Searches for QCD Instantons with ALFA Detector - Theory and Data

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Abstract

This work explores quantum chromodynamics (QCD) instantons, focusing on their potential experimental detection at the LHC. Instantons represent non-perturbative phenomena, enabling tunnelling transitions between QCD vacua. Motivated by the study of Khoze et al., we reevaluate his results and try to gain a better understanding. The valley approximation is applied to simplify theoretical calculations, aiming to enhance our understanding of instanton-induced processes and their observability.

Contents

1 Introduction

The study of quantum chromodynamics (QCD) instantons offers insights into non-perturbative processes that are otherwise inaccessible through standard Feynman diagrams.

It was shown by Belavin, Polyakov, Schwartz and Tyupkin in 1975 that there exists a topologically non-trivial solution to the Yang-Mills equations [1] - the BPST solution. Jackiw and Rebbi showed in their pivotal work that this non-trivial solution implies the tunnelling transitions between different vacua [2]. Later work by 't Hooft in 1976, he inferred that the instanton solution leads to new physics [3]. Explicitly, he argued that the tunnelling from different vacua to another leads to chiral symmetry violation - which is not expected in perturbative QCD but needed to explain the nucleon masses. In his later publishment, he found the effective interaction Lagrangian for the QCD instanton [4].

Instantons are not restricted to QCD. The electroweak instanton processes are associated with B + L violation [5]. In QCD, they provide a possible solution to the axial $U(1)$ problem or imply chiral symmetry violation [5].

However, even though the anomalous fermion-number violation in electroweak interactions holds significant theoretical interest, they are challenging to detect experimentally and are ruled out for any future analysis [5]. This work aims to better understand the theoretical predictions and possible experimental observations of QCD instanton-dominated processes at the Large Hadron Collider (LHC). A thorough summary of the instanton mechanism and nonperturbative topological phenomena in QCD, in general, was given by E. Shuryak in [6].

Motivated by the work of Khoze, Khoze, Milne, and Ryskin in "Hunting for QCD Instantons at the LHC in Events with Large Rapidity Gaps" from 2021 [7], this project aims to check whether data from the ALFA detector at the LHC [8] could contain evidence of such instanton events. Previous studies suggested that the acceptance of the ATLAS Forward Proton (AFP) detector was insufficient for meaningful detection [9], prompting a reevaluation using the ALFA detector, which could offer better sensitivity to instanton-induced processes. From the experimental standpoint, there is a theoretical reason why we are interested in QCD instantons. As we will see later, the total cross section for an instanton process is exponentially suppressed by the action $\propto \mathrm{e}^{-S_I}$ [5]. This factor is referred to as the Hooft suppression factor. In the electroweak case, the action is given by $\frac{4\pi}{\alpha_w}$, and thus $\mathrm{e}^{-4\pi/\alpha_w} \lll 1$ [5]. In the QCD case, however, the factor $\mathrm{e}^{4\pi/\alpha_s}$ gets larger in the strongly coupling region [5].

This work delves into several theoretical and methodological frameworks needed for understanding and estimating the number of instanton events detectable at the LHC. These include the total cross-section estimates for anomalous fermion-number violation provided by Khoze and Ringwald from 1991 [10] and the valley approach for handling quasi-zero modes as developed by Balitsky and Yung in 1986 [11]. Balitsky and Yung based their work on Coleman [12] and provided an efficient method to correctly approximate the integration over collective coordinates, which posed a problem in physics for a long time. The valley method is particularly important to simplify the complex functional integrals involved in instanton calculations by focusing on paths of least action, which is then used by Khoze and Ringwald.

The core objective of this work is to gain a better understanding of applying the valley approximation and other theoretical tools to estimate instanton-induced total cross-section. By verifying and extending the theoretical results of Khoze and colleagues and comparing them with experimental data, this project aims to further our understanding of non-perturbative QCD effects and their potential observability at high-energy colliders.

This summer student project scope was to repeat the analysis done for the AFP detector [9] for ALFA. For that, the results of Khoze, Khoze, Milne and Ryskin [7] needed to be checked. Checking the results independently has the advantage that reservations about the theory can be resolved. It is important to understand that the idea of instantons is a subtle consequence of the theory, which still has no experimental evidence. Furthermore, we need to deal with the following questions: a) What an instanton configuration is and how to choose the dominant one? b) Around which vacuum does one quantise? c) How the projection operator is used in the optical theorem. The answers to these questions are far from being trivial.

As Khoze and Ringwald gave the theoretical concepts in [10], understanding this paper is of key interest. As we will see later, integrating over zero and quasi-zero modes comes with some intricacies, which were dealt with in the "Collective-coordinate method for quasi-zero modes" by Balitsky and Yung [11]. This work, therefore, starts with a review of Balitsky and Yung [11]. After that, this framework is then extended to understand the total cross-sections calculated in [10]. This understanding will then be directly applied to derive the results in [7]. After verifying the results and calculating the first theoretical predictions, this project will conclude by focusing on the potential observability of instanton-dominated processes at the LHC.

2 Collective-Coordinate Method

As mentioned in the introduction, we start by going through calculations done by Balitsky and Yung [11]. In this paper, we look at the case of a double well potential in quantum mechanics. In the next chapter, we will see the complete QCD vacua to have multiple vacua through which tunnelling is possible. Therefore it makes sense to study the tunneling in the double well first. In the case of a double well potential, we have one zero mode due to the symmetry of the system and one quasi-zero mode due to the non-zero tunnelling probability [11]. The existence of zero-modes and quasi-zero modes has an important implication. Namely, in the zero mode case, there exists a direction in the functional space along which the action does not change. In the quasi-zero mode case, there exists a direction in the functional space along which the action is slower than compared to the non-zero case. This fact will be employed in the next sections. Also, the transfer of these results from quantum mechanics to field theory is dealt with in the next section. Suppose we want to calculate the partition function:

$$Z = \langle \exp(-HT) \rangle = N^{-1} \int \mathcal{D}\phi \, \exp(-S(\phi)). \tag{1}$$

Where the action S is given by

$$\mathrm{S}(\phi) = \int dt \left(\frac{\dot{\phi}^2}{2} + V(\phi) \right). \tag{2}$$

We can find the saddle point solution ϕ_c for this action using classical mechanics. Now expanding ϕ into a small deviation $\delta\phi$ around ϕ_c, we have:

$$S(\phi) = \int dt \left(\frac{(\dot{\phi}_c + \delta\dot{\phi})^2}{2} + V(\phi_c + \delta\phi) \right). \tag{3}$$

Now, performing a Taylor expansion of V up to second order, we get:

$$S(\phi) = \int dt \left(\frac{\dot{\phi}_c^2}{2} + \dot{\phi}_c \delta\dot{\phi} + \frac{\delta\dot{\phi}^2}{2} + V(\phi_c) + V'(\phi_c)\delta\phi + \frac{1}{2}V''(\phi_c)\delta\phi^2 + O(\delta\phi^3) \right). \tag{4}$$

Next, assuming that ϕ decays rapidly enough so that the boundary terms vanish, integrating by parts gives:

$$S(\phi) = \int dt \left(\frac{\dot{\phi}_c^2}{2} + V(\phi_c) + \left(-\ddot{\phi}_c + V'(\phi_c) \right) \delta\phi + \frac{1}{2}\delta\phi \left(-\partial^2 + V''(\phi_c) \right) \delta\phi \right). \tag{5}$$

As ϕ_c is a saddle point solution to the action, the variational terms must vanish, leading to the condition:

$$\left. \frac{\delta S}{\delta\phi} \right|_{\phi=\phi_c} = -\ddot{\phi}_c + V'(\phi_c) = 0, \tag{6}$$

and,

$$\left. \frac{\delta^2 S}{\delta\phi^2} \right|_{\phi=\phi_c} = -\partial^2 + V''(\phi_c) = \Box(\phi_c) = 0. \tag{7}$$

By differentiating Eq. (6), i.e. differentiating the equation of motion after time, we find that $\dot{\phi}_c(t)$ is a zero mode. Additionally, $\Box(\phi_c)$ contains a single parametrically small mode - the quasi-zero mode, while the other modes are Gaussian. This allows us to study different cases for the integration. In other words, we have:

$$\Box(\phi_c)\phi_n = \lambda_n \phi_n, \tag{8}$$

where $\phi_0 = \dot{\phi}_c$ corresponds to the zero mode with $\lambda_0 = 0$. For the remaining modes, we impose the condition $\lambda_1 \ll \lambda_n$, where λ_1 represents the quasizeromode. This condition is there to separate the integration into three different cases. In the functional integration, the three cases are: integrating over the zero-mode, quasizeromode and the non-zero modes. Since the non-zero modes are expected to have large eigenvalues, their integration will be nearly Gaussian. We begin by addressing the integration over the zero mode.

2.1 Explaining the Key Concept of the Valley Approach

Before delving into the mathematical intricacies, it is good to get a brief overview of the idea behind the method. First, as mentioned before, the solutions are characterised by zero and quasi-zero modes. The idea is to employ this fact by finding the path along which the action changes the slowest. This path is defined to be the stream-line. The exact condition this stream-line has to satisfy is given in Eq.(66). By going into the eigenmode space, we can write separate the integration into the respective mode cases and deal with each of them individually. In the figure below, the functional space for the quasi-zero mode and one of the non-zero ones is drawn. c_i is the coefficient of the respective mode.

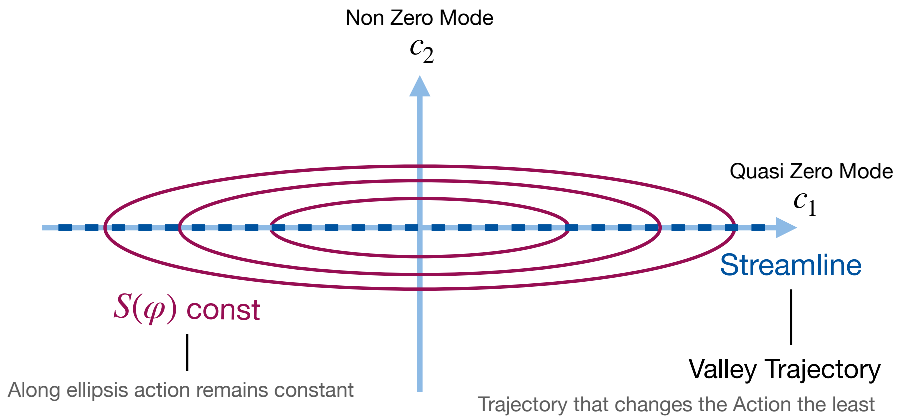


Figure 1: Sketch of Valley Approach for the case of the quasi-zero mode and one non-zero mode. On the x-axis, the coefficient of quasi-zero modes is drawn, and on the y-axis, the coefficient of the non-zero modes is drawn. In purple, ellipses along which the action remains the same. The dashed blue line depicts the streamline.

We can see that along the streamline, here along c_1, the action changes slower than along c_2. Integrating over the non-zero modes gives us integrals that are Gaussian and finite. The quasi-zero mode yields an integral that would diverge as the eigenvalue can be chosen parametrically small. However, in the streamline equation (see Eq.(66)), we will impose a boundary condition on the integration. This makes the divergence explicit by being an integral over collective coordinates. Now that we have an idea of the process, we move on to the mathematical part of the integration.

The mathematical part consists of two cases. First, we start by integrating over zero and non-zero modes. Here, we assume that there exists no quasi-zero mode. After that, we assume there exists no zero mode and go through the case of the quasi-zero and non-zero modes. Dealing with the integration of the zero and quasi-zero modes would make the derivation more confusing. After that, both results will be combined, leaving us with a result on how to approach the integration over the full mode space.

2.2 Integration of Zero Modes

As we will see later, the solution ϕ_c inherits a translational invariance, characterized by τ, such that $\phi_c(t-\tau) = \phi_\tau$ (see [11]). This fact can be used to find the closest instanton for every function ϕ over which we need to integrate. Mathematically, this can be written as

$$\frac{\partial}{\partial\tau}\frac{1}{2}\|\phi - \phi_\tau\|^2 = \frac{1}{2}\frac{\partial}{\partial\tau}(\phi-\phi_\tau, \phi-\phi_\tau) = (\phi-\phi_\tau, \dot{\phi}_\tau) = 0, \tag{9}$$

where we have defined the scalar product between two functions as $(f(t), g(t)) = \int dt f(t)g(t)$. Eq.(9) implies that the deviation from the classical solution is orthogonal to the zero mode. Additionally, close to the minimum where $\tau = \tau^*$, we can expand the latter scalar product as

$$(\phi-\phi_\tau, \dot{\phi}_\tau) = \underbrace{(\phi-\phi_{\tau^*}, \dot{\phi}_{\tau^*})}_{=0,\text{ at min}} + (\tau-\tau^*)\,\frac{d}{d\tau}(\phi-\phi_\tau, \dot{\phi}_\tau)\Big|_{\tau=\tau^*} + O\left((\tau-\tau^*)^2\right) \tag{10}$$

Focusing first on the derivative, we have

$$\begin{aligned}\frac{d}{d\tau}(\phi-\phi_\tau,\dot{\phi}_\tau) &= \frac{d}{d\tau}(\phi-\phi_c(t-\tau),\dot{\phi}_c(t-\tau))\\ &= (\dot{\phi}_c(t-\tau),\dot{\phi}_c(t-\tau))-(\phi-\phi_c(t-\tau),\ddot{\phi}_c(t-\tau))\\ &= \|\dot{\phi}_c\|^2-(\phi-\phi_\tau,\ddot{\phi}_\tau).\end{aligned} \tag{11}$$

Multiplying $\dot{\phi}_c$ by Eq.(6), we get

$$0=-\ddot{\phi}_c\dot{\phi}_c+V'(\phi_c)\dot{\phi}_c=\frac{d}{dt}\left(-\frac{\dot{\phi}_c^2}{2}+V(\phi_c)\right), \tag{12}$$

which implies the conservation of total energy. In the potential V, the minima of the double well are at $V=0$. For instantons, later we will see that the solution is found in Euclidean spacetime. So in imaginary time, the minima become maxima due to the flip into $-V$, however, $-V$ is still 0. Therefore the particle stops at the maxima, such the total energy is $E=0$. Conservation of E thus gives

$$\frac{\dot{\phi}_c^2}{2}-V(\phi_c)=E=0\Leftrightarrow\frac{\dot{\phi}_c^2}{2}=V(\phi_c). \tag{13}$$

Substituting this into the action yields

$$S(\phi_c)=\int dt\left(\frac{\dot{\phi}_c^2}{2}+V(\phi_c)\right)=\int dt\,\dot{\phi}_c^2=(\dot{\phi}_c,\dot{\phi}_c)=\|\dot{\phi}_c\|^2. \tag{14}$$

Returning to Eq.(11), we obtain

$$\frac{d}{d\tau}(\phi-\phi_\tau,\dot{\phi}_\tau)=S(\phi_c)-(\phi-\phi_\tau,\ddot{\phi}_\tau)=\Delta(\phi). \tag{15}$$

Since $S(\phi_c)\gg 1$, the action is a semiclassical parameter [11]. Substituting this result into Eq.(10) leads to

$$(\phi-\phi_\tau,\dot{\phi}_\tau)=(\tau-\tau^*)\Delta(\phi). \tag{16}$$

Using the properties of the delta distribution, we have

$$1=|\Delta(\phi)|\int d\tau\,\delta\left((\tau-\tau^*)\Delta(\phi)\right) \tag{17}$$

Now inserting Eq.(15) and Eq.(16) into the equation above, we have

$$1=\Big(S(\phi_c)-(\phi-\phi_{\tau^*},\ddot{\phi}_{\tau^*})\Big)\int d\tau\,\delta\Big((\phi-\phi_\tau,\dot{\phi}_\tau)\Big). \tag{18}$$

Again using $S(\phi_c)\gg 1$, such that

$$1=\int d\tau S(\phi_c)\delta\Big((\phi-\phi_\tau,\dot{\phi}_\tau)\Big). \tag{19}$$

Now, focusing on calculating the partition function, we have

$$Z=\langle\exp(-HT)\rangle=N^{-1}\int\mathcal{D}\phi\,\exp(-S(\phi)). \tag{20}$$

Inserting 1 into this equation and interchanging the order of integration, we find

$$Z=N^{-1}\int d\tau\int\mathcal{D}\phi\,S(\phi_c)\delta\Big((\phi-\phi_\tau,\dot{\phi}_\tau)\Big)\exp\left(-S(\phi)\right). \tag{21}$$

Next, we expand $S(\phi)$ around the classical solution:

$$S(\phi) = S(\phi_c) + \int dt \left(\frac{1}{2}(\phi - \phi_\tau)(-\partial^2 + V''(\phi_\tau))(\phi - \phi_\tau) + O\left((\phi - \phi_\tau)^3\right)\right) \tag{22}$$

$$= S(\phi_c) + \frac{1}{2}(\phi - \phi_\tau, (-\partial^2 + V''(\phi_\tau))(\phi - \phi_\tau)) + O\left((\phi - \phi_\tau)^3\right). \tag{23}$$

The first-order term can be set to zero due to the delta distribution. Substituting this into the partition function for the one-instanton contribution, we get

$$Z_1 = N^{-1} \int d\tau \int \mathcal{D}\phi \, S(\phi_c)\delta\left((\phi - \phi_\tau, \dot{\phi}_\tau)\right)$$
$$\times \exp\left(-S(\phi_c) - \frac{1}{2}(\phi - \phi_\tau, (-\partial^2 + V''(\phi_\tau))(\phi - \phi_\tau))\right). \tag{24}$$

Evaluating this integral is non-trivial. We will make use of the eigenvalues of the second-order operator as defined in Eq.(8). Then, the fields ϕ can be expressed in terms of the eigenmodes as

$$\phi(t) = \phi_c(t) + \sum_{n=0}^{\infty} c_n \phi_n(t). \tag{25}$$

As shown by Coleman, the integration measure becomes

$$\mathcal{D}\phi = \prod_{n=0}^{\infty} \frac{dc_n}{\sqrt{2\pi}}. \tag{26}$$

To normalise, consider the instanton solution, where

$$S(\phi_c) = \int dt \, \dot{\phi}_c^2 \quad \Leftrightarrow \quad 1 = \int dt \left(\frac{\dot{\phi}_c}{\sqrt{S(\phi_c)}}\right)^2.$$

Thus, the zero mode is given by

$$\phi_0 = \frac{\dot{\phi}_c}{\sqrt{S(\phi_c)}}. \tag{27}$$

Therefore

$$\frac{dc_0}{\sqrt{2\pi}} = d\tau \sqrt{\frac{S(\phi_c)}{2\pi}} \quad \Rightarrow \quad \int d\tau \sqrt{\frac{S(\phi_c)}{2\pi}} = T\sqrt{\frac{S(\phi_c)}{2\pi}}.$$

Now inserting Eq.(25) into Eq.(24) leaves us with

$$Z_1 = N^{-1} S(\phi_c) \exp(-S(\phi_c)) \int d\tau \int \prod_{n=0}^{\infty} \frac{dc_n(\tau)}{\sqrt{2\pi}} \, \delta\left(\left(\sum_{n=0}^{\infty} c_n(\tau)\phi_n(t-\tau), \dot{\phi}_\tau\right)\right) \tag{28}$$

$$\times \exp\left(-\frac{1}{2}\left(\sum_{n=0}^{\infty} c_n(\tau)\phi_n(t-\tau), (-\partial^2 + V''(\phi_\tau)) \sum_{n=0}^{\infty} c_n(\tau)\phi_n(t-\tau)\right)\right). \tag{29}$$

Now with

$$\Box(\phi_\tau)\phi_n(t-\tau) = \lambda_n \phi_n(t-\tau)$$

the scalar product in the exponential can be simplified into

$$\left(\sum_{n=0}^{\infty} c_n(\tau)\phi_n(t-\tau), \left(-\partial^2 + V''(\phi_\tau)\right) \sum_{n=0}^{\infty} c_n(\tau)\phi_n(t-\tau)\right) \tag{30}$$

$$= \left(\sum_{n=0}^{\infty} c_n(\tau)\phi_n(t-\tau), \sum_{n=0}^{\infty} c_n(\tau)\lambda_n\phi_n(t-\tau)\right) \tag{31}$$

$$= \sum_{n=0}^{\infty} c_n(\tau)^2 \lambda_n. \tag{32}$$

Additionally in

$$\delta\left(\left(\sum_{n=0}^{\infty} c_n(\tau)\phi_n(t-\tau), \dot{\phi}_\tau\right)\right) \tag{33}$$

$\dot{\phi}_\tau$ is the non normalized zero mode. Due to orthogonality, the only contribution in the sum that remains is ϕ_0. Or in other words with Eq.(27)

$$=\delta\left(\left(c_0(\tau)\phi_0(t-\tau), \dot{\phi}_\tau\right)\right) \tag{34}$$

$$=\delta\left(\left(c_0(\tau)\phi_0(t-\tau), \phi_0\sqrt{S(\phi_c)}\right)\right). \tag{35}$$

Inserting these results into Z_1 to get

$$Z_1 = N^{-1}S(\phi_c)\exp(-S(\phi_c))T\int\frac{dc_0(\tau)}{\sqrt{2\pi}} \tag{36}$$

$$\times\,\delta\left(\left(c_0(\tau)\phi_0(t-\tau), \dot{\phi}_\tau\right)\right)\prod_{n=1}^{\infty}\frac{dc_n(\tau)}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\sum_{n=0}^{\infty}c_n(\tau)^2\lambda_n\right) \tag{37}$$

$$=N^{-1}S(\phi_c)\exp(-S(\phi_c))T\int\frac{dc_0(\tau)}{\sqrt{2\pi}} \tag{38}$$

$$\times\,\delta\left(\left(c_0(\tau)\phi_0(t-\tau), \sqrt{S(\phi_c)}\phi_0(t-\tau)\right)\right)\prod_{n=1}^{\infty}\frac{dc_n(\tau)}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\sum_{n=0}^{\infty}c_n(\tau)^2\lambda_n\right) \tag{39}$$

$$=N^{-1}S(\phi_c)\exp(-S(\phi_c))T\int\frac{dc_0(\tau)}{\sqrt{2\pi}} \tag{40}$$

$$\times\,\delta\left(\left(c_0(\tau)\phi_0(t-\tau), \sqrt{S(\phi_c)}\phi_0(t-\tau)\right)\right)\prod_{n=1}^{\infty}\frac{dc_n(\tau)}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\sum_{n=0}^{\infty}c_n(\tau)^2\lambda_n\right) \tag{41}$$

$$\tag{42}$$

First, we evaluate the integral over the non-zeromodes. Keeping in mind that $\lambda_0 = 0$, we have

$$\int\prod_{n=1}^{\infty}\frac{dc_n(\tau)}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\sum_{n=0}^{\infty}c_n(\tau)^2\lambda_n\right) = \prod_{n=1}^{\infty}\left(\frac{1}{\lambda_n}\right)^{1/2} = \left(\det{}'\Box(\phi_\tau)\right)^{-1/2} \tag{43}$$

(the prime denotes the determinant without the zero mode). Next, we simplify the delta distribution

$$\int\frac{dc_0(\tau)}{\sqrt{2\pi}}\delta\left(\left(c_0(\tau)\phi_0(t-\tau), \sqrt{S(\phi_c)}\phi_0(t-\tau)\right)\right) \tag{44}$$

$$=\int\frac{dc_0(\tau)}{\sqrt{2\pi}}\delta\left(c_0(\tau)\sqrt{S(\phi_c)}\right) \tag{45}$$

$$=\frac{1}{\sqrt{2\pi S(\phi_c)}}. \tag{46}$$

These simplifications can be inserted into Z_1 to obtain

$$Z_1 = N^{-1}S(\phi_c)\exp(-S(\phi_c))T\frac{1}{\sqrt{2\pi S(\phi_c)}}\left(\det{}'(-\partial^2 + V''(\phi_\tau))\right)^{-1/2} \tag{47}$$

$$= N^{-1}T\sqrt{\frac{S(\phi_c)}{2\pi}}\exp(-S(\phi_c))\left(\det{}'\Box(\phi_\tau)\right)^{-1/2} \tag{48}$$

2.3 Integration of Quasi-Zero Modes

Having addressed the case with zero modes, we now move on to the quasi-zero ones. The assumption is that, aside from the zero mode, $\Box(\phi_\tau)$ has one parametrically small quasi-zero eigenvalue. Since we remain in the valley, the action in the vicinity of ϕ_c is given by

$$S(\phi_c + \delta\phi) = S(\phi_c) + \frac{1}{2}\left(\delta\phi, \Box\delta\phi\right). \tag{49}$$

In Eq.(32), the scalar product can be rewritten by choosing an appropriate functional space. Thus, we obtain

$$S(\phi_c + \delta\phi) = S(\phi_c) + \sum_{n=0}^{\infty} c_n(\tau)^2 \lambda_n. \tag{50}$$

We can now parameterise the streamline for the quasi-zero mode. Using Lagrange multipliers, we have

$$\xi(\alpha)\frac{\partial \phi(\alpha)}{\partial \alpha} = \left.\frac{\delta S}{\delta \phi}\right|_{\phi=\phi(\alpha)}, \tag{51}$$

where α parametrizes the streamline [11]. The boundary conditions are

$$\left.\frac{\delta S}{\delta \phi}\right|_{\phi=\phi(\alpha=0)} = \left.\frac{\delta S}{\delta \phi}\right|_{\phi_c} = 0$$

and

$$\left.\frac{\partial \phi(\alpha)}{\partial \alpha}\right|_{\alpha=0} \sim \phi_1.$$

The first boundary condition ensures that at $\alpha = 0$, we recover the classical solution, which is a local extremum of the action. The second condition implies that at $\alpha = 0$, the derivative of the field does not vanish, which leads to the requirement.

$$\xi(\alpha = 0) = 0.$$

Differentiating Eq.(51) with respect to α at $\alpha = 0$ gives us another condition on ξ:

$$\xi'(0) = \lambda_1.$$

To integrate over the segment of functional space near the streamline ϕ_α, we employ the same method as before. For every ϕ, we locate the nearest point on the streamline ϕ_{α^*}. This approach yields

$$\frac{\partial}{\partial \alpha}\frac{1}{2}\left\|\phi - \phi_\alpha\right\|^2 = \frac{\partial}{\partial \alpha}\frac{1}{2}(\phi - \phi_\alpha, \phi - \phi_\alpha) = -\left(\phi - \phi_\alpha, \frac{\partial \phi_\alpha}{\partial \alpha}\right) = 0. \tag{52}$$

From here, we deduce that the insertion of a 1 is given by

$$1 = \int d\alpha \left(-\frac{\partial}{\partial \alpha}(\phi - \phi_\alpha, \frac{\partial \phi_\alpha}{\partial \alpha})\right) \delta\left((\phi - \phi_\alpha, \frac{\partial \phi_\alpha}{\partial \alpha})\right). \tag{53}$$

Substituting this into the partition function given in Eq.(20), and after interchanging the integration variables, we obtain

$$Z = N^{-1}\int d\alpha \int \mathcal{D}\phi \left(-\frac{\partial}{\partial \alpha}(\phi - \phi_\alpha, \frac{\partial \phi_\alpha}{\partial \alpha})\right) \delta\left((\phi - \phi_\alpha, \frac{\partial \phi_\alpha}{\partial \alpha})\right) \exp\left(-S(\phi)\right). \tag{54}$$

Expanding S around ϕ_α leaves us with

$$S(\phi) = S(\phi_\alpha) + \left(\phi - \phi_\alpha, \frac{\delta S}{\delta \phi_\alpha}\right) + \frac{1}{2}\left(\phi - \phi_\alpha, \frac{\delta^2 S}{\delta \phi_\alpha^2}(\phi - \phi_\alpha)\right) + O((\phi - \phi_\alpha)^3). \tag{55}$$

However, using Eq.(51) and Eq.(52), the second term vanishes when integrated.

Thus, for the one-instanton contribution, we obtain

$$Z_1 = N^{-1}\int d\alpha \exp(-S(\phi_\alpha)) \int \mathcal{D}\phi \left[-\frac{\partial}{\partial \alpha}\left(\phi - \phi_\alpha, \frac{\partial \phi_\alpha}{\partial \alpha}\right)\right] \delta\left[\left(\phi - \phi_\alpha, \frac{\partial \phi_\alpha}{\partial \alpha}\right)\right] \tag{56}$$

$$\times \exp\left(-\frac{1}{2}(\phi - \phi_\alpha, \Box(\phi_\alpha)(\phi - \phi_\alpha))\right). \tag{57}$$

In the term $-\frac{\partial}{\partial \alpha}\left(\phi - \phi_\alpha, \frac{\partial \phi_\alpha}{\partial \alpha}\right)$, there is a part $\left(\frac{\partial \phi_\alpha}{\partial \alpha}, \frac{\partial \phi_\alpha}{\partial \alpha}\right) = \left\|\frac{\partial \phi_\alpha}{\partial \alpha}\right\|^2$ and a part $-\left(\phi - \phi_\alpha, \frac{\partial^2 \phi_\alpha}{\partial \alpha^2}\right)$. The latter can be neglected in the semi-classical approximation, which gives us

$$Z_1 = N^{-1}\int d\alpha \left\|\frac{\partial \phi_\alpha}{\partial \alpha}\right\|^2 \exp\left(-S(\phi_\alpha)\right) \int \mathcal{D}\phi\, \delta\left[\left(\phi - \phi_\alpha, \frac{\partial \phi_\alpha}{\partial \alpha}\right)\right] \tag{58}$$

$$\times \exp\left(-\frac{1}{2}(\phi - \phi_\alpha, \Box(\phi_\alpha)(\phi - \phi_\alpha))\right). \tag{59}$$

This integral closely resembles Eq.(24). However, the integration over α is significantly more complex than that over τ. Once again, we use eigenmodes to simplify the integral. Differentiating the streamline equation Eq.(51) with respect to α gives us

$$\xi'(\alpha)\partial_\alpha\phi_\alpha + \xi(\alpha)\frac{\partial^2\phi_\alpha}{\partial\alpha^2} = \frac{\delta^2 S}{\delta\phi_\alpha^2}\partial_\alpha\phi_\alpha. \tag{60}$$

Assuming the second derivative of ϕ_α is negligible, $\partial_\alpha\phi_\alpha$ becomes an eigenvector of the operator $\Box(\phi_\alpha)$. Thus, we can again represent each ϕ in the eigenmode space as

$$\phi = \phi_\alpha + \sum_{n=1}^{\infty} c_{(\alpha,n)}\phi_{(\alpha,n)}, \tag{61}$$

with

$$\Box(\phi_\alpha)\phi_{(\alpha,n)} = \lambda_{(\alpha,n)}\phi_{(\alpha,n)}, \quad n \geq 1. \tag{62}$$

From Eq.(60), we see that for the quasi-zero mode, the eigenmode and eigenvalue are given by

$$\frac{\partial\phi_\alpha}{\partial\alpha} \sim \phi_{(\alpha,1)}, \quad \xi'(\alpha) = \lambda_{(\alpha,1)}. \tag{63}$$

Also, eigenmode can be normalised using its norm. With the same logic as before in the zero mode case, we go into eigenmode space to obtain

$$\begin{aligned}
Z_1 &= N^{-1}\int d\alpha \left\|\frac{\partial\phi_\alpha}{\partial\alpha}\right\|^2 \exp\left(-S(\phi_\alpha)\right)\int\prod_{n=1}^{\infty}\frac{dc_{\alpha,n}}{\sqrt{2\pi}}\delta\left[\left(\sum_{n=1}^{\infty}c_{\alpha,n}\phi_{\alpha,n}, \frac{\partial\phi_\alpha}{\partial\alpha}\right)\right] \\
&\quad\times\exp\left(-\frac{1}{2}\left(\sum_{n=1}^{\infty}c_{\alpha,n}\phi_{\alpha,n}, \Box(\phi_\alpha)\sum_{m=1}^{\infty}c_{\alpha,m}\phi_{\alpha,m}\right)\right) \\
&= N^{-1}\int d\alpha \left\|\frac{\partial\phi_\alpha}{\partial\alpha}\right\|^2 \exp\left(-S(\phi_\alpha)\right)\int\frac{dc_{\alpha,1}}{\sqrt{2\pi}}\delta\left[\left(c_{\alpha,1}\phi_{\alpha,1}, \left\|\frac{\partial\phi_\alpha}{\partial\alpha}\right\|\phi_{\alpha,1}\right)\right] \\
&\quad\times\exp\left(-\frac{1}{2}c_{\alpha,1}^2\lambda_{\alpha,1}\right)\prod_{n=2}^{\infty}\frac{dc_{\alpha,n}}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\sum_{n=2}^{\infty}c_{\alpha,n}^2\lambda_{\alpha,n}\right) \\
&= N^{-1}\int d\alpha \left\|\frac{\partial\phi_\alpha}{\partial\alpha}\right\|\frac{1}{\sqrt{2\pi}}\exp\left(-S(\phi_\alpha)\right)\prod_{n=2}^{\infty}\left(\frac{1}{\lambda_{\alpha,n}}\right)^{1/2}.
\end{aligned} \tag{64}$$

To write this similarly to the result from Eq.(48), the product of the eigenvalues can be rewritten as

$$\begin{aligned}
\prod_{n=2}^{\infty}\left(\frac{1}{\lambda_{(\alpha,n)}}\right)^{\frac{1}{2}} &= \left(\det\Box(\phi_\alpha)\right)^{-\frac{1}{2}}\left(\frac{1}{\lambda_{(\alpha,1)}}\right)^{-\frac{1}{2}} \\
&= \left(\det\Box(\phi_\alpha)\right)^{-\frac{1}{2}}\left(\phi_{(\alpha,1)}, \left(\det\Box(\phi_\alpha)\right)^{-1}\phi_{(\alpha,1)}\right)^{-\frac{1}{2}} \\
&= \left(\det\Box(\phi_\alpha)\right)^{-\frac{1}{2}}\left\|\frac{\partial\phi_\alpha}{\partial\alpha}\right\|\left(\frac{\partial\phi_\alpha}{\partial\alpha}, \left(\det\Box(\phi_\alpha)\right)^{-1}\frac{\partial\phi_\alpha}{\partial\alpha}\right)^{-\frac{1}{2}}.
\end{aligned}$$

Thus

$$Z_1 = N^{-1}\int d\alpha \left\|\frac{\partial\phi_\alpha}{\partial\alpha}\right\|^2\frac{1}{\sqrt{2\pi}}\exp(-S(\phi_\alpha))\left(\det\Box(\phi_\alpha)\right)^{-\frac{1}{2}}\left(\frac{\partial\phi_\alpha}{\partial\alpha}, \left(\det\Box(\phi_\alpha)\right)^{-1}\frac{\partial\phi_\alpha}{\partial\alpha}\right)^{-\frac{1}{2}}. \tag{65}$$

2.4 Combining the Results

Now that we have integrated the zero mode and the quasi-zero mode individually, in this subsection, we will take into account the zero mode in the latter calculation. The key idea is to promote ϕ_α to also have the τ variable, thus generalizing Eq. (51) to

$$\xi(\alpha)\partial_\alpha\phi_{(\alpha,\tau)}(t) = \left.\frac{\delta S}{\delta\phi}\right|_{\phi=\phi_{(\alpha,\tau)}}, \tag{66}$$

with appropriate boundary conditions. Similar to before, we will insert a 1 for both the zero mode case and the quasi-zero mode, i.e.,

$$
\begin{aligned}
1 &= 1 \times 1 \\
&= \int d\tau \left(\frac{\partial}{\partial \tau} (\phi - \phi_{(\alpha,\tau)}, \dot{\phi}_{(\alpha,\tau)}) \right) \delta \left((\phi - \phi_{(\alpha,\tau)}, \dot{\phi}_{(\alpha,\tau)}) \right) \\
&\quad \times \int d\alpha \left(-\frac{\partial}{\partial \alpha} (\phi - \phi_{(\alpha,\tau)}, \phi'_{(\alpha,\tau)}) \right) \delta \left((\phi - \phi_{(\alpha,\tau)}, \phi'_{(\alpha,\tau)}) \right) \\
&= \int d\alpha \, d\tau \left(\|\dot{\phi}_{(\alpha,\tau)}\|^2 - (\phi - \phi_{(\alpha,\tau)}, \ddot{\phi}_{(\alpha,\tau)}) \right) \\
&\quad \times \left(\|\phi'_{(\alpha,\tau)}\|^2 - (\phi - \phi_{(\alpha,\tau)}, \phi''_{(\alpha,\tau)}) \right) \\
&\quad \times \delta \left((\phi - \phi_{(\alpha,\tau)}, \phi'_{(\alpha,\tau)}) \right) \delta \left((\phi - \phi_{(\alpha,\tau)}, \dot{\phi}_{(\alpha,\tau)}) \right) .
\end{aligned}
$$

Following the same logic as before, after expanding the action around $\phi_{(\alpha,\tau)}$ and limiting ourselves to quadratic order, we have

$$
\begin{aligned}
Z_1 = N^{-1} &\int d\alpha \, d\tau \, \|\dot{\phi}_{(\alpha,\tau)}\|^2 \|\phi'_{(\alpha,\tau)}\|^2 \exp(-S(\phi_{(\alpha,\tau)})) && (67) \\
&\times \int \mathcal{D}\phi \, \delta^{(2)} \left((\phi - \phi_{(\alpha,\tau)}, \dot{\phi}_{(\alpha,\tau)}), (\phi - \phi_{(\alpha,\tau)}, \phi'_{(\alpha,\tau)}) \right) && (68) \\
&\times \exp \left(-\frac{1}{2} \left(\phi - \phi_{(\alpha,\tau)}, \Box(\phi_{(\alpha,\tau)})(\phi - \phi_{(\alpha,\tau)}) \right) \right) . && (69)
\end{aligned}
$$

Again, using eigenmodes, including the zero and quasi-zero modes, we can write any ϕ as

$$
\phi = \phi_{(\alpha,\tau)} + \sum_{n=0}^{\infty} c_{(\alpha,\tau,n)} \phi_{(\alpha,\tau,n)} . \tag{70}
$$

Thus,

$$
\begin{aligned}
Z_1 = N^{-1} &\int d\alpha \, d\tau \, \|\dot{\phi}_{(\alpha,\tau)}\|^2 \|\phi'_{(\alpha,\tau)}\|^2 \exp(-S(\phi_{(\alpha,\tau)})) && (71) \\
&\times \int \left[\frac{dc_{(\alpha,\tau,0)}}{\sqrt{2\pi}} \frac{dc_{(\alpha,\tau,1)}}{\sqrt{2\pi}} \delta^{(2)} \left((c_{(\alpha,\tau,0)} \phi_{(\alpha,\tau,0)}, \dot{\phi}_{(\alpha,\tau)}), (c_{(\alpha,\tau,1)} \phi_{(\alpha,\tau,1)}, \phi'_{(\alpha,\tau)}) \right) \right. && (72) \\
&\times \left. \prod_{n=2}^{\infty} \frac{dc_{(\alpha,\tau,n)}}{\sqrt{2\pi}} \exp \left(-\frac{1}{2} \sum_{n=2}^{\infty} c^2_{(\alpha,\tau,n)} \lambda_{(\alpha,\tau,n)} \right) \right] . && (73)
\end{aligned}
$$

Now, combining the results of the last two subsections, we have

$$
\begin{aligned}
Z_1 = N^{-1} &\int d\alpha \, d\tau \, \|\dot{\phi}_{(\alpha,\tau)}\| \|\phi'_{(\alpha,\tau)}\| \exp(-S(\phi_{(\alpha,\tau)})) && (74) \\
&\times \int \left[\frac{dc_{(\alpha,\tau,0)}}{\sqrt{2\pi}} \frac{dc_{(\alpha,\tau,1)}}{\sqrt{2\pi}} \delta^{(2)} \left((c_{(\alpha,\tau,0)} \phi_{(\alpha,\tau,0)}, \frac{\dot{\phi}_{(\alpha,\tau)}}{\|\dot{\phi}_{(\alpha,\tau)}\|}), (c_{(\alpha,\tau,1)} \phi_{(\alpha,\tau,1)}, \frac{\phi'_{(\alpha,\tau)}}{\|\phi'_{(\alpha,\tau)}\|}) \right) \right. && (75) \\
&\times \left. \prod_{n=2}^{\infty} \left(\frac{1}{\lambda_{(\alpha,\tau,n)}} \right)^{1/2} \right] . && (76)
\end{aligned}
$$

Thus, we can simplify the result to

$$
Z_1 = N^{-1} T \int \frac{d\alpha}{2\pi} \, \|\dot{\phi}_\alpha\| \|\phi'_\alpha\| \exp(-S(\phi_\alpha)) \prod_{n=2}^{\infty} \left(\frac{1}{\lambda_{(\alpha,n)}} \right)^{1/2} , \tag{77}
$$

or if the product is rewritten as before, we have

$$
\begin{aligned}
Z_1 = N^{-1} T &\int \frac{d\alpha}{2\pi} \, \|\dot{\phi}_\alpha\|^2 \|\phi'_\alpha\|^2 \exp(-S(\phi_\alpha)) \left(\det \Box(\phi_\alpha) \right)^{-1/2} && (78) \\
&\times \left(\left(\dot{\phi}_\alpha, (\Box(\phi_\alpha))^{-1} \dot{\phi}_\alpha \right) \right)^{-1/2} \left((\phi'_\alpha, (\Box(\phi_\alpha))^{-1} \phi'_\alpha) \right)^{-1/2} . && (79)
\end{aligned}
$$

Now that we know how to approach the integration over the full mode space, we focus on the main part of this project. Precisely, using these results to understand the paper written by Khoze, Khoze, Milne, and Ryskin [7].

3 Chiral Symmetry Violation due to QCD Instantons

Now that we have discussed the intricacies of integrating zero and non-zero modes, we want to use the method to derive the total cross section for QCD Instantons. We start by looking at the QCD Lagrangian, i.e. [13]

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a + \bar{\psi}_{q,i} \left(i \not{D}_{ij} - m_q \, \delta_{ij} \right) \psi_{q,j} \tag{80}$$

where $\psi_{q,i}$ depicts the spinor of the q-quark of color i, $\not{D}_{ij} = \gamma^\mu \left(D_\mu\right)_{ij} = \partial_\mu \delta_{ij} - ig \left(T_a\right)_{ij} A^a_\mu$ is the covariant derivative, and $G^a_{\mu\nu}$ the Field strength tensor of color a defined as

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu. \tag{81}$$

The process of interest is the instanton-dominated QCD process with two gluons in the initial state [7],

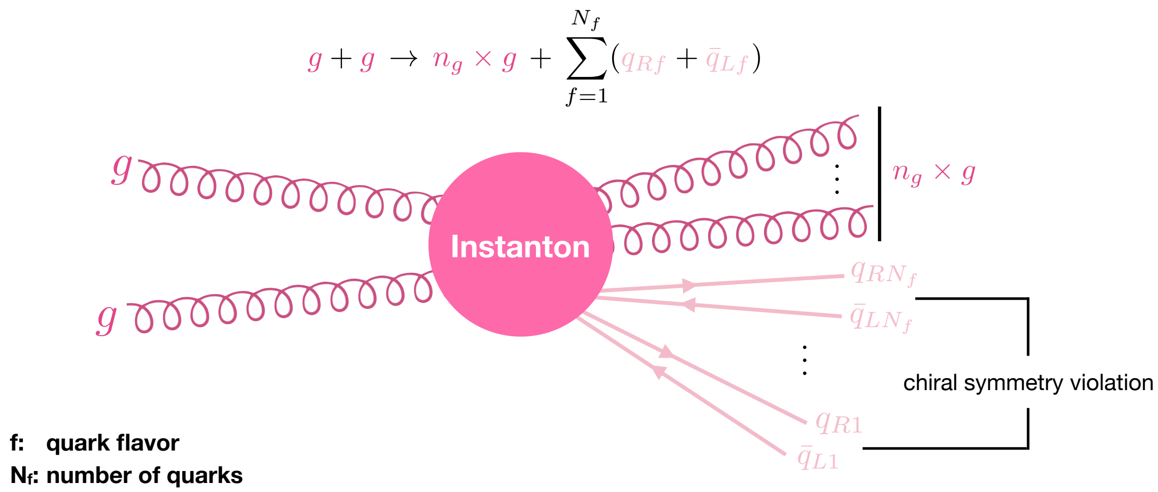


Figure 2: Instanton process of two gluons in the initial state and n_g in the final state. Additionally, $2N_f$ chiral symmetry violating quarks in the final state

Herein, the number of gluons n_g in the final state is not fixed and can become large even in leading order. Contrary to ordinary perturbation theory, here this process will be examined in the picture of instantons, in which the instanton will only couple to righthanded fermions q_{Rf} and lefthanded antifermions $\bar{q}_{Lf}$.

In quantum mechanics, instantons can be used to calculate the transition probability for a particle tunnelling through a potential barrier. In QFT, the tunnel process is more subtle. To understand the idea of tunnelling for a system described by Eq.(80), one should point out that in QCD, there is not one fixed vacuum. As the symmetry associated with strong interactions is $SU(3)$, i.e. non-Abelian, this implies a non-trivial vacuum structure. Here, we characterise the different vacua with the winding number N_{CS} (see Fig.(3)).

The instanton process now describes the tunnelling between the degenerate vacua differing by $\Delta N_{CS} = 1$ in Minkowski spacetime. According to t'Hooft, in the QCD case, this tunnelling process will lead to a violation of the axial charge [3]. Analogous to ordinary QM, the instanton solution for strong interactions, i.e. Eq.(80), can be found by looking at the classical solution of the Yang-Mills action in Euclidean space-time. The classical solution was found by Belavin et al. as [1]:

$$A^{a\,\text{inst}}_\mu(x) = \frac{2\rho^2}{g} \frac{\bar{\eta}^a_{\mu\nu}(x - x_0)_\nu}{(x - x_0)^2((x - x_0)^2 + \rho^2)}. \tag{82}$$

When inserted into the covariant derivative, we can see that this solution corresponds to the so-called zero mode, or in other words

$$\gamma^\mu D_\mu [A^{a\,\text{inst}}_\mu(x)] \psi_{q,i}(x) = 0. \tag{83}$$

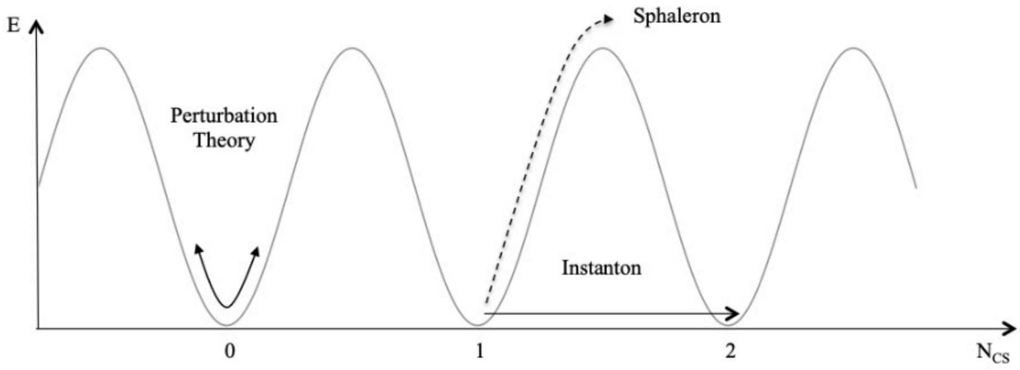


Figure 3: Image from [14], Non-trivial vacuum structure of QCD and how the instanton process insinuates a tunnelling process

To calculate the process, we will employ the optical theorem. The idea is to simplify the calculation of the instanton process, given in fig.(2), by calculating the forward scattering amplitude's imaginary part.

$$\sigma_{\text{tot}}^{\text{inst}} = \frac{1}{E^2}\text{Im}\{\mathcal{A}^{I\bar{I}}(p_1, p_2, -p_1, -p_2)\} \tag{84}$$

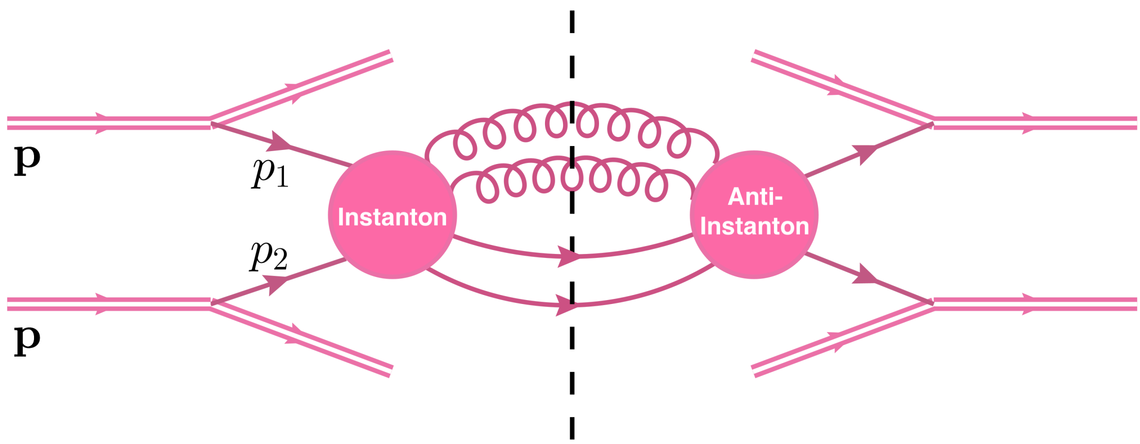


Figure 4: Optical theorem approach: Forward scattering of two partons with momentum p_1, p_2. The dashed line depicts along which the final state one is interested in is mirrored to apply optical theorem

To calculate the Amplitude $\mathcal{A}^{I\bar{I}}(p_1, p_2, -p_1, -p_2)$, we need to calculate the following Green's Function

$$G\left(p_1, p_2, p_3, p_4\right) = \int \mathcal{D}A_\mu [\mathcal{D}\psi\mathcal{D}\bar{\psi}]^{N_f}\, A_{\text{LSZ}}\left(p_1\right) A_{\text{LSZ}}\left(p_2\right) A_{\text{LSZ}}\left(p_3\right) A_{\text{LSZ}}\left(p_4\right) e^{-S_E[A_\mu,\psi,\bar{\psi}]} \tag{85}$$

Herein the momentum p_3, p_4 will be fixed by p_1, p_2. Evaluating this integral will be done using the valley approach. The valley approach is based on the fact that (quasi) zeromodes arise when the system has an (approximate) symmetry, which leaves the action S_E invariant. A common illustration of a zero mode is the centre of the instanton. The associated collective coordinate, x_0, representing translation, does not alter the instanton action, indicating translational symmetry. More broadly, any symmetry broken by the background field configuration (like the instanton in our example) corresponds to a collective coordinate, τ. This coordinate is linked to a zero mode, $\frac{\partial A_\tau}{\partial \tau}$, where A_τ represents the background field, i.e. the instanton solution. The integration over the functional space in the valley method employs these symmetries by separating the integration along the valley and orthogonal to it. We characterise systems with more than one symmetry by the collective coordinates τ_i. Due to the translational invariance of the instanton for each function A, we will find the closest function $A_{\tau *}=A_{cl}(t-\tau *)$ where A_{cl} is the classical solution to the Euclidean action. Analogous to Eq.(52), the closest function would then satisfy by definition

$$\frac{1}{2}\|A-A_{\tau *}\|^2=\min \Rightarrow \frac{\partial}{\partial \tau}\frac{1}{2}\|A-A_\tau\|^2\big|_{\tau=\tau *}=0. \tag{86}$$

However, in contrast to the section before, we define the scalar product as

$$\langle f,g\rangle_w=\int d^4x g_\tau(x)w(x;\tau)f_\tau(x), \tag{87}$$

with $w(x;\tau)$ as a weighting function. The weight function is necessary to maintain gauge invariance [7]. With this definition, Eq.(86) can be written explicitly as

$$\left\langle A-A_{\tau *},\frac{\partial A_\tau}{\partial \tau}\bigg|_{\tau=\tau *}\right\rangle=0. \tag{88}$$

Next, we again define a streamline equation. In difference of Eq.(51), here we need to take into account the weight function. Thus, within the valley, the following relation between the collective coordinates and the change in action holds

$$\left.\frac{\delta S}{\delta A}\right|_{A=A_\tau}=\epsilon^2(\tau)\,w(x;\tau)\,\frac{\partial A_\tau}{\partial \tau}\,. \tag{89}$$

In the equation above, $\epsilon^2(\tau)$ is the Lagrange multiplier. Regarding the Lagrange multiplier ϵ, one should point out that Eq.(89) characterises the quasizeromodes, while for the zeromodes ϵ is exactly 0, such that one is left with the Euler Lagrange equation. In the previous chapter, it was mentioned that a boundary condition is used to make the integration of the quasi-zero mode explicit. Here, this boundary condition can be understood physically, namely at large separations of the instanton and antiinstanton, there is no interaction between them. From here, the steps are analogous to how we got to Eq.(79). However, in the previous case, we only had one zero mode and one quasi-zero mode therefore, in Eq.(79), we had the term $\|\dot{\phi}_\alpha\|\|\phi'_\alpha\|$. Here, we need to replace it by the following determinant $\det\left(\left\langle\frac{\partial A_\tau}{\partial \tau_i},\frac{\partial A_\tau}{\partial \tau_j}\right\rangle_w\right)$. This makes sure that we take all contributions into account.[1]

Leaving us with

$$\begin{aligned}G=&N\int\prod_i d\tau_i \det\left(\left\langle\frac{\partial A_\tau}{\partial \tau_i},\frac{\partial A_\tau}{\partial \tau_j}\right\rangle_w\right)\det{}^{-1/2}\left(\left\langle\frac{\partial A_\tau}{\partial \tau_i},(\Box(A_\tau))^{-1}\frac{\partial A_\tau}{\partial \tau_j}\right\rangle_w\right)\\&(\det(\Box(A_\tau)))^{-1/2}\prod_{m=1}^4 A_{\mathrm{LSZ}}(p_m)\,e^{-S(A_\tau)}\end{aligned} \tag{90}$$

with $\Box(A_\tau)=\frac{\delta^2 S(A)}{\delta A^2}|_{A=A_\tau}$. Herein $(\det(\Box(A_\tau)))^{-1/2}$, is the functional determinant and $(\Box(A_\tau))^{-1}$, the Greens function to $\Box(A_\tau)$. As the $\frac{\partial A_\tau}{\partial \tau}$ are the (quasi)zeromodes of the action, they are the eigenfunctions of $\Box(A_\tau)$, so [11]

$$\Box(A_\tau)\,\frac{\partial A_\tau}{\partial \tau_i}=\lambda_i\frac{\partial A_\tau}{\partial \tau_i}\,. \tag{91}$$

Now employing this in Eq.(90), more specifically only on the part containing determinants, the determinants can be written as

$$\det{}^{1/2}\left(\left\langle\frac{\partial A_\tau}{\partial \tau_i},\frac{\partial A_\tau}{\partial \tau_j}\right\rangle\right)\left(\det{}^{(2p)}\left(\Box\left(A_\tau\right)\right)\right)^{-1/2}. \tag{92}$$

The $(2p)$ stands for the determinant calculated without the $2p$ (quasi)zeromodes. Here p (quasi)zeromodes stem from the first instanton and the other from the antiinstanton. This removal is needed as the (quasi)zeromodes would give us a diverging term for the $\det(...)^{-1/2}$ term. In the leading order approximation, i.e. in which we take $\Box(A_{I\bar{I}})=\Box(A_I+A_{\bar{I}})$ the determinants over two instantons separate into two determinates, each evaluated over one instanton. Physically, this makes sense as, in the leading order, the interaction between the instantons is negligible. This must be, as for $R=x_I-x_{\bar{I}}$

[1] see Appendix A for a more thorough deriavtion

going to infinity, there should be no physical interaction, and the quasizeromodes become zero ones. Or explicitly

$$N\det^{(2p)}\left(\Box\left(A_{I\bar{I}}\right)\right) \approx N\det^{(p)}\left(\Box\left(A_{I}\right)\right) N\det^{(p)}\left(\Box\left(A_{\bar{I}}\right)\right). \tag{93}$$

Of course, the determinantes are now calculated only over p (quasi)zeromodes. All this in Eq.(90), leaves us with

$$G = \int d\mu_1 d\mu_2 \prod_{m=1}^{4} A_{LSZ}\left(p_m\right) e^{-S\left(A^{\mathrm{cl}(\tau)}\right)}, \tag{94}$$

where

$$d\mu_a = N \prod_{i=1}^{p} d\tau_{a,i} \det{}^{1/2}\left(\left\langle \frac{\partial A_a}{\partial \tau_{a,i}}, \frac{\partial A_a}{\partial \tau_{a,j}} \right\rangle\right) \left(\det^{(p)}\left(\Box\left(A_a\right)\right)\right)^{-1/2}. \tag{95}$$

Now that we have a term for our Greens function, we need to find the valley trajectory $(A_a^{I\bar{I}})_\mu$ for the instanton anti-instanton valley. For a $SU(2)$ theory, this was calculated explicitly in [10], for $SU(3)$, the result was given in [15]. It is shown in [16] that the trajectory can be used to calculate the action:

$$S_{I\bar{I}}(z) \;=\; \frac{16\pi^2}{g^2}\left(3\frac{6z^2-14}{(z-1/z)^2} \;-\; 17 \;-\; 3\log(z)\left(\frac{(z-5/z)(z+1/z)^2}{(z-1/z)^3} - 1\right)\right), \tag{96}$$

where z depends on the collective coordinates in the following way

$$z \;=\; \frac{R^2+\rho^2+\bar{\rho}^2+\sqrt{(R^2+\rho^2+\bar{\rho}^2)^2-4\rho^2\bar{\rho}^2}}{2\rho\bar{\rho}}. \tag{97}$$

Till now, we have found an expression for the Green's function of said instanton process. To compute the cross-section, one needs to go back to Minkowski space. After that, Fourier transforming G, i.e. the 4-point Green function, will lead us to the total cross-section. This, however, is outside the scope of this summer student project. As a reference, in [16] the total cross section is given in Eq.(2.24) as

$$\begin{aligned}\hat{\sigma}_{\mathrm{tot}}^{\mathrm{inst}} \;\simeq\; & \frac{1}{E^2}\,\mathrm{Im}\,\frac{\kappa^2\pi^4}{36\cdot 4}\int\frac{d\rho}{\rho^5}\int\frac{d\bar{\rho}}{\bar{\rho}^5}\int d^4R\int d\Omega\left(\frac{2\pi}{\alpha_s(\mu_r)}\right)^{14}(\rho^2E)^2(\bar{\rho}^2E)^2\,\mathcal{K}_{\mathrm{ferm}}(z)\\ & \times(\rho\mu_r)^{b_0}(\bar{\rho}\mu_r)^{b_0}\,\exp\left(R_0E \;-\; \frac{4\pi}{\alpha_s(\mu_r)}\,S_{I\bar{I}}(z) \;-\; \frac{\alpha_s(\mu_r)}{16\pi}(\rho^2+\bar{\rho}^2)\,E^2\,\log\frac{E^2}{\mu_r^2}\right).\end{aligned} \tag{98}$$

This equation needs some explanation. First, as mentioned before, we integrate over all the collective coordinates:

1. ρ and $\bar{\rho}$ - the instanton and anti-instanton sizes
2. $R_\mu = (R_0, \vec{R})$ - the separation of I and $\bar{I}$ positions in Euclidean space
3. Ω - the 3×3 matrix of relative $I\bar{I}$ orientations in the $SU(3)$ colour space

The factor $\mathcal{K}_{\mathrm{ferm}}(z)$ appearing in the equation above comes from calculating the overlap between the instanton and anti-instanton fermion zero modes,

$$\omega_{\mathrm{ferm}} = \int d^4x\psi^{\bar{I}}_{0,\mathrm{ferm}}\left(x\right) i\not{D}\psi^{I}_{0,\mathrm{ferm}}\left(x\right). \tag{99}$$

This term is then raised to the power of $2N_f$, the number of fermions:

$$\mathcal{K}_{\mathrm{ferm}} \;=\; \left(\omega_{\,\mathrm{ferm}}\right)^{2N_f}, \tag{100}$$

The $2N_f$ arises from the number of fermions in the final state of the process given in fig.(2).

The following terms are left

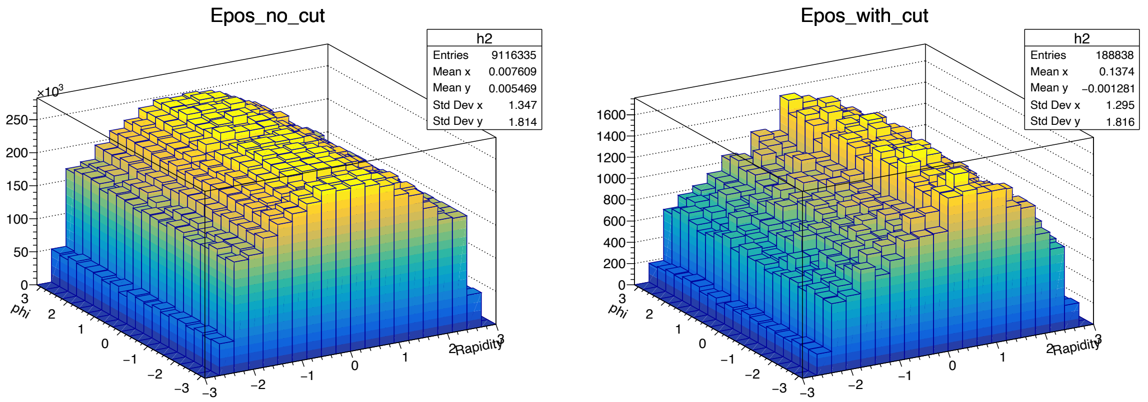
1. μ_r - renormalization scale
2. κ - normalisation constant of instanton density in $\overline{\mathrm{MS}}$ scheme
3. $b_0 = (11/3)N_c - (2/3)N_f$ - term arising from renormalisation scheme

4 Data Analysis

V. A. Khoze, V. V. Khoze, D. L. Milne and M. G. Ryskin [7] have stated that by imposing large rapidity gaps in diffractive processes, we can separate background processes such as multiple parton interactions from instanton signal. Furthermore, in the paper, the following cuts are suggested

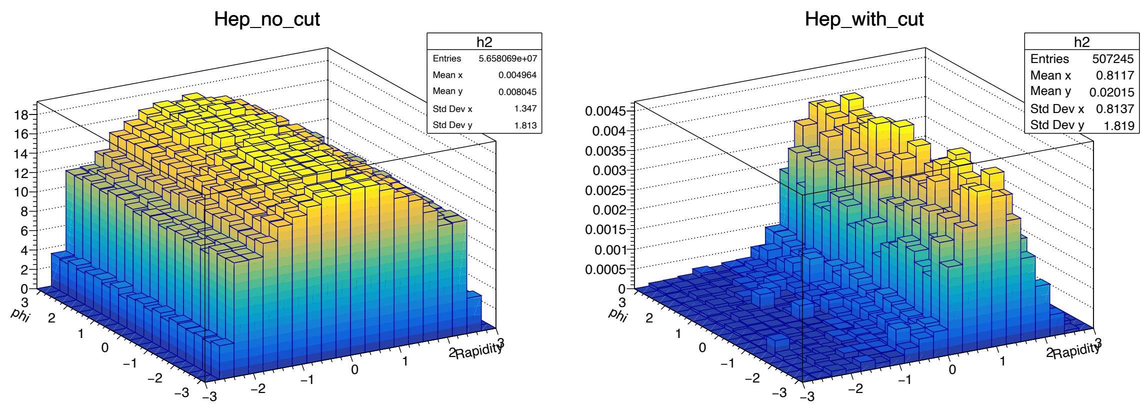
1. $N_{ch} > 20$ and $\sum E_{T,i} > 15$ GeV
2. only sum over particles in the rapidity region of $0 < \eta < 2$ with $p_{T,i} > 0.5$ GeV
3. discard full event if any particle has $p_{T,i} > 2$ GeV.

Here N_{ch} depicts the number of charged particles, $E_{T,i}$, the transverse energy of the i-th particle in the final state, and $p_{T,i}$ aforementioned transverse momentum. These cuts were performed for this project on data samples generated by EPOS, Sherpa (HepMC), and PYTHIA. Additionally, these cuts were also applied to real data taken. However, the samples used were not specifically for the diffractive processes. The application of the cut on this data is to show that the ROOT file is indeed working and can be applied for any further studies. In the following figures, the samples before the cut and after the cuts are plotted.



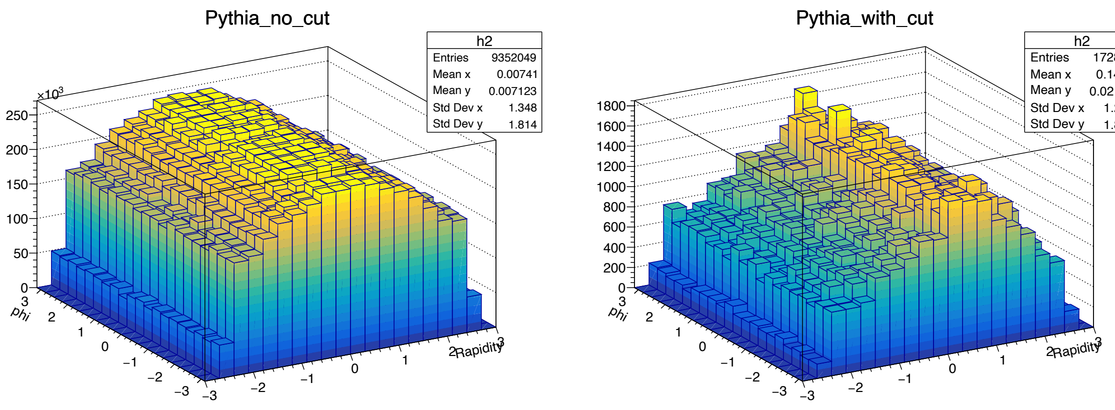
(a) EPOS data without cuts (b) EPOS data with cuts

Figure 5: EPOS data plotted against rapidity η and angle ϕ. On the left without cuts and on the right with the cuts mentioned.



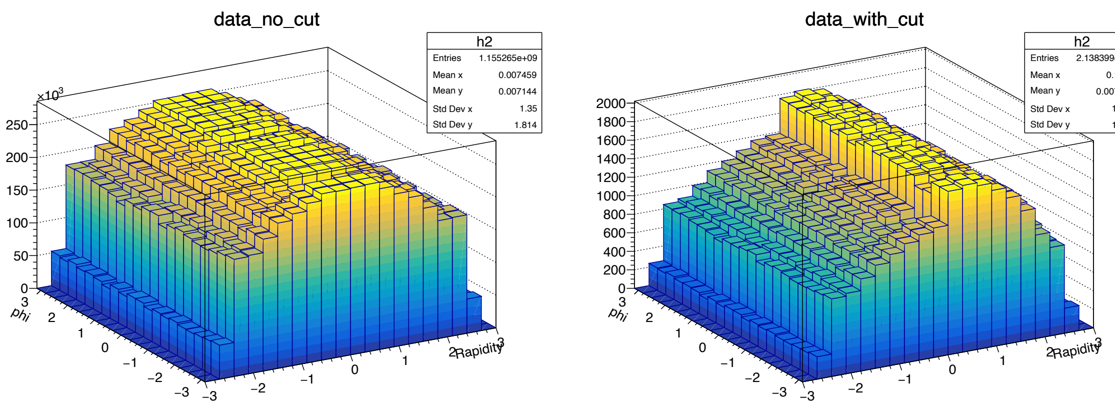
(a) Sherpa (HepMC) data without cuts (b) Sherpa (HepMC) data with cuts

Figure 6: Sherpa (HepMC) data plotted against rapidity η and angle ϕ. On the left without cuts and on the right with the cuts mentioned.



(a) PYTHIA data without cuts (b) PYTHIA data with cuts

Figure 7: PYTHIA data plotted against rapidity η and angle ϕ. On the left without cuts and on the right with the cuts mentioned.



(a) Data without cuts (b) Data with cuts

Figure 8: Data plotted against rapidity η and angle ϕ. On the left without cuts and on the right with the cuts mentioned.

To compare the cut algorithm for the different data samples, we look at the cut efficiency. The cut efficiency is given by the ratio between the entries left after the cut and before the cut. For the different data samples, they are given in the table below.

Data Set	Cut Efficiency
EPOS	2.071%
Sherpa (HepMC)	0.896 %
PYTHIA	1.849 %
Real Data	1.851 %

Table 1: Cut Efficiencies for the data samples generated by EPOS, Sherpa (HepMC), PYTHIA and Real Data.

In the table, we can see that the efficiency for the different samples is of the same order (around $1-2\%$).

Again, it is important to mention the fact that the samples used were not for diffractive processes. This analysis was to check if the ROOT file is working properly. As this is the case as the next step of this project, we should apply this code to the diffractive data samples taken by ALFA.

5 Conclusion and Outlook

In conclusion, this project started with getting a better understanding of QCD instantons, focusing on their theoretical background and the possibility of detecting them experimentally at the LHC. Instantons represent an aspect of quantum chromodynamics, allowing transitions between different QCD vacua and leading to effects such as chiral symmetry violation, first explained by standard perturbation theory. And later linked to non-perturbative effects (Instantons) by 't Hooft [3]. By revisiting the work of previous researchers and using the valley approximation, this project tried to gain a better understanding of these non-perturbative processes. However, it is important to acknowledge that the work is far from complete. A key area that still requires attention is the computation of the instanton-induced cross section, which has not yet been fully carried out. This step is crucial for making direct comparisons between theoretical predictions and experimental data. Additionally, while the study focused on using data from the ALFA detector, the connection between theoretical models and experimental results has not yet been fully established. One important aspect of the current analysis is that the data used were not only for diffractive events. Moving forward, it will be essential to use the code for diffractive processes to more accurately filter the data. Thus, while this project has made progress in reevaluating the theoretical framework and identifying experimental possibilities, there are still open questions left. Future work will need to focus on completing the cross-section calculations, refining the theoretical-experimental link, and analysing diffractive data.

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A Explicit Derivation of Greens function

To solve Eq.(85), we make use of the Fadeev-Popov procedure, i.e.

$$1 = \int d\tau \left|\det\left(\frac{\partial}{\partial\tau}\left\langle A - A_\tau, \frac{\partial A_\tau}{\partial\tau}\right\rangle_w\right)\right| \delta\left(\left\langle A - A_\tau, \frac{\partial A_\tau}{\partial\tau}\right\rangle_w\right)$$

or generalised for more than one collective coordinates

$$1 = \int \prod_i d\tau_i \left|\det\left(\frac{\partial}{\partial\tau_j}\left\langle A - A_\tau, \frac{\partial A_\tau}{\partial\tau_i}\right\rangle_w\right)\right| \delta\left(\left\langle A - A_\tau, \frac{\partial A_\tau}{\partial\tau_i}\right\rangle_w\right) \tag{101}$$

Furthermore

$$\begin{aligned}\frac{\partial}{\partial\tau_j}\left\langle A - A_\tau, \frac{\partial A_\tau}{\partial\tau_i}\right\rangle_w &= \left\langle \frac{\partial A_\tau}{\partial\tau_j}, \frac{\partial A_\tau}{\partial\tau_i}\right\rangle_w + \left\langle A - A_\tau, \frac{\partial^2 A_\tau}{\partial\tau_i\partial\tau_j}\right\rangle_w \\ &\approx \left\langle \frac{\partial A_\tau}{\partial\tau_j}, \frac{\partial A_\tau}{\partial\tau_i}\right\rangle_w\end{aligned} \tag{102}$$

as in the first RHS, the first part dominates [11]. Additionally expanding the action around the valley up to the second order:

$$\begin{aligned}S(A) = S(A_\tau) \quad &+ \quad \left\langle \left.\frac{\delta S(A)}{\delta A}\right|_{A=A_\tau}, (A - A_\tau)\right\rangle_w \\ &+ \quad \frac{1}{2}\left\langle (A - A_\tau), \left.\frac{\delta^2 S(A)}{\delta A^2}\right|_{A=A_\tau} (A - A_\tau)\right\rangle_w .\end{aligned} \tag{103}$$

Eq.(102) in Eq.(101), together with Eq.(103) can be used to write Eq.(85) as

$$\begin{aligned}G =& N \int \prod_i d\tau_i \det\left(\left\langle \frac{\partial A_\tau}{\partial\tau_i}, \frac{\partial A_\tau}{\partial\tau_j}\right\rangle_w\right) \int \mathcal{D}A \prod_i \delta\left(\left\langle A - A_\tau, \frac{\partial A_\tau}{\partial\tau_i}\right\rangle_w\right) \\ &\prod_{m=1}^{4} A_{\text{LSZ}}(p_m)\, e^{-S(A_\tau) - \left\langle \frac{\delta S(A)}{\delta A}|_{A=A_\tau}, (A-A_\tau)\right\rangle_w - \frac{1}{2}\left\langle (A-A_\tau), \frac{\delta^2 S(A)}{\delta A^2}|_{A=A_\tau}(A-A_\tau)\right\rangle_w}\end{aligned} \tag{104}$$

the proportionality implied in Eq.(89), together with the delta distribution can be used to set the linear fluctuation in the exponent to zero.

$$\begin{aligned}G =& N \int \prod_i d\tau_i \det\left(\left\langle \frac{\partial A_\tau}{\partial\tau_i}, \frac{\partial A_\tau}{\partial\tau_j}\right\rangle_w\right) \int \mathcal{D}A \prod_i \delta\left(\left\langle A - A_\tau, \frac{\partial A_\tau}{\partial\tau_i}\right\rangle_w\right) \\ &\prod_{m=1}^{4} A_{\text{LSZ}}(p_m)\, e^{-S(A_\tau) - \frac{1}{2}\left\langle (A-A_\tau), \frac{\delta^2 S(A)}{\delta A^2}|_{A=A_\tau}(A-A_\tau)\right\rangle_w}\end{aligned} \tag{105}$$

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