[\frac{i}{k^2 - m_\phi^2}\sum_i U_{ei}\bar{v}_i(p_4)(-ig_\phi)v_i^C(p_3)\right] \\
&\bar{u_L}(p_1)e^{ix_1p_1}\gamma_\alpha\left[-\frac{i}{(2\pi)^4}\sum_k U_{ek}\int \mathrm{d}^4q e^{-iq(x_1)}\hat{P}_L\frac{\not{q} + m_k}{q^2 - m_k^2}\hat{C}\right]\hat{C} \\
&(-ig_\phi\delta_{kl})\left[-i\sum_l U_{el}e^{i(q-k)x_2}\frac{\not{q} - \not{k}_2 + m_l}{(q-k)^2 - m_l^2}\hat{P}_L\hat{C}\right]\gamma_\beta^T\bar{u_L}^T(p_2)e^{ix_2p_2} - (p_1 \rightleftharpoons p_2).
\end{aligned} \tag{17}
$$

With $\hat{P}_L\gamma_\mu = \gamma_\mu\hat{P}_R$ terms proportional to only $\not{q}$ vanish. The nominator in the integral over d^4q can then be expressed as

$$
\not{q}(\not{q} - \not{k}) + m_k^2 = \frac{1}{2}\left(q^2 - m_k^2 + (q-k)^2 + m_k^2\right) + \frac{1}{2}\left([\not{k}, \not{q}] - k^2 + 4m_k^2\right). \tag{18}
$$

The neutrino masses m_k^2 are small compared to k^2 and $[\not{k}, \not{q}]$ is small compared to q^2, so both can be neglected. The approximation is justified, as neglecting the k^2 term in the nominator only leads to a difference of about 5%.
The amplitude can then be written as

$$\begin{aligned}\langle f|\, S^2\, |i\rangle = &-\frac{4i}{2!}\left(\frac{G_F\cos(\theta_C)g_\phi}{\sqrt{2}}\right)^2 N_{p_1}N_{p_2}N_{p_4}\left[\frac{1}{k^2-m_\phi^2}\sum_i U_{ei}\bar{v}_i(p_4)v_i^C(p_3)\right] \\ &\int \mathrm{d}^4x_1\mathrm{d}^4x_2 e^{ix_1p_1}e^{ix_2p_2}\,\langle N_f|\, T\{J^\alpha(x_1)J^\beta(x_2)\}\,|N_i\rangle \\ &\left[\sum_k U_{ek}^2\int\frac{\mathrm{d}^4q}{(2\pi)^4}e^{-iqx_1+i(q-k)x_2}\frac{\frac{1}{2}\left(q^2-m_k^2+(q-k)^2-m_k^2\right)+\frac{1}{2}\left(-k^2\right)}{(q^2-m_k^2)((q-k)^2-m_l^2)}\right] \\ &\bar{u}_L(p_1)\gamma_\alpha\hat{P}_L\gamma_\beta\hat{C}\bar{u}_L^T(p_2)-(p_1\rightleftharpoons p_2).\end{aligned} \tag{19}$$

with $G_F=\sqrt{2}g^2/(8M_W^2)$ and $C^{-1}=C^\dagger=-C$ and $\gamma_\beta^T=-C\gamma_\beta C^{-1}$.
To evaluate the Feynman diagram with the switched electrons, it can be observed, that

$$[\bar{u}_L(p_1)\gamma_\alpha\hat{P}_L\gamma_\beta\hat{C}\bar{u}_L^T(p_2)]^T=-\bar{u}_L(p_2)\gamma_\beta\hat{P}_L\gamma_\alpha\hat{C}\bar{u}_L^T(p_1). \tag{20}$$

Since $T\{J^\alpha(x_1)J^\beta(x_2)\}=T\{J^\beta(x_2)J^\alpha(x_1)\}$, the amplitude for the switched electrons is just minus the amplitude and the term $-(p_1\rightleftharpoons p_2)$ leads to a factor 2 in the amplitude.

3.1.2 Evaluation of the integrals of the internal propagators

To evaluate the integrals in the amplitude, first the integrals over $\mathrm{d}q^0$ and then $\mathrm{d}x_1^0$ and $\mathrm{d}x_2^0$ will be calculated.
The integral over d^4q can be split in 2 parts

$$\begin{aligned}&\frac{1}{2}\int\frac{\mathrm{d}^4q}{(2\pi)^4}e^{-iqx_1+i(q-k)x_2}\frac{\left(q^2-m_k^2+(q-k)^2-m_k^2\right)+\left(-k^2\right)}{(q^2-m_k^2)((q-k)^2-m_l^2)} \\ &=\frac{1}{2}\int\frac{\mathrm{d}^4q}{(2\pi)^4}e^{-iqx_1+i(q-k)x_2}\left[\frac{1}{q^2-m_k^2}+\frac{1}{(q-k)^2-m_k^2}\right] \\ &-\frac{1}{2}\int\frac{\mathrm{d}^4q}{(2\pi)^4}e^{-iqx_1+i(q-k)x_2}\frac{k^2}{(q^2-m_k^2)((q-k)^2-m_l^2)}.\end{aligned} \tag{21}$$

The second summand in the first integral in equation (21) can be transformed with $q\to q-k$ resulting in

$$\begin{aligned}&\frac{1}{2}\int\frac{\mathrm{d}^4q}{(2\pi)^4}e^{-iqx_1+i(q-k)x_2}\left[\frac{1}{q^2-m_k^2}+\frac{1}{(q-k)^2-m_k^2}\right] \\ &=\frac{1}{2}\int\frac{\mathrm{d}^4q}{(2\pi)^4}e^{-iq(x_1-x_2)}\frac{1}{q^2-m_k^2}(e^{-ikx_1}+e^{-ikx_2}).\end{aligned} \tag{22}$$

The integral over $\mathrm{d}q^0$ for this can then be evaluated with the Cauchy's residue theorem, where the areas $x_1>x_2$ and $x_1<x_2$ need to be separated.
After the integration over $\mathrm{d}q^0$ the integration over $\mathrm{d}x_1^0$ and $\mathrm{d}x_2^0$ is made. For these integrations the invariance under time translations of the hadronic currents is used

$$J^\alpha(x)=e^{iHx^0}J^\alpha(\vec{x})e^{-iHx^0}. \tag{23}$$

For the second integral in equation (21) the integration over $\mathrm{d}q^0$ can be solved by either integrating over the 4 poles or with Feynman parametrization.

3.1.3 Approximations and Nuclear Matrix Element

The remaining amplitude after the integrations over the energy components of the 4-momenta is

$$\langle f|S^2|i\rangle = -4i\left(\frac{G_F\cos(\theta_C)g_\phi}{\sqrt{2}}\right)^2 N_{p_1}N_{p_2}N_{p_4}\left[\frac{1}{k^2-m_\phi^2}\sum_i U_{ei}\bar{v}_i(p_4)v_i^C(p_3)\right] \tag{24}$$

$$\bar{u}_L(p_1)\gamma_\alpha\hat{P}_L\gamma_\beta\hat{C}\bar{u}_L^T(p_2)$$

$$\int \mathrm{d}^3x_1\mathrm{d}^3x_2 e^{-i\vec{x}_1\vec{p}_1-i\vec{x}_2\vec{p}_2}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\sum_k U_{ek}^2 e^{i\vec{q}(\vec{x}_1-\vec{x}_2)}\delta^0(p_1^0+p_2^0-k^0+E_f-E_i)$$

$$\{\frac{(-2\pi)}{4q^0}\sum_n(\frac{\langle N_f|J^\alpha(x_1)|N_n\rangle\langle N_n|J^\beta(x_2)|N_i\rangle}{E_n-E_i+p_2^0+q^0}e^{i\vec{k}\vec{x}_1}$$

$$+\frac{\langle N_f|J^\alpha(x_1)|N_n\rangle\langle N_n|J^\beta(x_2)|N_i\rangle}{E_n-E_i+p_2^0-k^0+q^0}e^{i\vec{k}\vec{x}_2}$$

$$+\frac{\langle N_f|J^\beta(x_2)|N_n\rangle\langle N_n|J^\alpha(x_1)|N_i\rangle}{E_n-E_i+p_1^0-k^0+q^0}e^{i\vec{k}\vec{x}_1}$$

$$+\frac{\langle N_f|J^\beta(x_2)|N_n\rangle\langle N_n|J^\alpha(x_1)|N_i\rangle}{E_n-E_i+p_1^0+q^0}e^{i\vec{k}\vec{x}_2})\}$$

$$+\frac{k^2}{2}\int_0^1\mathrm{d}y\frac{2\pi}{4(q_{ij}^0)^3}e^{iy\vec{k}\vec{x_1}+i(y+1)\vec{k}\vec{x_2}}$$

$$\sum_n\left[\frac{\langle N_f|J^\alpha(x_1)|N_n\rangle\langle N_n|J^\beta(x_2)|N_i\rangle}{E_n-E_i+p_2^0+q_{ij}^0+(y-1)k^0}+\frac{\langle N_f|J^\beta(x_2)|N_n\rangle\langle N_n|J^\alpha(x_1)|N_i\rangle}{E_n-E_i+p_1^0+q_{ij}^0-yk^0}\right].$$

With $q_{ij}^0=\sqrt{|\vec{q}|^2+yk^2(y-1)+y(m_j^2-m_i^2)+m_i^2}$ and $q^0=\sqrt{|\vec{q}|^2+m^2}$.
The following approximations can be made:

- $m_k^2 \ll |\vec{q}|^2$, since neutrino masses are very small. This leads to $q^0=|\vec{q}|$ and $q_{ij}^0=\sqrt{|\vec{q}|^2+yk^2(y-1)}$.
- Closure Approximation: $E_n=\bar{E}$, since the energy of the virtual neutrino is much bigger, than the energy of the intermediate states of the nucleon. $\bar{E}$ is the average energy of the intermediate states.
- Long-wave approximation: $e^{-i\vec{p_i}x_i}=1$ and $e^{-i\vec{k}x_i}=1$.

With energy conservation we also have, that $E_i+E_3=E_f+p_1^0+p_2^0+p_4^0$. Since the nucleons can be viewed as static at low energies, $E_i=M_i$ and $E_f=M_f$, so that the energy conservation becomes $M_i-M_f=p_1^0+p_2^0-k^0$. With energy conservation, the denominators can be simplified to

$$E_n-E_i+p_2^0+q^0=|\vec{q}|+\bar{E}-\frac{M_i+M_f}{2}+\frac{E_2-E_1+E_3-E_4}{2}. \tag{25}$$

For low energies ($E_3\le 100\,\mathrm{MeV}$), $|\vec{q}|\sim 100\,\mathrm{MeV}$ is much bigger than the differences of the energies of the external particles, so they can be neglected. The other denominators can be simplified analogously.
With this additional approximation the integrand becomes independent from the integration variable y, the integral over y becomes trivial.

With all approximations the cross section is

$$\langle f|S^2|i\rangle = 4i\left(\frac{G_F\cos(\theta_C)g_\phi}{\sqrt{2}}\right)^2 N_{p_1}N_{p_2}N_{p_4}\left[\frac{1}{k^2-m_\phi^2}\sum_i U_{ei}\bar{v}_i(p_4)v_i^C(p_3)\right] \tag{26}$$

$$\bar{u}_L(p_1)\gamma_\alpha \hat{P}_L \gamma_\beta \hat{C} \bar{u}_L^T(p_2)\delta^0(p_1^0+p_2^0-k^0+E_f-E_i)$$
$$\frac{\sum_k U_{ek}^2 2\pi}{4} \int \mathrm{d}^3x_1 \mathrm{d}^3x_2 \left\langle N_f \right| J^\alpha(x_1)J^\beta(x_2)+J^\beta(x_2)J^\alpha(x_1) \left| N_i \right\rangle$$
$$\int \frac{\mathrm{d}^3q}{(2\pi)^3} \frac{e^{i\vec{q}(\vec{x}_1-\vec{x}_2)}}{|\vec{q}|(\bar{E}+|\vec{q}|-\frac{M_i+M_f}{2})} \cdot \left(2-\frac{k^2}{2|\vec{q}|^2}\right).$$

Since $J^\alpha(x_1)J^\beta(x_2)+J^\beta(x_2)J^\alpha(x_1)$ is symmetric under exchange of α and β, but the spinor part of the amplitude $\gamma_\alpha \hat{P}_L \gamma_\beta = \gamma_\alpha \gamma_\beta \hat{P}_R$ is antisymmetric under exchange of the indices for $\alpha \neq \beta$, only the case $\alpha = \beta$ has a non-vanishing contribution. From the symmetric contribution of the nuclear currents, we get an additional factor 2.
The nuclear matrix element (NME) is defined as

$$M^{0\nu} = -\frac{4\pi R}{(2\pi)^3} \int \mathrm{d}^3x_1 \mathrm{d}^3x_2 \mathrm{d}^3q \frac{e^{i\vec{q}(\vec{x}_1-\vec{x}_2)}}{|\vec{q}|(\bar{E}+|\vec{q}|-\frac{M_i+M_f}{2})} \left\langle N_f \right| J^\alpha(x_1)J_\alpha(x_2) \left| N_i \right\rangle. \tag{27}$$

R is the radius of the nucleus.
If the $1/q^3$ term in the last equation is transformed into $1/(q\bar{q}^2)$ with $\bar{q}$ the average momentum of the virtual neutrino, the integrals can then be collected in the NME. The approximation is justified, as k^2 is assumed to be small next to q^2. Furthermore, the NME has a very high uncertainty, so this assumption should not significantly influence the result.
Extracting the NME the amplitude becomes

$$\left\langle f \right| S^2 \left| i \right\rangle = -i\left(\frac{G_F\cos(\theta_C)g_\phi}{\sqrt{2}}\right)^2 N_{p_1}N_{p_2}N_{p_4}\left[\frac{1}{k^2-m_\phi^2}\sum_i U_{ei}\bar{v}_i(p_4)v_i^C(p_3)\right] \tag{28}$$
$$\bar{u}_L(p_1)\hat{P}_R\hat{C}\bar{u}_L^T(p_2)\delta^0(p_1^0+p_2^0-k^0+E_f-E_i)$$
$$\sum_k U_{ek}^2\frac{M^{0\nu}}{R}\cdot\left(2-\frac{k^2}{2\bar{q}^2}\right).$$

3.1.4 Evaluating the phase space integrations

The formula for the cross-section of the $2 \to 4$ scattering process is (from [1])

$$\mathrm{d}\sigma = \frac{1}{2E_i 2E_3|v_i-v_3|}|\left\langle f \right| \bar{S}^2 \left| i \right\rangle|^2 F(E_1,Z+2)F(E_2,Z+2)\mathrm{d}^3p_1\mathrm{d}^3p_2\mathrm{d}^3p_4\mathrm{d}^3p_f. \tag{29}$$

$F(E,Z) = \frac{2\pi\eta}{1-e^{-2\pi\eta}}$ with $\eta = Z\alpha m_e/p$ is the Fermi-function of the nucleus, that describes the interaction of the outgoing electron with the remaining nucleus. Since the nucleus can be viewed as static, the integral over d^2p_f can be omitted. Because of that also the factor $1/(2E_i)$, which is the relativistic normalisation factor, is omitted since the nucleon is not relativistic with $v_i = 0$.
The amplitude squared is

$$|\left\langle f \right| S^2 \left| i \right\rangle|^2 = \left(\frac{G_F\cos(\theta_C)g_\phi}{\sqrt{2}}\right)^4 \frac{1}{(2\pi)^9}\frac{1}{2E_1 2E_2 2E_4}\left[\frac{1}{k^2-m_\phi^2}\right]^2 \tag{30}$$
$$\left(\sum_k U_{ek}^2\frac{M^{0\nu}}{R}\right)^2\cdot\left(2-\frac{k^2}{2\bar{q}^2}\right)^2(\delta^0(p_1^0+p_2^0-k^0+E_f-E_i))^2$$
$$8\cdot(p_1\cdot p_2)(p_3\cdot p_4).$$

It is again assumed, that the neutrino masses are small next to the momenta and can be neglected. The square of the spinor matrix multiplications was evaluated with the FeynCalc

package in Mathematica [8–10].
The differential cross section dependent on the 2 electron energies $\mathrm{d}\sigma/(\mathrm{d}E_1\mathrm{d}E_2)$ can be calculated analytically

$$\frac{\mathrm{d}\sigma}{\mathrm{d}E_1\mathrm{d}E_2} = \left(\frac{G_F\cos(\theta_C)g_\phi}{\sqrt{2}}\right)^4 \frac{1}{(2\pi)^7}\left(\frac{M^{0\nu}}{R}\right)^2 \frac{1}{4}E_4^2\sqrt{E_1^2 - me^2}E_1\sqrt{E_2^2 - me^2}E_2 \qquad (31)$$
$$\Omega F(E_1, Z+2)F(E_2, Z+2)\delta^0(E_1 + E_2 + E_4 - E_3 - Q_{if}).$$

$Q_{if} = E_i - E_f$ is the difference of the energy of the nucleon before and after the decay. Ω is the integral over the angular dependencies

$$\Omega = \int_0^\pi \left[\frac{1}{2E_3E_4(1-\cos(\theta_4)) + m_\phi^2}\right]^2 \left(2 + \frac{2E_3E_4(1-\cos(\theta_4))}{2\bar{q}}\right)^2 \sin\theta_4(1-\cos(\theta_4))\mathrm{d}\theta_4 \quad (32)$$
$$= \int_0^\pi \frac{1}{4\bar{q}^4}\frac{\left(1-\cos\theta_4 + \frac{2\bar{q}^2}{E_3E_4}\right)^2}{(1-\cos\theta_4 + \frac{m_\phi^2}{2E_3E_4})^2}\sin\theta_4(1-\cos\theta_4)\mathrm{d}\theta_4$$
$$= \frac{1}{4\bar{q}^4}\frac{1}{2+\frac{m_\phi^2}{2E_3E_4}}\{4 - 6(\frac{m_\phi}{2E_3E_4})^2 + 8\frac{2\bar{q}^2}{E_3E_4} - 2(\frac{2\bar{q}^2}{E_3E_4})^2 + (8\frac{2\bar{q}^2}{E_3E_4} - 6)\frac{m_\phi^2}{2E_3E_4}$$
$$-(2+\frac{m_\phi^2}{2E_3E_4})(3(\frac{m_\phi^2}{2E_3E_4})^4 - 4\frac{m_\phi^2}{2E_3E_4}\frac{2\bar{q}^2}{E_3E_4} + (\frac{2\bar{q}^2}{E_3E_4})^2)\log\left(\frac{m_\phi^2}{2E_3E_4}/(2+\frac{m_\phi^2}{2E_3E_4})\right)\}.$$

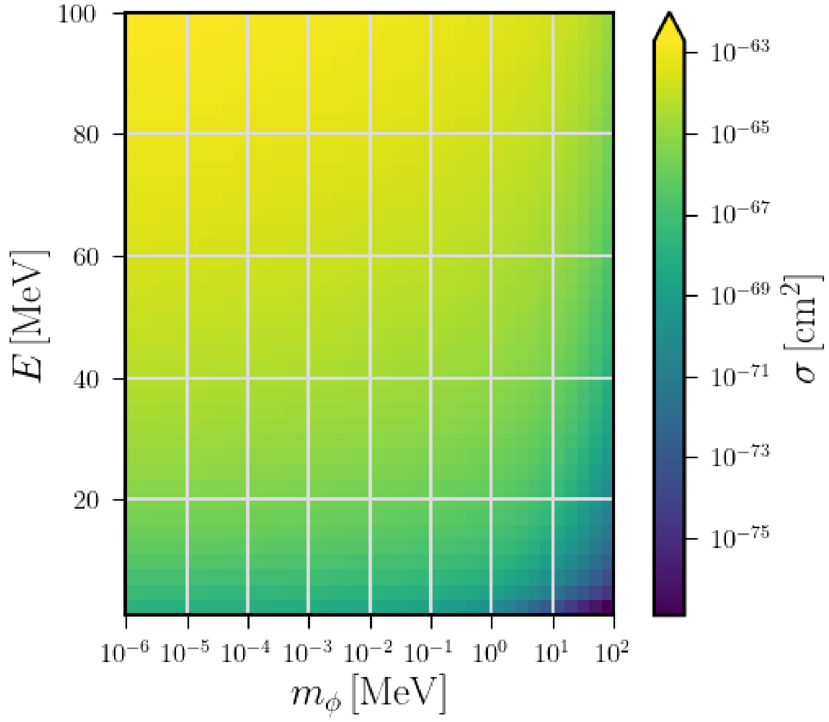
The remaining integral over the two energies can be evaluated numerically. For this report the python module SciPy [11] was used to integrate the cross sections.

3.1.5 *Numerical Evaluation for Germanium*

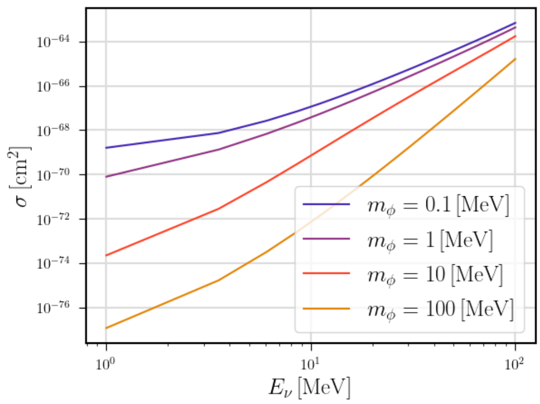
Germanium ${}^{76}_{32}\mathrm{Ge}$ is a nucleus, that decays under the double β decay and is therefore a potential candidate for the neutrinoless double β decay.
Different values for the NME from different calculations can be found in [1]. For this calculation the mean value of $M^{0\nu} = (10.39 - 3.59)/2$ was used.
The cross section can than be evaluated for variable low energy neutrinos with E_3 and variable Majoron mass $m_\phi > m_\nu$. The lower border of the Majoron mass is determined by the assumption that $m_\phi > m_\nu$. In [7] it can be seen, that for a $\mathcal{O}(1)$ coupling g_{ee} of the Majoron to the neutrinos, $m_\phi > 1\,\mathrm{MeV}$ is needed. This leads to the evaluated range of $10^{-6}\,\mathrm{MeV} < m_\phi < 100\,\mathrm{MeV}$. Since the Majoron neutrino coupling is just a prefactor, g_ϕ is set to 1 in the calculations.
The assumptions made in the low energy calculation are valid until about 100 MeV, so the energy range from $1 - 100\,\mathrm{MeV}$ is evaluated.

In order to compare the cross section with rates needed for experiments, the rate of the process is calculated. For this a neutrino flux from a reactor source with 1 GW thermal power is assumed. Neutrinos carry one third of the thermal power. The energy of the emitted reactor neutrino is the energy of the energy of the inducing neutrino, leading to $N_\nu = 1/3 \cdot 1\,\mathrm{GW} \cdot 1/E_3$. The distance to the reactor is 10 m and the target mass is 1 kg. The expected rate can then be calculated with (see [5])

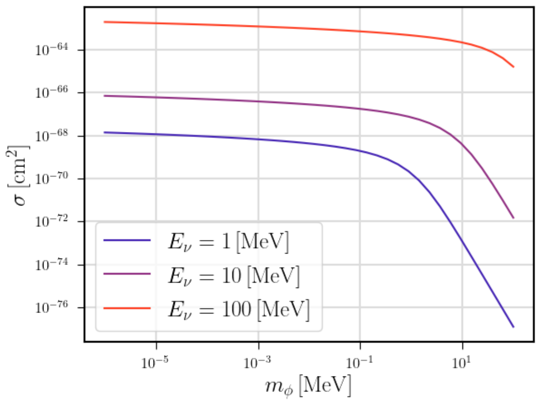
$$\Gamma = \phi\sigma N_{\mathrm{Atoms}} = \frac{N_{\mathrm{Atoms}}N_{\nu/s}}{A}\sigma. \qquad (33)$$



(a) Cross-section.



(b) Energy Scaling.



(c) Mass Scaling.

Figure 3: The cross-section of the induced neutrinoless double β decay for different masses and energies. The energies and masses given for the different curves in the scaling plot are approximates to give an impression of the scaling at different order of magnitude for E_3 and m_ϕ.

The calculated cross section for the induced decay is still very small, as can be seen in figure 3a. But it can be seen, that the cross section scales strongly with the energy, since it grows multiple orders of magnitude in the energy range of $1 - 100\,\text{MeV}$ for the same Majoron mass (see figure 3b). This scaling also depends on the mass of the Majoron, for higher masses a stronger energy scaling can be expected. It can also be observed that the cross section reduces for a growing Majoron mass. For Majoron masses lower than $1\,\text{MeV}$ the cross section is almost constant for different masses, but for $m_\phi > 1\,\text{MeV}$ it falls rapidly with growing majoron mass (see figure 3c).

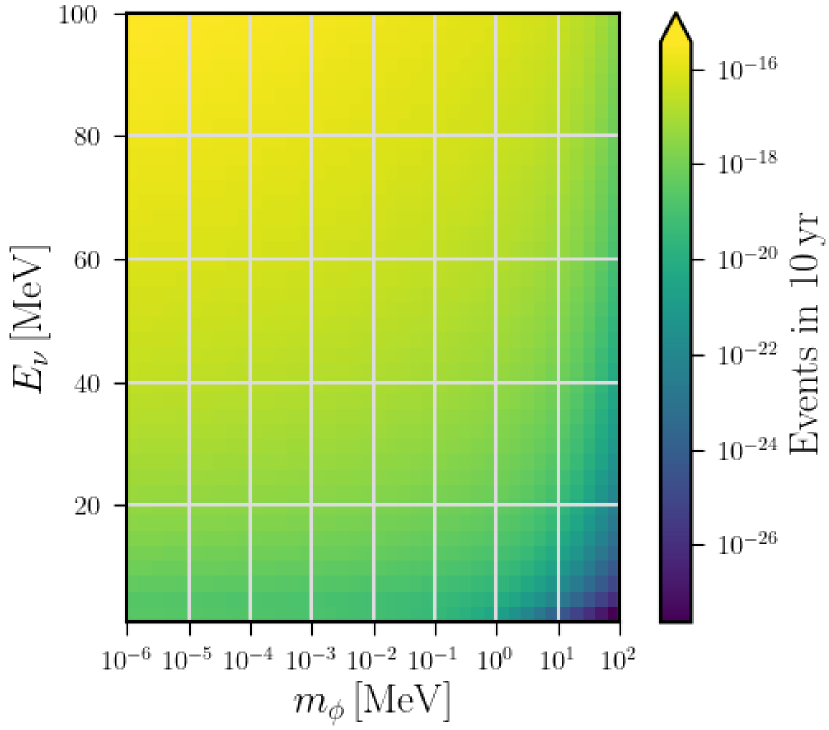


Figure 4: Expected number of events in 10 yr.

As can be seen in figure 4 the expected amounts of neutrinos in 10 years for the induced neutrinoless double β decay from a reactor source is too small to be able to be measured in an experiment over a feasible amount of time. However, it is also visible, that there is a relatively strong power-law energy dependence. Therefore, for higher energies it could be possible to get a measurable rate.

3.2 Dark matter induced neutrinoless double β decay

As seen in figure 4 inducing the neutrinoless double β decay with low energy neutrinos does not lead to a measurable rate. However, in [12] it was motivated, that a dark matter particle could also induce the neutrinoless double β decay via dark matter coupling to the Majoron. Therefore, the cross section for the dark matter induced decay is calculated in this section.
Here it is assumed, that the dark matter particle is a scalar particle with m_{DM} in the range $1 - 100\,\mathrm{MeV}$. These assumptions differ from [12], where the dark matter particle is a fermion.

3.2.1 Calculation of the cross section

As a dark matter particle a scalar particle is assumed, that couples to the Majoron. Therefore, the coupling g_{DM} has Dimension [MeV] as it connects two scalar particles.

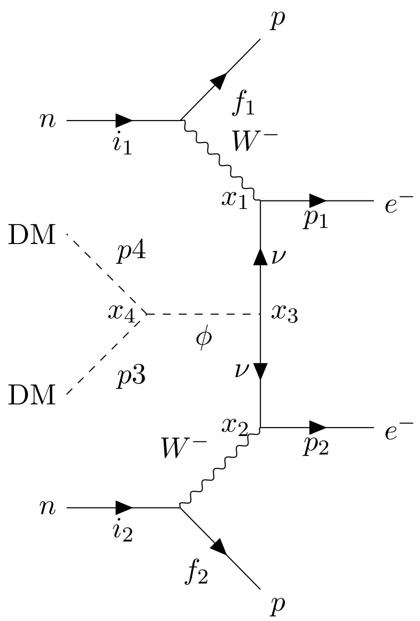


Figure 5: Feynman diagram of the dark matter induced neutrinoless double β decay.

The amplitude for the Feynman diagram for the dark matter particle (see figure 5) is in position space

$$\begin{aligned}\langle f|\, S^2\,|i\rangle = \frac{1}{2!}\left(\frac{ig}{2\sqrt{2}}\cos\theta_C\right)^2\left(\frac{ig}{\sqrt{2}}\right)^2\left(\frac{i}{M_W}\right)^2 N_{p_1}N_{p_2}N_{p_4} \\ \int \mathrm{d}^4x_1\mathrm{d}^4x_2\mathrm{d}^4x_3\mathrm{d}^4x_4\,\langle N_f|\,T\{J^\alpha(x_1)J^\beta(x_2)\}\,|N_i\rangle \\ \bar{u_L}(p_1)e^{ix_1p_1}\gamma_\alpha\,\langle 0|\,T\{v_{e,L}(x1)v_k^T(x_3)\}\,|0\rangle\,\hat{C} \\ \left[\langle 0|\,T\{\phi(x_3)\phi^\dagger(x_4)\}\,|0\rangle\,e^{ip_4x_4}(-ig_{DM})e^{-ip_3x_4}\right] \\ (-ig_\phi\delta_{kl})\,\langle 0|\,T\{v_l(x3)v_{e,L}^T(x_2)\}\,|0\rangle\,\gamma_\beta^T\bar{u_L}^T(p_2)e^{ix_2p_2} - (p_1 \rightleftharpoons p_2).\end{aligned} \tag{34}$$

The difference between the amplitudes is mainly in the absence of the spinors for the incoming and outgoing neutrinos, which are replaced by the incoming and outgoing dark matter particles. With the assumption, that $m_{\mathrm{DM}} \ll m_\phi$ the amplitude can be evaluated similar to the amplitude of the decay induced with neutrinos.
The amplitude in momentum space is then

$$\begin{aligned}\langle f|S^2\,|i\rangle = -2i\left(\frac{G_F\cos(\theta_C)}{\sqrt{2}}\right)^2 g_\phi g_{\mathrm{DM}}N_{p_1}N_{p_2}N_{p_4}\left[\frac{1}{k^2-m_\phi^2}\right] \\ \bar{u}_L(p_1)\hat{P}_R\hat{C}\bar{u}_L^T(p_2)\delta^0(p_1^0+p_2^0-k^0+E_f-E_i) \\ \sum_k U_{ek}^2\frac{M^{0\nu}}{R}\cdot\left(1-\frac{k^2}{4\bar{q}'^2}\right).\end{aligned} \tag{35}$$

The differential cross section for this process is

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}E_1\mathrm{d}E_2\mathrm{d}\theta} = \frac{2}{(2\pi)^7}\frac{(M^{0\nu})^2}{R^2}(G_F\cos\theta_C)^4 g_\phi^2 g_{\mathrm{DM}}^2\frac{F(E_1,Z+2)F(E_2,Z+2)}{16\bar{q}^4|\vec{p}_3|} \\ \cdot\left[\frac{4\bar{q}^4-2m_{\mathrm{DM}}^2+E_3E_4-|\vec{p}_3||\vec{p}_4|\cos\theta}{2m_{\mathrm{DM}}^2-E_3E_4+|\vec{p}_3||\vec{p}_4|\cos\theta-m_\phi^2}\right]^2\cdot E_1|\vec{p}_1|E_2|\vec{p}_2||\vec{p}_4|\sin\theta.\end{aligned} \tag{36}$$

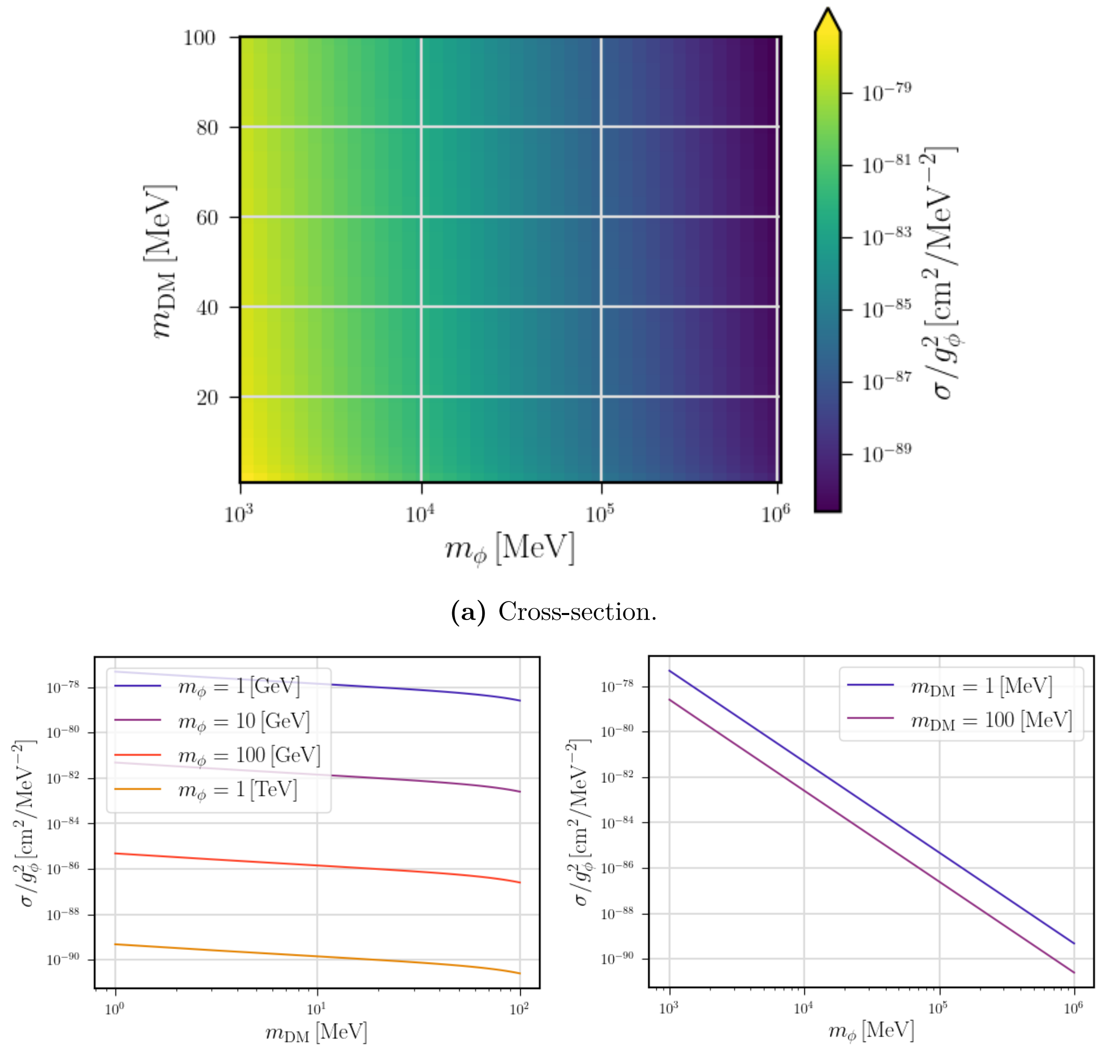
with $E_4 = M_i - M_f + E_3 - E_1 - E_2$.
The remaining cross section for the dark matter induced decay can also be evaluated numerically. The python module SciPy was again used for the numerical integration [11].

3.2.2 Numerical evaluation for Germanium

The cross section and the rate was calculated for 1 kg of Germanium as the β decay material and for variable m_{DM} and m_ϕ. As m_{DM} is in the range $1 - 100\,\mathrm{MeV}$, the Majoron mass is chosen in the range $1\,\mathrm{GeV} - 1\,\mathrm{TeV}$ so that $m_{\mathrm{DM}} \ll m_\phi$. The energy of the incoming dark matter particle is given by m_{DM} and the dark matter velocity $v_{\mathrm{DM}} = 220\,\mathrm{km/s}$

$$E_{\mathrm{DM}} = \sqrt{m_{\mathrm{DM}}^2 + \left(m_{\mathrm{DM}} \cdot \frac{v_{\mathrm{DM}}}{c}\right)^2}. \tag{37}$$

c is the speed of light.



(a) Cross-section.

(b) Energy Scaling.

(c) Mass Scaling.

Figure 6: The cross section of the dark matter induced neutrinoless double β decay for different masses and energies. The masses given for the different curves in the scaling plot are approximate to give an impression of the scaling at different order of magnitude for m_{DM} and m_ϕ.

As can be seen in figure 6a the cross section for the dark matter induced case is multiple orders of magnitude smaller, than for the neutrino induced case. The cross section has only a small dependence on the energy E_{DM}, as can be seen in figure 6b. This is to be expected, since $k^2 \ll \vec{q}^2$ and $E_{\text{DM}} \ll m_\phi$.
The cross section also still depends on the Majoron dark matter coupling g_{DM} [MeV].

The experimental rate $\Gamma_{\text{DM}} = nvN_A\sigma$ can be evaluated with N_T the number of atoms in the target material, $n = \rho/m_{\text{DM}}$ with the local dark matter density $\rho = 0.3\,\text{GeV/cm}^3$ and the dark matter speed $v = 220\,\text{km/s}$.

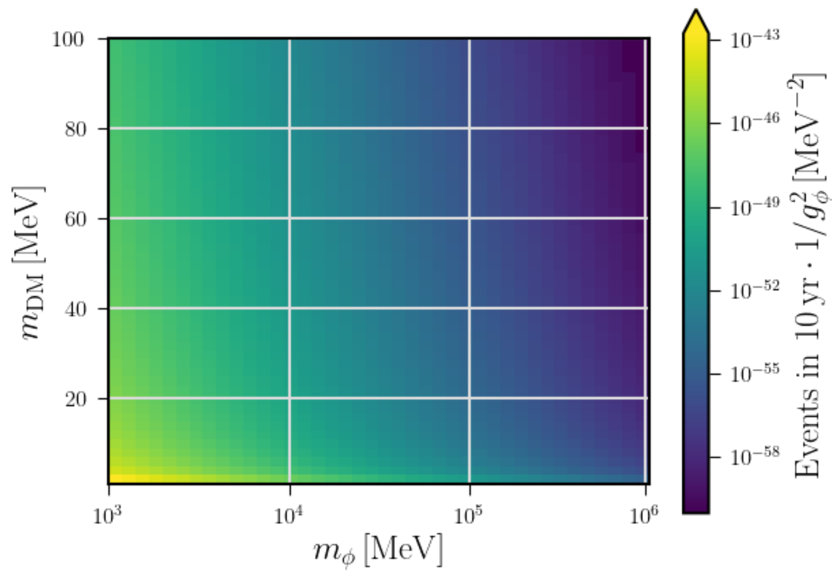
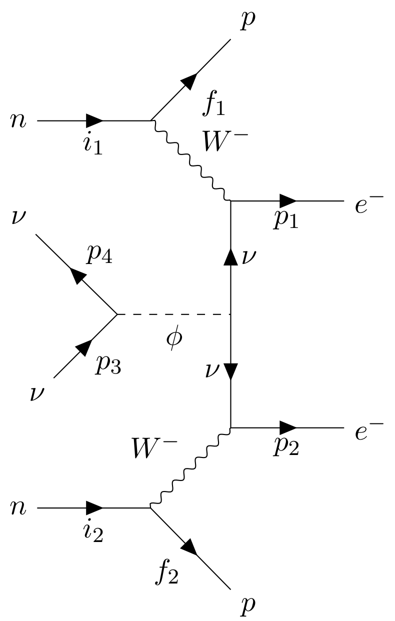


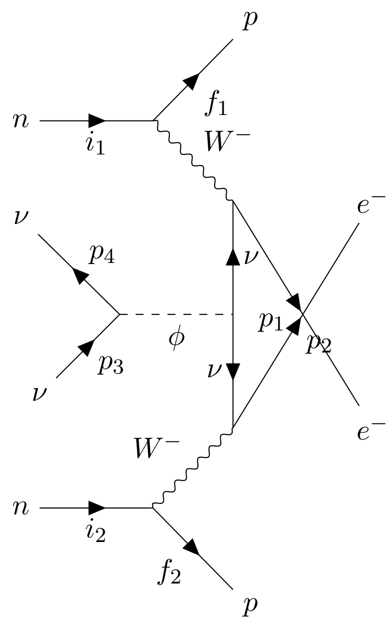
Figure 7: Expected number of events in 10 yr for the dark matter induced neutrinoless double β decay.

As seen in figure 7 the event numbers for this process are very small for the considered dark matter masses m_{DM}. Therefore, an experiment to measure the decay induced by a dark matter particle is not viable.
A measurable number of events could be achieved by going to ultra-light dark matter, as the dark matter rate could be big enough, so that this decay is measurable, as motivated in [13].
The rate is still dependent on the coupling strength between the Majoron and the dark matter particle, but the coupling is not expected to be strong enough to reach a measurable number of events.

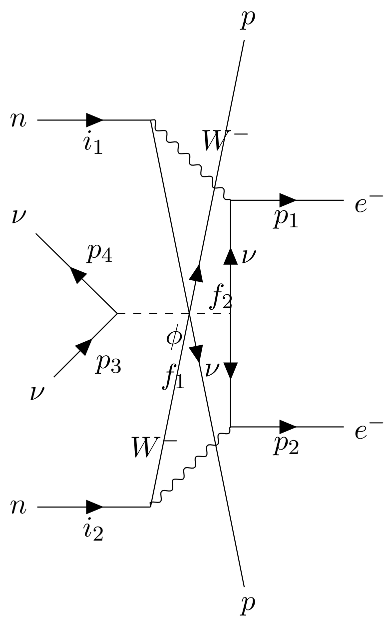
4 Quasi-elastic scattering



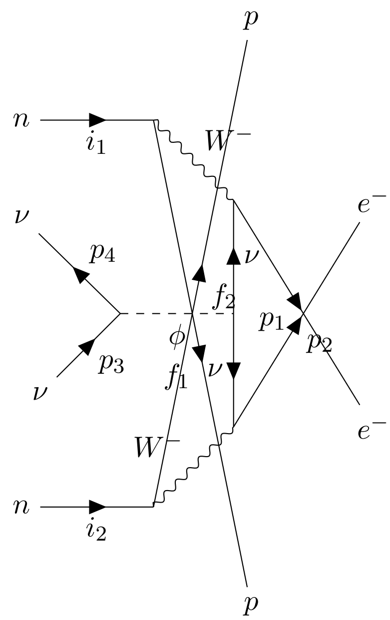
(a) No switched vertices.



(b) Switched electron vertices.



(c) Switched nucleon vertices.



(d) Switched electron and nucleon vertices.

Figure 8: All Feynman Diagrams for the induced $0\nu\beta\beta$ with different switched vertices. The initial neutrons are still bound in a nucleus, but the final state protons are not necessarily bound anymore.

For higher energy the nucleus, that the neutrino scatters off, can't be treated as a static object anymore. The neutrinos are energetic enough to interact directly with the neutrons in the nucleus.
The induced decay is then calculated with the neutrons and protons involved in the decay.
The neutrons are still bound in the nucleus and the possible neutron energies are therefore

determined by the distribution of a Fermi-gas at zero temperature in the given nucleus.
The final protons are treated as separate final state particles next to the electrons.
The Feynman diagram for this process is displayed in figure 8a.

4.1 Calculation of the amplitude

The amplitude for this process can be determined with the Feynman rules in momentum space and the hadronic current

$$\bar{\Psi}\gamma^{\mu}(1-1.27\gamma^{5})\Psi \tag{38}$$

for the current between the neutron and the proton. This leads to the proton neutron interaction vertex $\frac{ig}{\sqrt{2}}\gamma^{\alpha}(1-1.27\gamma^{5}) = \frac{ig}{\sqrt{2}}J^{\alpha}$. Since there are 2 identical final state electrons and 2 identical final state protons, the symmetry factor is 1/(2!2!). This also leads to 3 additional diagrams with exchanged vertices. All Feynman diagrams for this process are displayed in 8.
The full matrix element is then

$$\begin{aligned}\langle f|S^{2}|i\rangle = \frac{4}{2!2!}\left(\frac{G_F\cos(\theta_C)g_\phi}{\sqrt{2}}\right)^{2} N_{f_1}N_{f_2}N_{p_1}N_{p_2}N_{p_4}(2\pi)^4\left[\sum_i U_{ei}\frac{\bar{v}_i(p_4)v_i^C(p_3)}{k^2-m_\phi^2}\right]\delta^{(4)} \qquad (39)\\
\{(\bar{p}(f_1)J^{\alpha}n(i_1))\left(\bar{p}(f_2)J^{\beta}n(i_2)\right)\bar{e}(p_1)\gamma_\alpha\frac{(\not{f}_1+\not{p}_1-\not{i}_1)(\not{f}_2+\not{p}_2-\not{i}_2)}{(f_1+p_1-i_1)^2\cdot(f_2+p_2-i_2)^2}\hat{P}_L\gamma_\beta\hat{C}\bar{e}^T(p_2)\\
-(\bar{p}(f_1)J^{\alpha}n(i_1))\left(\bar{p}(f_2)J^{\beta}n(i_2)\right)\bar{e}(p_2)\gamma_\alpha\frac{(\not{f}_1+\not{p}_2-\not{i}_1)(\not{f}_2+\not{p}_1-\not{i}_2)}{(f_1+p_2-i_1)^2\cdot(f_2+p_1-i_2)^2}\hat{P}_L\gamma_\beta\hat{C}\bar{e}^T(p_1)\\
-(\bar{p}(f_2)J^{\alpha}n(i_1))\left(\bar{p}(f_1)J^{\beta}n(i_2)\right)\bar{e}(p_1)\gamma_\alpha\frac{(\not{f}_2+\not{p}_1-\not{i}_1)(\not{f}_1+\not{p}_2-\not{i}_2)}{(f_2+p_1-i_1)^2\cdot(f_1+p_2-i_2)^2}\hat{P}_L\gamma_\beta\hat{C}\bar{e}^T(p_2)\\
+(\bar{p}(f_2)J^{\alpha}n(i_1))\left(\bar{p}(f_1)J^{\beta}n(i_2)\right)\bar{e}(p_2)\gamma_\alpha\frac{(\not{f}_2+\not{p}_2-\not{i}_1)(\not{f}_1+\not{p}_1-\not{i}_2)}{(f_2+p_2-i_1)^2\cdot(f_1+p_1-i_2)^2}\hat{P}_L\gamma_\beta\hat{C}\bar{e}^T(p_1)\}.\end{aligned}$$

Here p is the proton spinor, n the neutron spinor and e the electron spinor. The neutrino mass in the propagators is small enough to be neglected.
The square of the amplitude is evaluated with FeynCalc in Mathematica [8–10].

4.2 Evaluation of the cross section

With the given matrix element, the cross section for the 5-particle final state is calculated

$$\mathrm{d}\sigma(p_3,i_1,i_2) = \frac{|\langle f|S^2|i\rangle|^2}{2E_3(2M_{\text{initial}})^2}F(E_1,Z+2)F(E_2,Z+2)\mathrm{d}^3p_1\mathrm{d}^3p_2\mathrm{d}^3p_4\mathrm{d}^3f_1\mathrm{d}^3f_2. \tag{40}$$

The initial energy of the neutrons inside the nucleons is not known. Therefore, it is assumed, that the neutrons form a Fermi-gas at zero temperature inside the nucleus. The energy distribution of the neutrons is then a Fermi-Dirac distribution at zero temperature

$$f(E) = \Theta(E_F + m_n - E) \tag{41}$$

where Θ is the Heaviside step function, E_F is the Fermi energy and m_n is the neutron mass. The cross section is then integrated over all initial neutron states weighted by the Fermi-Dirac distributions of the initial nuclei

$$\sigma(E_3) = \int \mathrm{d}\sigma(p_3,i_1,i_2)f(E(i_1))f(E(i_2))\mathrm{d}i_1\mathrm{d}i_2\cdot V_{\text{Nucleus}}. \tag{42}$$

The integral is normalised with the volume of the nucleus.
The normalisation parameter is derived with the total rate

$$\Gamma \propto V(\int \mathrm{d}^3p_\nu f_\nu)\frac{1}{2E_\nu}(\int \Pi_f|\langle f|S^2\frac{\mathrm{d}^3p_f}{2E_f}|i\rangle|^2F(E_1,Z+2)F(E_2,Z+2)f(E_{i_1})f(E_{i_2}). \tag{43}$$

The cross section is related to the total rate via $\sigma = \Gamma/j_0$ with the neutrino rate $j_0 = n_\nu v_0$. If the neutrino distribution is now a very sharply peaked distribution around E_3, the integral over the neutrino distribution is just the rate $j_0 = \int \mathrm{d}^3 p_\nu f_\nu$ and the neutrino energy is E_3 in the cross section. This leads to the normalisation factor of V_{nucleus} in the cross section.

4.3 Approximations in the numerical Integration

The integral is calculated numerically with the CUBA library in C++ [14] and also the VEGAS library in python [15]. The python module SymPy [16] was used to convert Mathematica expressions into python functions.
However, the amplitude does not converge, as the expression is large and also displays resonances. Therefore, the following approximations are made:

- The initial momenta of the neutrons are assumed to be 0, the initial energy of the neutrinos is just the rest energy $E_i = m_n$. The integrals over the Fermi-Dirac distributions can therefore be evaluated analytically $\int \mathrm{d}i_1 f(E(i_1)) = N/V_{Nucleus}$ with N the number of neutrons in the nucleus.
- The proton and neutron mass are assumed to be equal $m_n = m_p$.
- To take care of the resonances, a regulator is introduced to the neutrino propagators $\frac{\not{p}}{p^2} \to \frac{\not{p}}{p^2 + i/R^2}$. $1/R$ is the radius of the nucleus. It is chosen as a regulator, since the neutrinos are expected to stay in the nucleus and the momenta should therefore not be smaller than $1/R$.

The cross section does not converge with the first two approximations and there was not enough time to attempt to achieve convergence with the regulator.
Therefore, it is not possible to attempt give an estimate for the cross section of the quasi-elastic scattering process at this point. The evaluation will be resumed elsewhere.

5 Deep inelastic scattering

For energies higher than 3 GeV the neutrino is energetic enough to scatter directly with the partons inside the nucleons. Therefore, the neutrino scattering has to be treated as a deep inelastic process.
The Feynman diagrams for this process do not change significantly, only the neutrons are now d-quarks and the protons are u-quarks. Therefore, the Feynman diagrams in figure 8 are used. The nucleon current $J^\alpha = \gamma^\alpha(1 - 1.27\gamma^5)$ is replaced by the quark current $J^\alpha = \gamma^\alpha(1 - \gamma^5)/2$. However, the W-Boson propagator cannot be integrated out anymore, since the requirement $M_W^2 \gg (i_1 - f_1)^2$ or $M_W^2 \gg (i_2 - f_2)^2$ is not necessarily satisfied anymore. i_1, i_2, f_1 and f_2 are the initial or final state momenta of the nucleus as defined in figure 8. Therefore, the W-Boson propagator is written out explicitly leading to the amplitude:

$$\begin{aligned}\langle f|\, S^2\, |i\rangle =& 9\frac{4}{2!2!}\left(\frac{g_w^2\cos(\theta_C)g_\phi}{8}\right)^2 N_{f_1}N_{f_2}N_{p_1}N_{p_2}N_{p_4}(2\pi)^4\left[\sum_i U_{ei}\frac{\bar{v}_i(p_4)v_i^C(p_3)}{k^2-m_\phi^2}\right]\delta^{(4)} \\ &\{\frac{(\bar{u}(f_1)J^\alpha d(i_1))}{(m_W^2-(i_1-f_1)^2)^2}\frac{\left(\bar{u}(f_2)J^\beta d(i_2)\right)}{(m_W^2-(i_2-f_2)^2)^2} \\ &\cdot\bar{e}(p_1)\gamma_\alpha\frac{(\not{f}_1+\not{p}_1-\not{i}_1)(\not{f}_2+\not{p}_2-\not{i}_2)}{(f_1+p_1-i_1)^2\cdot(f_2+p_2-i_2)^2}\hat{P}_L\gamma_\beta\hat{C}\bar{e}^T(p_2) \\ &-\frac{(\bar{u}(f_1)J^\alpha d(i_1))}{(m_W^2-(i_1-f_1)^2)^2}\frac{\left(\bar{u}(f_2)J^\beta d(i_2)\right)}{(m_W^2-(i_2-f_2)^2)^2}\end{aligned} \tag{44}$$

$$\cdot \bar{e}(p_2)\gamma_\alpha \frac{(\not{f}_1 + \not{p}_2 - \not{i}_1)(\not{f}_2 + \not{p}_1 - \not{i}_2)}{(f_1 + p_2 - i_1)^2 \cdot (f_2 + p_1 - i_2)^2} \hat{P}_L \gamma_\beta \hat{C} \bar{e}^T(p_1)$$

$$- \frac{(\bar{u}(f_2) J^\alpha d(i_1))}{(m_W^2 - (i_1 - f_2)^2)^2} \frac{(\bar{u}(f_1) J^\beta d(i_2))}{(m_W^2 - (i_2 - f_1)^2)^2}$$

$$\cdot \bar{e}(p_1)\gamma_\alpha \frac{(\not{f}_2 + \not{p}_1 - \not{i}_1)(\not{f}_1 + \not{p}_2 - \not{i}_2)}{(f_2 + p_1 - i_1)^2 \cdot (f_1 + p_2 - i_2)^2} \hat{P}_L \gamma_\beta \hat{C} \bar{e}^T(p_2)$$

$$+ \frac{(\bar{u}(f_2) J^\alpha d(i_1))}{(m_W^2 - (i_1 - f_2)^2)^2} \frac{(\bar{u}(f_1) J^\beta d(i_2))}{(m_W^2 - (i_2 - f_1)^2)^2}$$

$$\cdot \bar{e}(p_2)\gamma_\alpha \frac{(\not{f}_2 + \not{p}_2 - \not{i}_1)(\not{f}_1 + \not{p}_1 - \not{i}_2)}{(f_2 + p_2 - i_1)^2 \cdot (f_1 + p_1 - i_2)^2} \hat{P}_L \gamma_\beta \hat{C} \bar{e}^T(p_1)\}.$$

Since the process is a parton level process, the initial momenta and energies are momentum fractions of the nucleon energy $i_1 = x_1 I$ and $i_2 = x_2 I$, where I is the neutron momenta. For the neutron momenta, the mean momentum as given by the Fermi-Dirac distribution of the neutron in the nucleus at zero temperature is being used.
The integration over all final state momenta and the 2 initial state momenta i_1 and i_2 is also still needed in this case to get the cross section. However, the differential cross section does not depend on i_1 and i_2 anymore, since only momentum fractions of the mean momentum are used. Therefore these 2 integrals are just integrals over the Fermi-Dirac distributions and lead to the particle number density N_{neutrons}/V.
To get the full cross section, the cross section dependent on the Bjorken variables x_1, x_2 needs to be integrated over all possible x_1, x_2 weighted by the parton distribution function of both partons in one nucleon. Therefore, a double parton distribution function (double pdf) of dd quark combination in a neutron is needed. As a first approximation the uu-quark combination in a proton is used. The double pdf is taken from figure 5.a in [17].
The total cross section for a given neutrino energy E_3 is then

$$\sigma(p_3) = \int \frac{N_{\text{neutrons}}^2/V}{2E_3(2E_i)^2 x_1 x_2} |\langle f| S^2 |i\rangle|^2 F(E_1, Z+2) F(E_2, Z+2) \mathrm{d}^3p_1 \mathrm{d}^3p_2 \mathrm{d}^3p_4 \mathrm{d}^3f_1 \mathrm{d}^3f_2 \mathrm{d}x_1 \mathrm{d}x_2. \tag{45}$$

For this interaction there is also no estimation of the cross section available since the numerical integration did not converge and we will return to this problem elsewhere.

6 Conclusion

The amplitude for the induced neutrinoless double β decay was calculated for low energies and the cross section of this process was evaluated numerically.
The number of events for the decay induced with low energy fission neutrinos over 10 years is smaller than 10^{-15} for energies from 1 MeV to 100 MeV and Majoron masses from 10^{-6} MeV to 10^2 MeV. Therefore, it is not possible to measure this process for low energies.
The cross section does however grow more than linearly with the energy. This could lead to measurable number of events at higher energies.

The cross section of the induced decay was also evaluated for scalar dark matter as the inducing particle. For the considered dark matter masses m_{DM} in the range $1-100$ MeV and Majoron masses m_ϕ in the range 1 GeV $-$ 1 TeV the expected number of events in 10 yr is smaller than 10^{-43}. Therefore, an experiment to measure the dark matter induced decay is not feasible.

For higher energies the amplitude for quasi-elastic scattering and deep inelastic scattering of the neutrinos with the neutrons in the nucleus or the d-quarks in a neutron is evaluated. But since the integrals did not converge, no result was obtained for both cases. The evaluations at higher energies will be resumed elsewhere.

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