



TECHNISCHE
UNIVERSITÄT
WIEN
Vienna University of Technology



Dissertation

Exact Solutions of Indirect Transverse Field Effects in Elongated Structures with Applications to CERN LHC and PS

Supervision

Michael Benedikt

Vienna University of Technology
Atomic and Subatomic Physics

Massimo Giovannozzi

CERN
BE-ABP-HSS

Vienna University of Technology
Faculty of Physics

Author

Simon Hirländer

Vienna, February 2020

*To all the people who believed in me and gave
me support.*

Abstract

The understanding of the electromagnetic interaction of the particle beam with the surrounding elements - so-called indirect space charge driven (ISCD) effects - in particle accelerators, is crucial for stable high-intensity performance. It is addressed and applied at the CERN accelerator complex.

An appropriate quantitative explanation for the ISCD tune-shift which must be corrected during the operation of the Large Hadron Collider (LHC) was missing. This work developed an approach based on complex Green functions (solving the arbitrary Dirichlet and Neumann boundary problem) which matches measurements with unprecedented accuracy. As the primary origin of the ISCD tune-shift, the electric interaction with the beam-screen is identified. A closed-form model is obtained, that is also applicable to future accelerator projects as the High Luminosity (HL)-LHC, where these effects will be at least a factor two higher. During the Multi-Turn Extraction in the Proton Synchrotron (PS), the beam is split into the main beam and four islands so-called beamlets. In measurements, an intensity dependence in the beamlet position and tune was observed. ISCD effects are the cause as shown in calculations and numerical simulations based on closed-form expression acquired in this thesis. To obtain simple mathematical expressions, a novel Lorentz operator of the Green functions, on the Riemann-sphere (RS), and from it, the so-called image operators for arbitrary beam distributions are derived. These operators allow estimating the ISCD tune-shift of complex accelerator models. A novel method to approximate the fundamental electrostatic field (the Green function) of arbitrary simply-connected domains, including an error bound, is proven. It is used to approximate the rect-elliptical LHC beam-screen. Additionally, a new integral representation of the Neumann function on the RS for smooth bounded simply-connected domains is derived. It allows for classifying domains concerning the solution of the Neumann function into bounded, unbounded star-like and exterior solutions. A novel method is presented, to obtain closed-forms in the case of unbounded star-like domains. Consequently, several novel closed-form solutions for the magnetic interaction of essential shapes as the n -poles or the combined-function magnets of the PS are obtained (so far only parallel plates were used).

Finally, several new off-axis image tensors for standard geometries are provided.

Acknowledgements

I have to thank all the people who accompanied my journey. This work is the fruit of a long history, which started long before my PhD program and has not ended yet. If my mother had not been so supportive during my education, for sure, I would not have ended in this field. Along my educational way, I was supported by many exceptional persons, especially one teacher, who taught me how to do mathematics and physics, how to have patience and keep up the hope, *Bernhard Schnizer*.

All along the tough time of my PhD, I was guided and supported, not only in technical aspects but also in human concerns by *Bernhard Holzer*. Thank you, Bernhard, for your endless patience and for repetitively reading my thesis drafts. I want to appreciate *Detlef Kuechler*, who gave me feedback on the chapter on potential theory. Rigorous feedback on mathematical physics was given by *Klaus Heinemann* and *Jim Ellison*, which clearly helped me to improve the quality of the work. Without doubt, I also deeply thank my university supervisor *Michael Benedikt*, who, apart from being an excellent physicist, showed extraordinary management skills, which made this work possible. I have to be grateful for being allowed to work on these topics by my CERN supervisor *Massimo Giovannozzi*.

Many colleagues and friends helped me in understanding the subject and developing a critical and sharp scientific mind. Among them is *Alexander Huschauer*, who gave me advice on PS domain-specific topics and *Michaela Schaumann*, who supported me on the LHC domain-specific topics.

Special thanks go to my friend *Daniela Laskovski*, who gave me field external advice on my thesis and corrected my English while having a small kid *Chloé*.

Last but not least, I want to thank my wife, *Daniela Hirlander*, who always supported me without asking many questions. She held my hand in the darkest nights of doubts and bitterness during this time. Thank you, Daniela, I love you.

Diese Arbeit wurde vom österreichischen Bundesministerium für Wissenschaft, Forschung und Wirtschaft unterstützt und im Rahmen des österreichischen Doktorandenprogramms am CERN durchgeführt.

Contents

Abstract	vii
Acknowledgements	ix
List of Figures	xv
List of Tables	xix
Part I Motivation	
1 Introduction	3
1.1 Definition of the Problem	3
1.2 Objectives of the Work	6
1.3 Content of the Thesis	7
1.4 Key Results and Relevance of the Present Work	8
1.5 Structure of the Thesis	9
Part II Theoretical Foundations	
2 Theory of Accelerators	13
2.1 Setting the Scene	13
2.1.1 Background of CERN	13
2.2 The Formulation of Physics of Accelerators	14
2.2.1 The Challenge	15
2.3 The Circular Accelerator as a Hamiltonian System	15
2.3.1 Stability of Repetitive Systems: LA Theorem and Invariant KAM Tori	15
2.3.2 The Invariant of Courant and Snyder	17
2.3.3 Methods of Perturbing the Linear System	18
2.3.4 Canonical Perturbations through Indirect Fields	19
2.4 Summary	21

3	Potential Theory on the Riemann-sphere	23
3.1	The Fundamental Importance of the Logarithmic Singularity	23
3.1.1	A Novel Approach	23
3.1.2	Prelude: The Theorem of Green-Riemann	26
3.1.3	The Logarithmic Singularity	27
3.1.4	The Complex Green Function	29
3.1.5	The Connection to Harmonic Functions	30
3.2	Topology, Conformal Mapping and Boundary Behaviour	32
3.2.1	Topology and the Riemann-sphere	32
3.2.2	Conformal Mapping	32
3.2.3	Boundary Behaviour	34
3.3	The Physical Aspects	37
3.3.1	The Magnetostatic Case in the Limit of Perfect Permeability	37
3.3.2	The Electrostatic Case in the Limit of Perfect Conductivity	40
3.4	The Green Function of the Classical Dirichlet Boundary Problem	42
3.4.1	A Novel Approximation Method for Simply-connected Domains including an Error-bound	43
3.4.2	How to find the Explicit Green Function	45
3.5	Novel solutions of the Generalized Neumann Problem	47
3.5.1	Dini's Formula	50
3.5.2	A Novel Integral Representation of the Neumann Function	53
3.5.3	First Consequence of eq. (3.57): the Neumann Function as a Conformal Invariant	56
3.5.4	Second Consequence of eq. (3.57): the Neumann Function for Unbounded Regulated Simply-connected Domains	58
3.5.5	Some Comments on the Source Boundary Behaviour	60
3.6	A New Classification of the Neumann function on the Riemann-sphere	61
3.6.1	Unbounded Domains: ∞ Non-degenerated	61
3.6.2	Regulated Domains: ∞ Degenerated	66
3.6.3	Exterior Domains $M : \infty \in M$	71
4	Conformal Mappings	75
4.1	The Schwarz-Christoffel-Transformation	75
4.1.1	Elementary Operations	76
4.1.2	Construction of the Schwarz-Christoffel-transformation	77
4.1.3	The Parameter Problem	78
4.1.4	A Variant of the Schwarz-Christoffel-transformation	78
4.2	Polygonal solutions	79
4.2.1	The Strip and the Half-Strip	79
4.2.2	The Rectangle	80
4.2.3	The Regular Polygon	82
4.2.4	Arbitrary Polygons	84
4.3	Composed Mappings	85
4.3.1	The Ellipse - Including a New Mapping	85

4.3.2	Novel n -Poles	90
4.4	Summary	91

Part III Applications

5	Image Operators	97
5.1	Classic Image Tensors	97
5.1.1	The Equations of Motion	97
5.1.2	The Form Tensors	100
5.1.3	Different Regimes	101
5.2	The Image Operators	103
5.2.1	Indirect Space Charge Effects in more Complex Models	105
5.3	Form Factors for Particular Cases	106
5.3.1	The Rectangular Cross-Section	108
5.3.2	The Parallel Plates	110
5.3.3	The Cut Parallel Plates - on-axis	112
5.3.4	The n -Pole Structure - on-axis	112
5.3.5	The Combined-Function Magnet - on-axis	113
5.3.6	The Octagonal Chamber (HL-LHC Beam Screen)	113
5.3.7	The Quadrupole - Hyperbolic Pole Shoes	115
5.3.8	The Circular Chamber	117
5.3.9	The Elliptic Chamber	118
5.4	Summary	121
6	Applications to the CERN Proton Synchrotron (PS)	123
6.1	The Machine	123
6.2	The PS Magnets	123
6.3	The Modelling of the Elements	124
6.3.1	The Vacuum Chamber	125
6.3.2	The Model of the Combined-Function Magnets	126
6.4	Results	129
6.4.1	The Open Magnetic Block	130
6.4.2	The Closed Magnetic Block	132
6.4.3	Multi-Turn Extraction Tune-Shift Closed-Form Model	133
6.4.4	Multi-Turn Extraction Intensity-Position-Dependence Studies	135
6.5	Summary	141
7	Applications to the CERN Large Hadron Collider (LHC)	143
7.1	The Machine	143
7.1.1	The Magnets	143
7.1.2	The LHC Beam-screen	145
7.2	The Modelling of the Elements	145
7.2.1	The LHC Beam-screen as a Polygon	147
7.2.2	Theoretical Convergence Studies	147
7.3	Measurements	148
7.4	Results	149

7.4.1	A Closed-Form Model	150
7.5	Future Application: CERN (H)igh-(L)uminosity - LHC	153
7.5.1	Results	155
7.6	Summary	156
Part IV Summary		
8	Summary and Outlook	159
Appendix A	The Complex Electro and Magnetostatic Equations	163
Appendix B	Dirac Sequences as Used for the Proof in theorem 3.11	165
Appendix C	Special Functions Used in the Text	169
C.1	Used Elliptic Integrals and Functions	169
C.1.1	Elliptic Integrals	169
C.1.2	Jacobi Elliptic Functions	170
C.1.3	The Parameter of the Rectangle and the Ellipse	170
C.1.4	Q-Pochhammer Symbol	171
C.1.5	Hypergeometric Function and Gamma Function	171
C.1.6	Heaviside Function	171
Appendix D	On Issues of the Method of Images	173
D.1	The Image Method Convergence Problem	173
D.1.1	The Problem of One Infinite Plate	173
D.1.2	The Problem of Two Infinite Plates	174
D.1.3	A new Representation for the Rectangular Shape	177
Appendix E	Closed-Form Potentials - PS Combined Function Magnets .	183
References		187

List of Figures

1.1a	The PS Tunnel	4
1.1b	The 3D Model of the PS Magnetic Unit	4
1.2	The Model of the PS Closed Combined-Function Magnet	5
1.3a	The LHC Tunnel	5
1.3b	The LHC Dipole Cross-section	5
1.4	Structure of the Beam Pipe Cross-section of the LHC	5
2.1	The CERN Accelerator Complex	14
2.2	Dividing the Accelerator into Sub-problems	18
2.3	Poincaré Section of the Physical Phase Space	18
2.4	Indirect Field Contributions of the Surrounding Elements	20
3.1	Schematic Configuration	23
3.2	Road Map of the Theory Chapter	24
3.3a	The Fundamental Source 1: Pure Flux Field	29
3.3b	The Fundamental Source 2: Pure Circulation Field	29
3.4	The Principle of a Conformal Map	33
3.5a	The Strip Domain	36
3.5b	The Bird's Eye View onto $\tilde{\mathbb{C}}$	36
3.6	Perfect Magnetic Boundary Conditions	39
3.7	Perfect Electric Boundary Conditions	41
3.8	Novel Concept of Domain Approximation with Known Error	45
3.9	The Interpretation of the Blaschke Factor on $\mathbb{D}$	46
3.10	The Interpretation of the Blaschke Factor on $\tilde{\mathbb{C}}$	46
3.11	Average Exterior Normal Vector for Piecewise Domains	48
3.12	Experimental Verification of the Neumann Function	49
3.13	Tangential and Normal Derivative under Conformal Mapping	51
3.14	Concept of the Radial Limit	52
3.15	Neumann Function: Novel Integral Representation of a Piecewise Continuous Domain	56

3.16	Neumann Function: Novel Integral Representation of a Smooth Domain	56
3.17	Conformal Invariant Neumann Functions	57
3.18	The Concept of a Star-like Domain Extended to Infinity	58
3.19	Source near the Boundary	60
3.20	Half-plane on Riemann-sphere	62
3.21	Half-strip on Riemann-sphere	64
3.22	C -Shaped Domains	65
3.23	Depression on Riemann-sphere	65
3.24	Bird's Eye View on the North-pole of $\bar{\mathbb{C}}$	66
3.25	Strip on Riemann-sphere	67
3.26	Quadrupole on Riemann-sphere - Magnetic	69
3.27	Quadrupole on Riemann-sphere - Electric	70
3.28	C-Shape on Riemann-sphere - Magnetic	72
3.29	C-Shape on Riemann-sphere - Electric	74
4.1	The Schwarz Christoffel Mapping	77
4.2	The Exterior Schwarz Christoffel Mapping	79
4.3a	The Green Function of a Fractal	86
4.3b	The Neumann Function of a Fractal	86
4.4	The Mapping of the Ellipse	88
4.5	The Green Function of the PS Chamber	89
4.6	The Conformal Mapping of Canonical Domains	92
4.7	Novel Solutions of the Green Function of the First and Second Kind of the n -Pole	93
5.1a	Center of Charge of the Beam at Equilibrium	98
5.1b	Center of Charge of the Beam at Non-equilibrium	98
5.2a	Incoherent Movement of a Test Particle	98
5.2b	Coherent Movement of the Beam	98
5.3a	The Coherent Electric Image Tensor of the Rectangular Chamber ...	108
5.3b	The Incoherent Electric Image Tensor of the Rectangular Chamber ..	108
5.4a	The Coherent Electric Image Tensor of the Octagonal Chamber	114
5.4b	The Incoherent Electric Image Tensor of the Octagonal Chamber ...	114
5.5	The Image Coefficients of the Quadrupolar Structure (Closed-Forms).115	
5.6a	The Coherent Electric Image Tensor of the Circular Chamber	117
5.6b	The Incoherent Electric Image Tensor of the Circular Chamber	117
5.7a	The Coherent Electric Image Tensor of the Elliptical Chamber	119
5.7b	The Incoherent Electric Image Tensor of the Elliptical Chamber	119
6.1a	The Model of the PS Closed Combined-Function Magnet	124
6.1b	The Model of the PS Open Combined-Function Magnet	124
6.2	Drawing of the Magnetic Units of the PS	125
6.3	The Magnetic Unit of the PS	125
6.4	The Model of the PS Vacuum Chamber	126

6.5	The Approximations of the Open PS Combined-Function Magnet . . .	127
6.6	The Approximations of the Closed PS Combined-Function Magnet . .	128
6.7	The Novel Mapping of the Combined-Function Magnet Structure . . .	129
6.8	The Mapping of the Half-Quadrupole	130
6.9	The Mapping using the Mirror Charge Trick	130
6.10	The Magnetic Coefficients of the Open Magnetic Block	131
6.11	The Magnetic Coefficients of the Closed Magnetic Block	132
6.12	The Orbit and β -Function of the MTE Case	135
6.13	MTE: Tune-Shift Caused by the Electric Boundaries	136
6.14	MTE: Tune-Shift Caused by the Magnetic Boundaries	137
6.15a	MTE: Phase Space - without Space-Charge	139
6.15b	MTE: Phase Space - including Space-Charge	139
6.16	MTE: Different Space Charge Contributions	140
6.17	MTE: Beam Position Depended on Different Geometries	140
6.18	MTE: Green Function Strip	140
6.19	MTE: Green Function Rectangle	141
7.1	A Schematic Layout of the LHC	144
7.2	The Main Magnets of the LHC	144
7.3	A Warm Magnet of the LHC	145
7.4	The Structure of the Beam Pipe Cross-section of the LHC	146
7.5	The LHC Beam-screen	147
7.6	The Polygonal Approximation Circular Case	148
7.7	The Polygonal Approximation Rectelliptical Case	149
7.8	Models vs. Measurements	151
7.9a	The Coherent Electric Image Tensor of the LHC Beam-screen	152
7.9b	The Incoherent Electric Image Tensor of the LHC Beam-screen	152
7.10	The Green Function of the Rectelliptical Shape	153
7.11	The Green Function of the Rectangular Shape	153
7.12	The Error of the Approximation of the LHC Beam-screen	154
7.13	The Octagonal Beam-screen of the LHC	155
B.1	The Transformation of an Unbounded Star-like Domain	166
C.1	The Parameter of the Elliptic Functions	171
D.1	The Classical Image Method	174
D.2	The Infinite Plates Problem: Infinite Series of Images	175
D.3	Mirror Method Rectangle: Infinite Image Array	178
D.4	Convergence of the Rectangular Configuration	181
E.1	Potential of the Closed Combined Magnet: Wide Range	183
E.2	Potential of the Closed Combined Magnet: Small Range	184
E.3	Potential of the Closed Combined Magnet: Wide Range	184
E.4	Potential of the Closed Combined Magnet: Small Range	185

List of Tables

6.1	Nomenclature of the Magnetic Units in the PS	124
6.2	PS: The Form Factors for the Different Elements - Centred	135
6.3	PS: Tune-shifts - Different Contributions	138
6.4	PS: Tune-shifts - Different Models	138
7.1	LHC: Tune-shifts - Different Studies	154
7.2	HL-LHC: Tune-shifts - Different Approximations	156

Part I

Motivation

Chapter 1

Introduction

Curiouser and curiouser!

— Lewis Carroll, *Alice in Wonderland*

1.1 Definition of the Problem

Studying the stability of repetitive systems such as circular particle accelerators is a fruitful and wide topic, touching many different fields like the subject of dynamical systems, mainly Hamiltonian systems. The goal of these studies is to find stable configurations to be able to run machines like the Large Hadron Collider (LHC) at the Conseil Européen de la Recherche Nucléaire, short CERN.

The subject of beam dynamics deals with the motion of ensembles of particles with similar coordinates in electromagnetic fields, called beam. Generally, the position and the momenta are sufficient to describe such a system, and due to the similarity of their behaviour, usually, a particle somewhere in the centre of the ensemble is picked out. The motion of the others is described relative to this particle. In a repetitive system, the trajectory described by this particle is termed closed orbit or equilibrium orbit. The ensemble of particles oscillates around the equilibrium orbit, and in most cases, the volume of this ensemble in phase space can be treated as constant, which is named emittance. The eigenfrequency of the repetitive system is named the tune, and it is chosen to avoid resonances, which cause the loss of particles of the beam under unavoidable small perturbations.

There are phenomena, which depend on the number of particles within the ensemble, called intensity-dependent. Special attention is paid to space charge effects, which are the electromagnetic interaction of the charged particles with each other and the environment. Space charge effects can perturb the system and cause in first order a shift of the eigenfrequency (the tune), which is termed tune-shift. The interaction of particles within the beam itself is termed direct space charge and the interaction of the particles with the close-by environmental elements of the accelerator structure, which are essential to building such machines, e.g. the vacuum chamber of the accelerator, is termed indirect space charge.

In this work, the indirect space charge effects are addressed, which are inherently hard to model and can influence the dynamics and hence, the stability of the particle beam gravely. For machines like the LHC, the impact is so big, that it has to be corrected

during the injection process and, so far, no accurate quantitative explanation was available. However, also for involved extraction processes, such as the multi-turn extraction (MTE), used in the CERN Proton-Synchrotron (PS), space charge effects play a key role. During this process the beam is split into sub-beams, a centred beam and four largely off-centred beams, termed beamlets, which are extracted over four turns. A quantitative understanding of the origin of effects as the empirically determined intensity-dependent position of the beamlets was missing and can be explained through indirect fields, which are developed in the context of this thesis. Two strategies are common to take indirect space charge effects into account:

- numerical simulations, which are very involved,
- (semi -) analytical studies, which give some insights, e.g. scaling laws.

While the first approach involves expensive simulations, the latter only costs the finding of analytical expressions, which then can be used for several studies to gather insights. Laslett [1] was the first one, providing a framework to estimate these effects, and consequently, the first Taylor series coefficients of the indirect field are termed Laslett coefficients. Based on his ideas and his successors [2–11], the present work explains the indirect space charge influence of the environmental elements, as the vacuum chamber or magnets, on the stability of the particle beam. It is done for different transverse cross-section profiles of common elements (their cross-sections are termed geometries or cross-sections) in terms of closed-forms, or, if not otherwise possible, as new approximative semi-analytic forms for arbitrary chamber geometries.

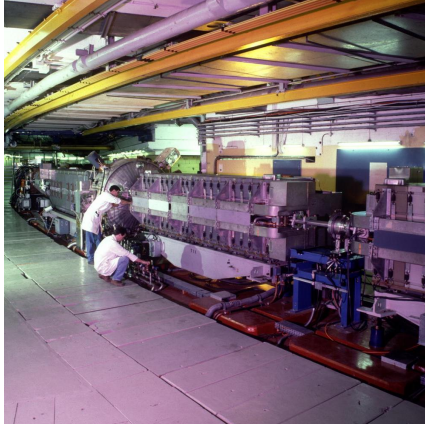


Fig. 1.1a The tunnel of the PS (see fig. 6.2). The elongated magnetic units and the beam pipe can be seen. Taken from [12].

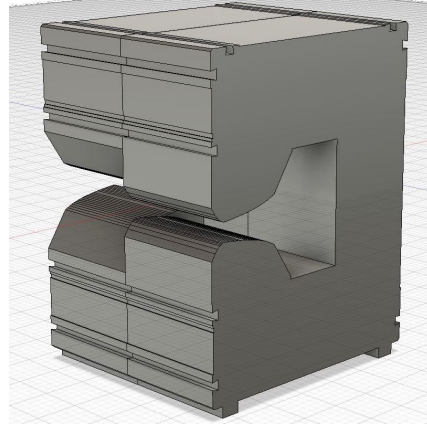


Fig. 1.1b The magnetic blocks (iron yokes) of the PS. This consists of a “closed” followed by an “open” combined function magnet. Taken from [13].

It was required, because, e.g. for the magnetostatic interaction, so far, only two parallel plates were considered. The fact that only limited geometries were available

Fig. 1.2 The cross-section of a “closed” block of the PS magnetic units (grey) as one half of the unit as show infig. 1.1b. The position of the vacuum chamber is shown (blue). A new approach is used to model the magnetic (Neumann) response cross-sections of the blocks with increasing complexity and accuracy in chapter 6.

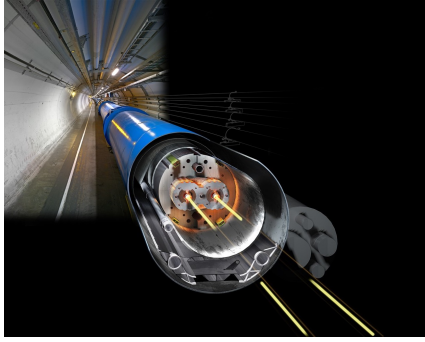
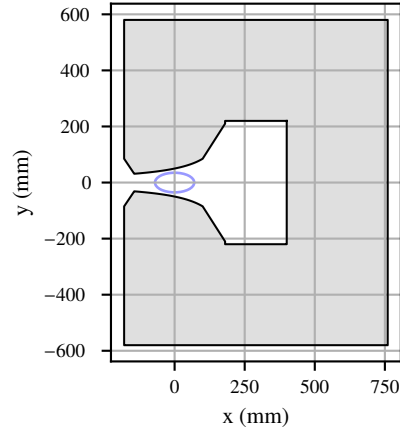


Fig. 1.3a The LHC tunnel. An elongated structure of a dipole magnet. Taken from [14].

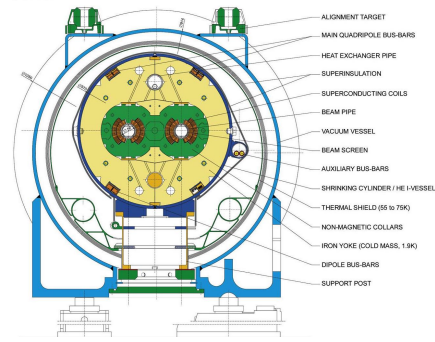
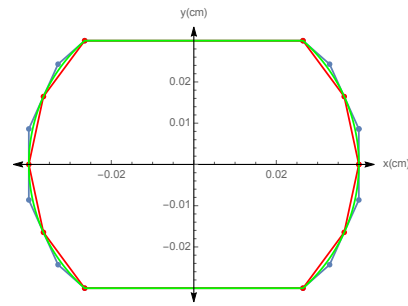


Fig. 1.3b The cross-section of an LHC dipole. The iron yoke and the beam-screen can be seen. Taken from [15]

Fig. 1.4 A geometrical configuration of the Beam Pipe Cross-Section of the LHC (green) and its approximative representations as in- (red) and out-scribed (blue) polygons. This element has the strongest impact on the beam as derived in chapter 7 and is of electric type (Dirichlet).



for off-symmetry fields is also addressed. These fields are expressed in terms of fundamental solutions, opening the possibility to calculate them up to arbitrary order for arbitrary charge distributions.

1.2 Objectives of the Work

As mentioned above, the modelling of indirect space charge effects is inherently complicated and demands heavy computations using numerical simulations. Numerical routines which estimate the region of the stability of the initial conditions of individual particle trajectories via numerical integrations and are termed tracking codes.

In this work, simple estimates, in the best case in terms of closed-forms, of the electromagnetic influence of the close-by surrounding elements for the tune-shift are provided. A photograph of such elements is provided in fig. 1.1a, where the PS tunnel can be viewed. One can see the vacuum chamber and magnetic units, which consist of combined-function magnet blocks. The structure these blocks is presented in fig. 1.1b. These blocks and the elliptical vacuum chamber are modelled in a 2D approximation, as depicted in fig. 1.2. This approximation is possible due to the elongated structure of the elements and no accurate closed-form solution was known before. For the LHC, the situation is illustrated in fig. 1.3a, where the the tunnel and the inner of a dipole element is shown. The cross-section of the LHC dipole with labelled elements is provided in fig. 1.3b. Closest to the beam and is the rect-elliptical vacuum chamber. It is modelled as drawn in fig. 1.4 and for the first time an accurate closed-form approximation is given. Other elements apart from the iron yoke do not interact significantly with the beam.

The obtained closed-form solutions are compared to semi-analytical cases. In the classical work of Laslett [1], the impact of the Laslett coefficients did not include coupling effects, which were later extended to tensors by Petracca [2]. The full fields for several new geometries and the corresponding tensors formulated as operators of underlying conformal mappings, which can be easily included in tracking codes, are derived in this thesis. Additionally, a method is presented to address arbitrary geometries of the involved elements, including an error bound. An example is the beam-screen of the LHC depicted in fig. 1.4. Here the rect-elliptical domain (no closed-form solution exists) is approximated using in- and out-scribing polygons, where the solution is known. It is proven in section 3.4 that the solution of the true rect-elliptical shape lies between these two solutions.

Estimations for the indirect intensity-dependent tune-shift of the LHC, which are, as stated, especially critical during the filling of the machine and have therefore to be considered, were provided by Ruggiero [10]. These estimates show a disagreement of a factor two compared to the measurements. This discrepancy is resolved in this work, providing a closed-form expression solution and a semi-analytical solution. For future projects like the (H)igh (L)uminosity LHC, the beam intensity increases (by a factor of two - in the HL-LHC case) and hence the impact of these effects increases as well (by a factor of two - in the HL-LHC case), which will become then even more critical. It is shown that the problem can be formulated through closed-forms and is closely related to the solution of the LHC.

The method of the MTE of the PS, which was going into operation in 2015 [16], showed a strong disagreement in simulations and measurements. It was shown [17] that indirect space charge effects could explain this. I provide closed-form expres-

sions for these phenomena used in tracking codes and calculate the tune-shift for the multi-turn extraction. To summarize the key problems addressed and solved in this study:

- Semi-analytic off-centred beam image field calculations are available for a limited number of geometries (circle, square).
- In the case of the magnetic interaction, only the parallel plates (the strip) and the circular chamber were available.
- The multi-turn extraction in the PS shows discrepancies in simulation and measurement concerning the intensity-dependent tune-shift and beamlet position.
- The indirect intensity-dependent tune-shift estimations of the LHC at injection deviate strongly from the measurements (by a factor of two).
- For future accelerators, such as the HL-LHC project, these effects increase by approximately a factor of two and have to be addressed.

1.3 Content of the Thesis

In this study, a solid theoretical framework is provided, highlighting the problems and some of the pitfalls arising when solving for the indirect field problem. The theoretical part is motivated by the fact that several new ideas, which are fundamental for the study and which are generally neglected in the literature, are discussed. Nevertheless, I tried to keep the amount of the mathematical derivations to the minimum, focusing on physical understanding. The entry point is the field, which is calculated for the 2D problems for different cross-sections for the electro- and magnetostatic case. To overcome some limitations, the problem is transferred to the complex plane to make use of the results of geometric function theory and complex analysis. The problem can be further simplified by leaving the plane and working on the Riemann-sphere. Here one profits from the results of mathematical physics in abstract and harmonic analysis and potential theory in the electrostatic case [18–30].

First, the electrostatic problem is attacked, where a new theorem is proved, allowing to find approximative solutions of simply-connected domains, including an error-bound. This is used in a later chapter to approximate the LHC vacuum chamber.

The magnetostatic case needs to be studied more carefully, and again, methods were developed in other domains. A publication of Van Bladel in 1961 [31] provides insights from experiments, which are extended here to the Riemann-sphere. A novel integral representation of the problem is deduced in this work, which provides a method to solve the problem for a new class of domains, yielding, in special cases closed-form solutions. It is done starting from Dini’s formula [32, 33] and the methods are presented in [24, 34–36].

The solution is finally expressed in terms of fundamental solutions, the Green functions of the first kind and the Green function of the second kind or Neumann function. These functions are in the used formulation conformal invariant and the problem is boiled down to finding appropriate mappings, namely conformal mappings [27, 35, 37–41].

After concentrating on the explicit construction of conformal mappings, the image tensors are formulated as operators of these mappings starting from a newly formulated complex Lorentz force operator. The Lorentz force operator is an operator of the Green functions, and consequently, the properties of the Green functions are inherited by all derived quantities. Based on this, the solutions for several new geometries for off-axis displaced beams are calculated explicitly. The verification of the results projected on the on-axis solution, as available in the literature [7–9], is included and should act as a reference.

Applications within the CERN accelerator complex include the PS and the LHC are treated in detail, where the importance of such theoretical studies becomes evident, and the concrete contributions of this work are listed in the next section.

1.4 Key Results and Relevance of the Present Work

Because of the variety of subjects the classification of the key results of the present work falls into several fields. The first part belongs to mathematical physics, namely potential theory and geometric function theory. The second more applied part belongs to the field of accelerator physics and therein to single-particle dynamics space charge within collective effects, in addition to that, these studies are essential to several aspects of operating an accelerator. First and most important, they explain the tune-shift and hence beam losses when crossing resonances as a function of intensity, which has to be taken into account, e.g. during injection of the CERN-LHC. Secondly, in the course of involved processes like the CERN PS multi-turn extraction (MTE), they explain the intensity dependence of the position of the beamlets, which is fundamental for this kind of operation. Accurate estimates of these effects are crucial for the successful performance of these machines. The explicit contributions are:

- A new method to calculate the approximative Green function of the first kind for arbitrary cross-sections including an error bound is provided (section 3.4).
- A novel integral representation of the Green function of the second kind (magnetostatic problems) for bounded domains yielding as a limiting closed-forms for unbounded star-like domains, e.g. for n -pole structures (magnets) is derived (section 3.5). The strategy of this work also permits to classify the Green function of the second kind for different domains in terms of their solution as bounded, unbounded star-like and exterior domains (section 3.6).
- A detailed treatment and a new solution of the converge issue of the mirror charge problem of two infinite plates is discussed in detail and a proof of the convergence is given appendix D.1. Furthermore, the problem is extended in this manner to the rectangle (appendix D.1), where also a proof of convergence is provided.
- Novel representations for classical problems like the conformal transformation of the ellipse onto the half-plane are explicitly provided (section 4.3.1).
- General off-axis image operators of the underlying conformal mapping are developed (section 5.2) and the Lorentz force is given as an operator of the Green

function of the first and the second kind (eq. (5.19)). The explicit computation of these operators for several new geometries including closed-forms for n -poles, elliptical, rectangular and octagonal cross-sections and their corresponding field, which can be easily implemented in beam dynamics simulation codes, are derived in detail, as the involved calculations are tedious and error-prone. It should serve as a reference because discrepancies were found in the literature (section 5.3). By employing the Schwarz-Christoffel transformation, the method naturally extends to image operators for arbitrary cross-sections for the electro and magnetostatic cases.

- Several novel closed-form models for the approximation of combined-function magnets, treated as parallel plates so far, are used in the context of the multi-turn extraction (MTE) in the CERN PS (section 6.3.2.2). The study also includes a (-in this context-) new approximative approach for arbitrary magnetic cross-sections, which is taken to verify the accuracy of the closed-form approximation. There was no solution before.
- With the help of closed-form expression of the transverse fields, the influence of indirect space charge effects on the position of the beamlets during the multi-turn extraction of the CERN PS is discussed. The indirect space charge effects mainly govern the position of the beamlets, as simulations in agreement with the measurement show, revealing the importance of simple, accurate expressions used in such situations (section 6.4.4). Additionally, the tune-shift during the MTE is calculated off-centred for the first time employing suitable closed-form models.
- Closed-form expressions based estimates for the indirect space charge driven tune-shifts of CERN LHC with excellent agreement with the measurements [42, 43] are derived, improving the accuracy of previously existing calculations by a factor of two. Several models with increasing complexity are compared, including measurements of the beam orbit and the relevant lattice functions (chapter 7). A novel approximative solution for the rect-elliptical beam screen of the CERN LHC, is given, showing that a rectangle, expressed in terms of closed-forms (elliptical functions), is sufficient to model the tune-shift caused by indirect space charge (section 7.4).
- The intensity for the HL-LHC project and therefore the tune-shift increases by a factor two. Accordingly, the indirect space charge driven tune-shift of the HL-LHC was calculated, providing exact solutions for the newly installed elements of an octupolar structure (section 7.5).

1.5 Structure of the Thesis

The thesis consists of four parts. The first part is the introductory part, which motivates the work. The second contains the theoretical topics providing a solid framework and introduces new concepts essential for this work. Readers who are primarily interested in the application of these studies can jump directly to chapter 5, where the results for image operators of several cross-sections are presented. The

third part addresses applications to the CERN accelerators PS and LHC. The thesis closes in the fourth part with a summary of the work and future applications will be bestowed.

Grey text boxes indent to highlight key findings and important statements throughout the theoretical chapters.

Part II
Theoretical Foundations

Chapter 2

Theory of Accelerators

Et harum scientiarum porta et clavis est Mathematica.
Mathematics is the door and key to the sciences. "
— Roger Bacon, *Opus Majus*

Abstract In this chapter, we want to motivate how the indirect fields affect the performance of accelerators. We mainly concentrate on the change of the eigenfrequency of the system, the tune, by neglecting the change of the equilibrium orbit and other parameters. This approach is justified since the change of the eigenfrequency is the most critical effect. To estimate the change of the eigenfrequency, we employ the methods of canonical perturbation theory.

Another application is addressing the multi-turn-extraction of the PS. The impact of the indirect fields is calculated within a numerical integration of the particle trajectories. These fields have a significant influence on the position of the beamlets. In this case, the effect cannot be treated solely using perturbation methods.

2.1 Setting the Scene

2.1.1 Background of CERN

The organization CERN, short for Conseil Européen de la Recherche Nuclaire, meaning European council for nuclear research, was founded after the Second World War as a peace-oriented scientific collaboration of countries of Europe. Because of its central position in Europe and the fact that several other international organizations were already located here, it was built in Geneva, Switzerland. In 1956 the first accelerator, the Synchro-Cyclotron started its operation. Three years later, the second accelerator, the CERN (P)roton (S)ynchrotron (PS) started to accelerate protons and is still running as the oldest operating accelerator of the world. After several fundamental discoveries, as the first proton-antiproton, the existence of W and Z bosons and the therewith connected installation of newly needed accelerators and experiments for this purpose (mainly in terms of energy and number of interactions), as e.g. the (S)uper (P)roton (S)ynchrotron (SPS), the missing piece, the origin of mass in the electroweak theory was finally confirmed in 2012 - the Higgs boson

was found. It was achieved with the largest existing accelerator the (L)arge (H)adron (C)ollider LHC, with a length of 27 km in its four years of operation. The facility hosts several other experiments ranging from neutrino experiments to antimatter research. For more information references [44,45] can be consulted.

An overview of the currently running accelerators and experiments is depicted in fig. 2.1. In the present work, we refer to the PS, which is currently part of the injector chain of the LHC and the LHC itself. In order to extend the potential of the LHC, an upgrade is currently ongoing, and in the 2020s, the rate of collisions of the machine will be a factor five higher than its design value [46]. Such improvement requires around ten years of effort, and the project is called (H)igh (L)uminosity LHC (HL-LHC).

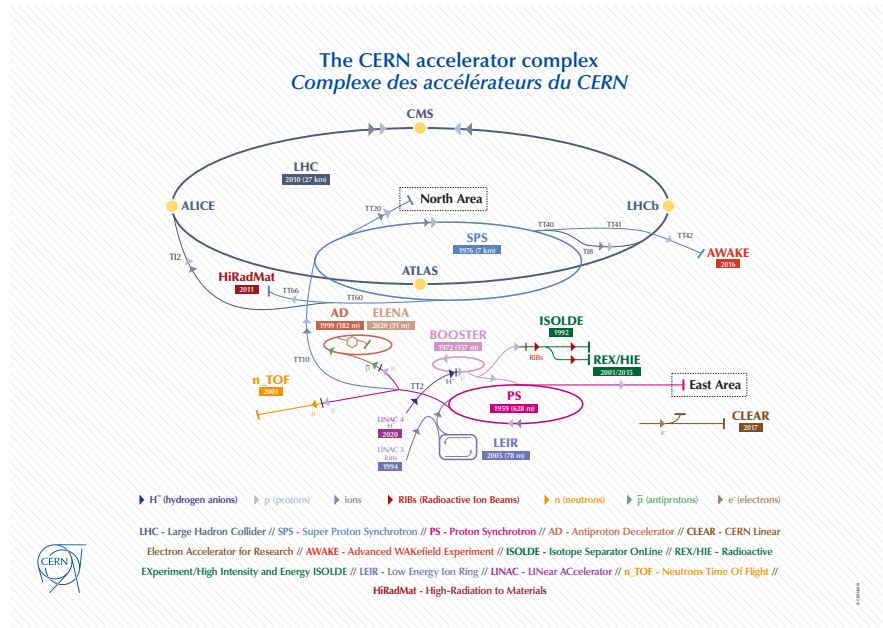


Fig. 2.1: An overview of the accelerators and some experiments currently located at CERN (2019). Taken from [47].

2.2 The Formulation of Physics of Accelerators

The formalism of physics of accelerators is mainly inherited from the theory of optics and celestial mechanics. The movement of a particle in a magnetic field is symplectic, so usually one works in the framework of Hamiltonian mechanics.

2.2.1 The Challenge

Within the high energy physics domain, charged particles are accelerated to very high energies. In a circular accelerator, the particles are forced by a magnetic field onto a circular trajectory. The magnetic field generates a spiral orbit without influencing the velocity parallel to the magnetic field. After a few turns, the particle would already leave the accelerator, and as a consequence, complicated arrangements of magnets need to be used to prevent the particle from leaving the accelerator. The question of how to set up such a system is studied within the domain of dynamical systems. In this work, the discussion of phenomena is restricted to deterministic and finite-dimensional systems, which can be described through systems of ordinary differential equations within the framework of Hamiltonian mechanics. Since one is merely interested in the trajectory of individual particles (which usually is computationally unfeasible - think of 10^6 to 10^{13} particles), the dynamics of ensembles of particles are regarded as a whole (the beam). Such a problem could be tackled with the Fokker-Planck approach [48] or the full Boltzmann equations. If the inter-collisions of the particles are neglected the equation becomes the Vlasov equation, which is classified as a Hamiltonian differential equation [49], and is heavily used in the so-called collective effects or intensity-dependent beams [50, 51].

In this work the efforts mainly concentrate on the understanding of small effects acting on the beam as a whole, which is usually referred to as single particle beam dynamics. The beam is only slightly perturbed around the equilibrium orbit. To some extent, we can take the environment around the equilibrium position as a continuum of stable orbits. However, with increasing complexity, the boundary of the stable region, which is called the dynamic aperture, has to be calculated via numerical integration around the ring for individual initial conditions, which is termed tracking.

2.3 The Circular Accelerator as a Hamiltonian System

2.3.1 Stability of Repetitive Systems: LA Theorem and Invariant KAM Tori

The theory of circular accelerators is mainly interested in the stability of an ensemble of particles over a sufficient amount of turns. The origin of the study of the stability of Hamiltonian systems is founded in celestial mechanics since we find many analogies, in the study of orbits of the planets in the solar system. Taking celestial mechanic as the entry point for our discussion we know the following two rigorous theorems are fundamental [52–54]. Cited from [55], (p. X):

- **LA-Theorem:** *If $N - 1$ (N is the number of degrees of freedom) global analytic, single-valued integrals exist, that are functionally independent and involution (the Poisson bracket of any two of them vanishes), the system is called completely*

integrable. It means the equations can be integrated by quadratures to single integral equations expressing the trajectories. These solutions generally lie on a N -dimensional tori and are either periodic or quasi-periodic functions of N incommensurate frequencies.

- **KAM theorem:** *If the Hamiltonian H can be written in the form $H = H_0 + \varepsilon H_1$ of a ε perturbation of a completely integrable Hamiltonian system H_0 , most quasi-periodic tori persist for sufficiently small ε . It means most of the near-integrable Hamiltonian systems are "globally stable" in the sense that most of their solutions around an isolated stable elliptic point or periodic orbit are "regular" or "predictable".*

In the case of an integrable system following the Hamilton-Jacobi theory [52, 53, 55] one obtains through canonical transformations a new simpler form of the Hamiltonian using angle-action variables, making use of the integrals of motion, which is nothing more than the equivalent formulation of integrability. The 3D normal form Hamiltonian of a particle can be written as:

$$H_0(I_1, I_2, I_3) = \omega_1 I_1 + \omega_2 I_2 + \omega_3 I_3, \quad (2.1)$$

where the i^{th} action-angle coordinate pair is denoted by (I_i, ϕ_i) and ω_i denotes the frequency. According to the KAM theorem the ratio of the independent oscillations has to be:

$$n_1 \omega_1 + n_2 \omega_2 + n_3 \omega_3 \neq 0, \quad n_i \in \mathbb{N} \quad (2.2)$$

in order to find stable orbits in the vicinity of the equilibrium orbit, these frequencies are called commensurate frequencies if eq. (2.2) is zero. Rewriting this yields:

$$n_1 \frac{\omega_1}{\omega_3} + n_2 \frac{\omega_2}{\omega_3} := n_1 \tilde{\omega}_1 + n_2 \tilde{\omega}_2 \neq n_3. \quad (2.3)$$

Equation (2.3) is known as tune relation. The objective is to choose eigenfrequencies of the system as far away as possible from a rational frequency. The numbers which have the largest distance to a rational number are $\frac{\sqrt{5} \pm 1}{2}$ [54, 56]. Choosing this eigenfrequency ratio should give the most stable orbits.

2.3.1.1 The Transverse Dynamics

In a first approximation, no coupling of the longitudinal and transverse plane is assumed¹, since the energy along the ring is considered as constant. In such a case the Hamiltonian can be written as:

$$H_0(I_1, I_2) = \tilde{\omega}_1 I_1 + \tilde{\omega}_2 I_2 \quad (2.4)$$

¹ It is justified since the frequency domains are clearly separated e.g. in the LHC the longitudinal frequency is in the order of 25 Hz, while the transverse oscillations are in the order of MHz and coupling fields are usually very small.

This is the starting point of the perturbation of the system, with the non-integrable part written as H_1 :

$$H_{\text{total}} = H_0(I_1, I_2) + \varepsilon H_1 \quad (2.5)$$

Since the system is for small perturbations ε near-integrable, the effect to the first order does not distort the invariants too strongly, and the main impact is the shift of the frequencies $\tilde{\omega}_i$ of the invariants. Therefore, if moving near an in-commensurable frequency shrinks the region of stability and the particles outside this region get lost. We primarily concentrate on the shift of these frequencies, also known as frequency pulling or tune-shift. If the tune-shift is large enough, it can cross a resonance condition, and the trajectory could become unstable.

2.3.2 The Invariant of Courant and Snyder

The Hamiltonian of the accelerator can be expressed in a way that the system along the position within the longitudinal coordinate s is divided into parts $\mathcal{M}_i$ at the positions s_i to s_{i+1} , according to the elements as indicated in fig. 2.2. The transport of the initial positions through the Hamiltonian flow can be interpreted as a canonical transformation itself [56], and the full Hamiltonian is build up as a sequence of consecutive canonical transformations of the individual elements $\mathcal{M}_i$. Details how to build up different elements can be found in [57, 58].

Since the design of an accelerator is much easier, we demand that there is no energy coupling in the orthogonal directions of the transverse plane. The circumference of the accelerator is C . Now, we introduce a longitudinal position-dependent focusing strength $\beta_i(s)$. The function β can be adjusted by the fields in the quadrupoles and does not depend on the beam. Therefore it is called a lattice function or an optics function obeying $\beta_i(s_0 + nC) = \beta_i(s_0)$, $n \in \mathbb{Z}$. By defining $\nu_i := \frac{1}{2\pi} \int_0^C \frac{ds}{\beta(s)}$ we can express the Hamiltonian of the transverse motion as:

$$H_0(I_1, I_2, s) = \frac{2\pi\nu_1}{C} I_1 + \frac{2\pi\nu_2}{C} I_2. \quad (2.6)$$

From the Hamiltonian equations we get the angle variables $\phi'_i = \frac{2\pi\nu_i}{C}$. The term $2I_i$ is called the Courant-Snyder invariant [59]. To understand the physical meaning of these quantities, we transform the angle-action variables into the phase space. Defining the quantity $\tilde{\phi}_i := \phi_i - \frac{2\pi s}{C} - \int_0^s \frac{ds'}{\beta_i(s')}$, we are able to express the system in the physical phase space coordinates as: $q_i(s) := \sqrt{2\beta_i(s)I_i} \cos(\tilde{\phi}_i)$ and $p_i(s) = \sqrt{\frac{2I_i}{\beta_i(s)}} (\sin(\tilde{\phi}_i) - \frac{\beta'_i}{2} \cos(\tilde{\phi}_i))$, as can be found in [60]. The function β_i for a given structure is determined via a non linear ordinary differential equation ([60], p.80). The physical meaning is depicted in fig. 2.3. A Poincaré cut at a specific position s along the accelerator shows that the cross points of a trajectory are confined onto an ellipse. The size of this ellipse is given by $2I_i$ and the equation for the ellipse

is given by $\frac{2I_i}{\beta_i} = \left(\frac{q_i}{\beta}\right)^2 + \left(p_i - \frac{\beta'_i q_i}{2}\right)^2$. The ellipse generated by the cross points is a dense curve in the non-resonant frequency ratio case (the KAM tori have to be dense), whereas in fig. 2.3 the first nine iterations are shown for the horizontal phase space. Since the size of the ellipse is constant, as it is an invariant, and its related to the size of the beam. It serves as an important parameter to determine the beam quality.

2.3.3 Methods of Perturbing the Linear System

As stated by the KAM theorem, the behaviour of the stability of the perturbed system for the correct ratio of the frequency is nearly integrable. There are several approaches to incorporate the effect of perturbations. We concentrate on the canonical perturbation theory, which allows us to estimate the change of the eigenfrequency - the tune - of systems in an analytical way.

Another approach to study the dynamics of more involved systems is based on the Poincaré sections or sections of the surface. Only a subspace of the system is analysed, keeping some coordinates constant. In this way, the dimensionality of the

Fig. 2.2 The Hamiltonian of the full ring is separated into smaller meaningful parts $\mathcal{M}_i$ as e.g. magnets and straight sections.

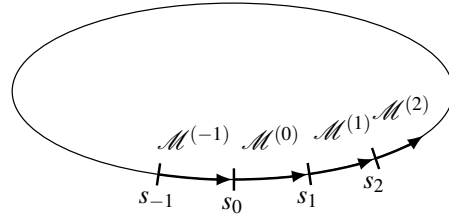
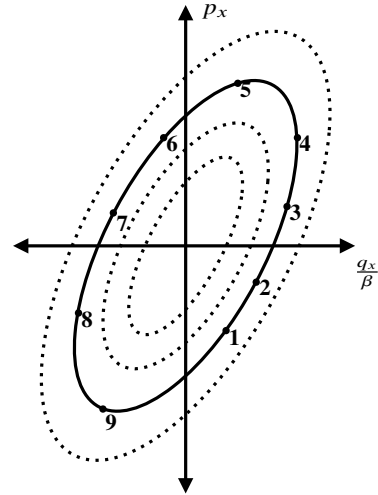


Fig. 2.3 A Poincaré cut of the phase space at a specific position along the accelerator at the repetition $i = 1, \dots, 9$.



system is reduced, which renders the problem usually simpler. Due to the symplectic nature of the Hamiltonian system many properties of the system are encoded in the Poincaré map (as a consequence of the Poincaré-Cartan theorem the map itself is symplectic [52, 61]), mapping the consecutive points following a trajectory which crosses the plane repetitively. The obtained dynamical system is discrete, and stability investigations are done by studying fixed points (invariant points of the Poincaré map). The Poincaré maps can be developed in series around these fixed points, and transformed order by order to normal forms, which are the invariants of motion of the truncated system. The technique of Birkhoff normal forms was used to track particles for the multi-turn extraction in the PS [16]. Details can be found in [62–64]. The focus here is not to provide an exhaustive overview of all methods used to design beam-lines but the idea is to preserve the symplecticity of the system. This leads to involved techniques like truncated Lie transformations using the Cremona map [65–67], exact explicit representations in the form of a power series with a finite number of terms [68, 69] and mixed-variable generating functions [70–73].

2.3.4 Canonical Perturbations through Indirect Fields

The main goal here is to estimate the influence of the electromagnetic interaction of the charged particles with the elements of the environment as, e.g. the vacuum chamber. Due to the elongated structure of the accelerator, the effect is approximated as a 2-D problem [74, 75].

The size of the beam is usually relatively small compared to the size of the surrounding elements as, e.g. the vacuum chamber. Consequently, it is sufficient to investigate the effect at the centre of charge of the beam as the cause. As shown in the next chapter the effect is divided into two different contributions: the interaction due to the electric field, which leads to Dirichlet boundary conditions and the interaction due to magnetic effects, leading to Neumann boundary conditions. Figure 2.4 illustrates schematically how the 3D problem is reduced to a 2D problem. It is possible due to the elongated structure of the elements. Electric effects, which do not penetrate the vacuum chamber of an accelerator interact at the blue boundary. Magnetic effects, which penetrate the vacuum chamber at low frequencies, interact at the red boundary. Details are given in chapter 5.

The Hamiltonian of the system H_0 is perturbed by the indirect fields, which depend on the longitudinal position s . In the following chapter, we see that H_1 can be derived from a potential V . The Hamiltonian can be formulated as:

$$H_{\text{total}} = \frac{2\pi\nu_1}{C}I_1 + \frac{2\pi\nu_2}{C}I_2 + V(I_1, I_2, \phi_1, \phi_2; s). \quad (2.7)$$

The potential is developed around the equilibrium position into a Taylor series and truncated at the second order $V \approx \tilde{V}$, which gives the approximation of the Hamiltonian:

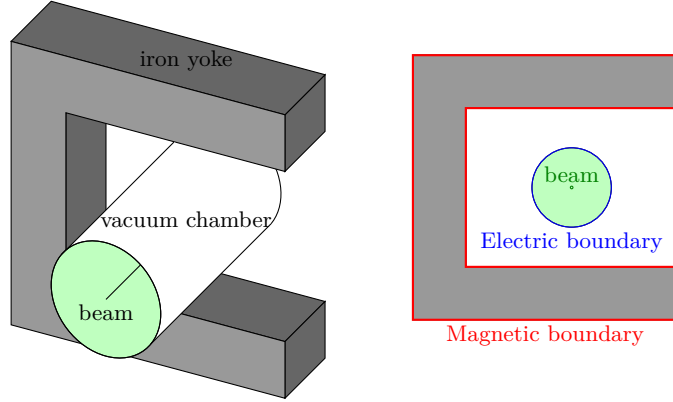


Fig. 2.4: Typical components of a circular accelerator in the 3D view (right) and the projection onto the plane (left).

$$\tilde{H}_{\text{total}} = \frac{2\pi\nu_1}{C}I_1 + \frac{2\pi\nu_2}{C}I_2 + \tilde{V}(I_1, I_2, \phi_1, \phi_2; s). \quad (2.8)$$

The approximated $\tilde{V}$ only changes the equilibrium orbit slightly and shifts the tune. The focus here is not to estimate the change of the equilibrium orbit since the effect is not critical in the current treatments except for the position shift in the PS during the multi-turn-extraction, which is discussed in detail in chapter 6. The strength of the potential V depends on the current of the beam. The so caused tune-shift is the most critical aspect causing beam losses. It also introduces coupling of the two planes. The coupling can lead to instabilities and tune-shifts at second-order perturbations, which will not be further treated in this thesis. Nevertheless, the full coupling terms are formulated.

Only one aspect will be mentioned here and that is: the coupling term which leads to the so-called difference resonances $\nu_1 - \nu_2 = n$, $n \in \mathbb{N}$, and the sum resonance $\nu_1 + \nu_2 = n$, $n \in \mathbb{N}$. The first does not cause instability but might lead to an increase of the oscillation amplitude, while the latter does and should be avoided in the case of strong coupling (details can be found in [71, 76, 77]).

In the case of an on-axis beam, where no coupling occurs due to symmetry reasons the tune-shift is derived using the canonical perturbation theory. The horizontal/vertical quadrupolar component of the potential is denoted by $\tilde{v}_{x1}(s)/\tilde{v}_{y1}(s)$, yielding a horizontal tune-shift of:

$$\Delta\nu_x = -\frac{1}{4\pi} \int_s^{s+C} \beta_x(s) \tilde{v}_{x1}(s). \quad (2.9)$$

Since the potential is harmonic ($\tilde{v}_{x1}(s) = -\tilde{v}_{y1}(s)$), the vertical tune-shift is:

$$\Delta\nu_y = \frac{1}{4\pi} \int_s^{s+C} \beta_y(s) \tilde{v}_{x1}(s). \quad (2.10)$$

$\tilde{v}(s)$ is intensity depended and in chapter 5 the potential in terms of Newtons equations is formulated, which have a more intuitive character.

2.4 Summary

The stability of orbits in an accelerator was discussed, and methods to estimate the impact of perturbations were shown. Surrounding elements can interact electromagnetically with the beam and cause unstable orbits, which can lead to beam losses. The final strength of the effect depends on the current of the beam - the intensity.

In chapter 3, it is shown how to calculate these fields for elongated structures in general and in section 5.2.1, explicit closed-forms formulas are given to calculate the intensity-dependent indirect tune-shifts for a set of new elements (expressed through the image operators as introduced in chapter 5).

Apart from estimating the change of the tune in the LHC (chapter 7) and during the MTE of the PS (section 6.4.3), numerical tracking simulations are performed, since a first-order perturbation of the transverse beam position in this case, due to the large deviations, is needed (section 6.4.4).

Chapter 3

Potential Theory on the Riemann-sphere

"Das also ist des Pudels Kern"

— Goethe, *Faust I*

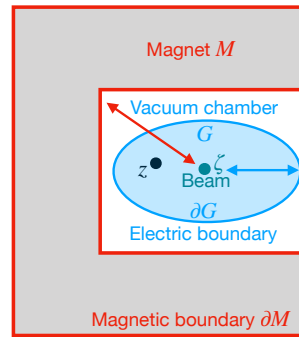
Abstract In this chapter, the abstract theoretical framework is introduced. It is indispensable to present the abstract theoretical framework in some detail in this chapter since it is the basis for all consecutive results of this thesis. The theoretical framework is treated in such a way that several novel insights will be revealed.

The mathematical concepts, used to derive the 2D electro and magnetostatic fields, are also developed in this chapter. Instead of working in the $x - y$ plane this thesis works on the Riemann-sphere which unveils some helpful properties to obtain new insights and closed-form solutions. The results are expressed in terms of the fundamental solution of the Poisson equation - the complex Green function.

3.1 The Fundamental Importance of the Logarithmic Singularity

3.1.1 A Novel Approach

Fig. 3.1 The schematic interaction of the elements and the beam. The vacuum chamber G generates the electric response. Its boundary ∂G is drawn in blue. The magnetic response is originated in the domain M , which indicates a magnet. Its boundary ∂M is drawn in red.



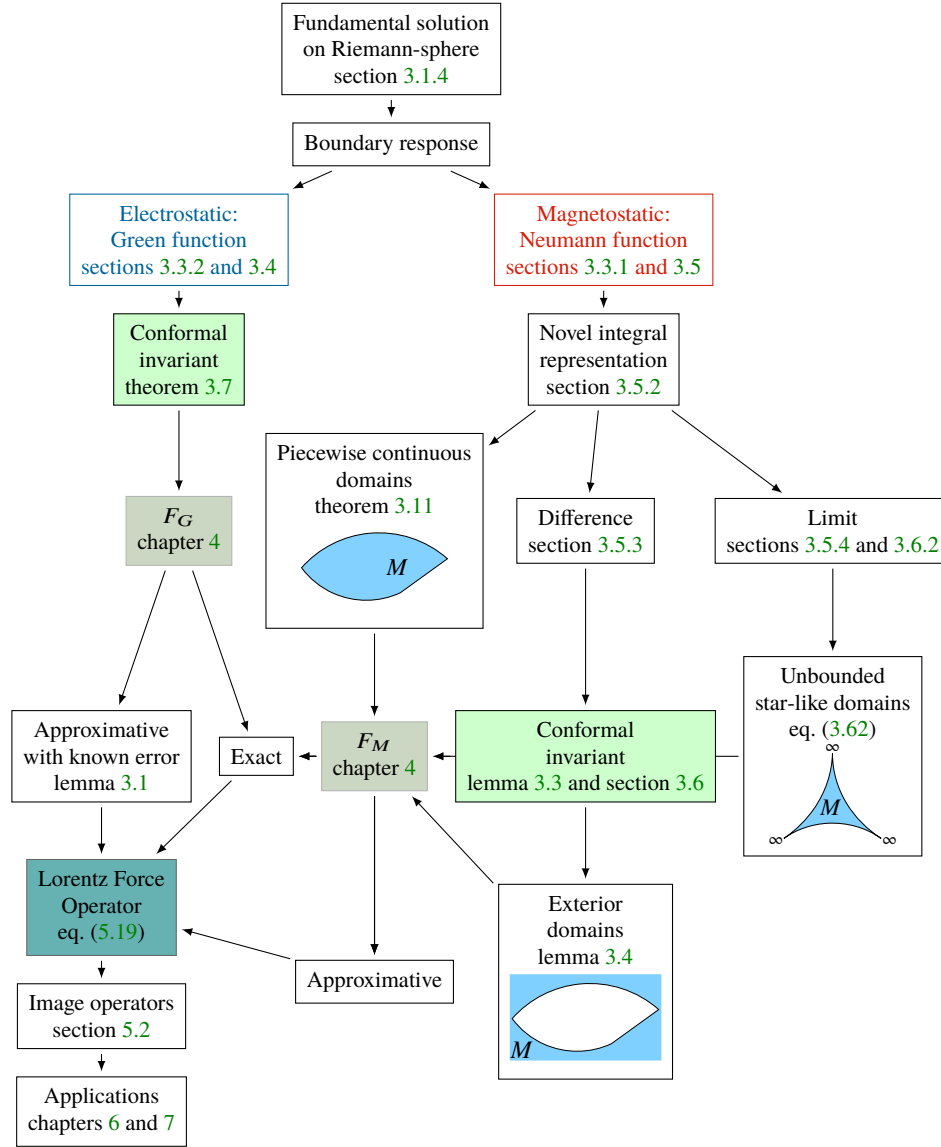


Fig. 3.2: The road map of the mathematics as used in this work.

A new, more extended framework is needed to overcome the limitations of the existing approaches as introduced in chapter 1. The goal is to find a mathematical framework to cover the phenomena observed in circular accelerator structures when the particle beam is interacting with its surrounding elements like the vacuum chamber or then magnets. As discussed, it is possible due to the elongated construction to

approximate the problem in a planar way. The plane is the so-called transverse plane, and the direction of the particle beam moving through the structures is perpendicular to it.

In fig. 3.1, the transverse plane is schematised. The problem naturally separates into the electric interaction of the particle beam with elements like the vacuum chamber, domain G , which is drawn in blue. G is bounded by ∂G . Depending on the physical situation, as described in section 5.1.3, the particle beam might also interact magnetically with magnetic elements of the accelerator, denoted as domain M , which is illustrated in red. The boundary of M is ∂M . Figure 3.1 also shows the beam located at ζ and the point of observation z .

We model the situation following the strategy that the problem is reduced to the most fundamental formulation. The response of the system to an elementary excitation of the system is calculated. This approach is of great generality and allows afterwards to compute arbitrary boundary problems by superposition. The result is the Green function or fundamental solution and the elementary excitation in the plane is the fundamental singularity (definition 3.1). The fundamental solution is the sum of the fundamental singularity and a response boundary function (eq. (3.14)).

An overview of the mathematics is provided in fig. 3.2. Starting from the fundamental solution, which differs by the response boundary function, its symmetries are investigated. For several practical reasons, we work in the framework of complex variables on the Riemann-sphere, which is discussed later on in this chapter. The strategy allows for finding new general solutions on an abstract level. The most important symmetry is conformal invariance. Conformal invariance is the invariance of characteristics under a conformal map (described section 3.2.2). As a well-established method, we diminish the technicalities of the problem using invariances and work on an abstract level, which reveals new insights. The technicalities are transferred to find the conformal mapping of a domain onto another, simpler domain, where the problem is understood and solved exactly. Then the original problem is solved, and if the mapping is expressed through closed-forms, the solution is exact. By the Riemann mapping theorem (theorem 3.1) the existence of such a mapping for simply connected domains is guaranteed. Chapter 4 is dedicated to discuss exact and approximative ways to find conformal mappings. It includes also entirely new mappings.

Figure 3.2 show that the electrostatic boundary response, formulated via the Green function (sections 3.3.2 and 3.4), is much simpler than the magnetostatic response, formulated via the Neumann function (sections 3.3.1 and 3.5). The reason is that the Green function is a conformal invariant (theorem 3.7) while the Neumann function is not. Nevertheless, it was possible to uncover a classification of the Neumann function, using an afresh derived integral representation (theorem 3.11), namely into piecewise continuous, exterior (lemma 3.4) and in the limiting case unbounded star-like domains sections 3.5.4 and 3.6.2. These will allow for the expression of the solution for several new standard geometries in accelerators, yielding exact or approximative solutions. Explicit solutions for the Neumann function on the Riemann-sphere for unbounded, and exterior domains are provided in section 3.6. These insights are used to model the combined function magnets (section 6.3.2).

As mentioned, if for the electric response a mapping can be given in a closed-form onto an exactly solved domain, the solution is exact. In most of the cases it is not possible to find an exact mapping. For this situation is another novel method proved. The Green function has the property that it grows with the size of the domain, which allows for the proof: the exact solution of a problem lies between two approximate solutions, one from an enclosed domain and the other from an enclosing domain (lemma 3.1), which is shown in the lower left part in fig. 3.2. Hence, the approximation error can be derived, which permits one to evaluate the validity of the model. This is used when approximating the LHC vacuum chamber in section 7.2.1.

Ultimately, the force created by the elementary excitation of the arrangement is condensed into a new Lorentz operator (eq. (5.19)). The Lorentz force operator is an operator of the Green and the Neumann function and, as these are formulated itself as operators of underlying conformal mappings, it acts as an operator of these mappings. Now it is possible to estimate the impact onto some critical parameters of the global accelerator system. The linearisation of the operator leads to the image operators (section 5.2) and a catalogue of closed-form solutions is provided in section 5.3.

For completeness a classical and common used method is mentioned here: the methods of images [74, 78, 79]. Its application is discussed in appendix D.1, the problem of the two infinite parallel plates, the strip. It leads to intrinsic convergence problems. The formulation yields a non-absolute convergent series. No clear convergence behaviour can be deduced. In fact, an arbitrary real number can be obtained by re-ordering the series terms. In standard treatments, this issue is not discussed and consequently here it is tackled explicitly. A novel solution to the problem is provided, arguing that there is a physical origin of the field in non-perfect materials, and this allows us to obtain convergence as a well-defined limiting case of perfect materials. Furthermore, the problem is extended to the rectangular case, where an upper bound for the convergence is found. In general, the method of images is inherently complicated and therefore not used further in this work. Ongoing research of this topic and the link to infinite product representations in mathematical physics can be found in [24].

3.1.2 Prelude: The Theorem of Green-Riemann

We begin with one of the most important theorems in terms of physical meaning in mathematics, namely the theorem of Green-Riemann [23, 80, 81]. In this context, we want to emphasize the physical meaning, but different presentations can be found [25, 28, 34]. We introduce some mathematical notations used in this work. The Riemann-sphere is denoted by $\bar{\mathbb{C}}$. Capital letter functions are complex-valued, and non-capital functions are real-valued with a complex-valued argument. A bar above a symbol or an asterisk as upper-script indicate its complex conjugation.

The Cauchy-Riemann operators are now given in the notation of Poincaré. If required, a subscript indicates the relevant variable; otherwise, it is clear out of context. The

operators are defined as (∂_x is the partial derivate w.r.t. x):

$$\partial := \frac{1}{2}(\partial_x - i\partial_y) \quad \bar{\partial} := \frac{1}{2}(\partial_x + i\partial_y) \quad (3.1)$$

assuming $M \subseteq \bar{\mathbb{C}}$ is a 1-dimensional connected complex manifold with ∂M as a Jordan contour (section 3.2.1) and F be a complex valued function of a complex argument. Here the theorem of Green-Riemann is written in its complex version ($\wedge$ is the wedge product see, e.g. [23] p. 5):

$$\int_{\partial M} F dz = \int_M \bar{\partial} F d\bar{z} \wedge dz. \quad (3.2)$$

If F satisfies $\bar{\partial} F = 0$, then the fundamental theorem of complex analysis, also named the Cauchy theorem follows and the function F is called holomorphic. Let ω denote the differential form $\omega := Fdz$ and $\bar{\partial} F = 0$, then one can write:

$$\int_{\partial M} \omega = 0. \quad (3.3)$$

This means ω is exact, hence F has a complex potential function G with $G' = F$. If we split the right side of eq. (3.2) into its real and imaginary parts, it can be shown that (appendix A):

$$\operatorname{Re} \left\{ \int_{\partial M} F dz \right\} \rightarrow \text{circulation of } F^*, \quad (3.4)$$

$$\operatorname{Im} \left\{ \int_{\partial M} F dz \right\} \rightarrow \text{flow of } F^*. \quad (3.5)$$

Holomorphic functions are exactly the vector fields, which are circulation and source free. By taking the real and imaginary parts of eq. (3.2) it can be shown that (appendix A):

$$\operatorname{Re} \{ \text{eq. (3.2)} \} \iff \text{classical Stokes theorem - Ampère's law}, \quad (3.6)$$

$$\operatorname{Im} \{ \text{eq. (3.2)} \} \iff \text{classical Gauss's theorem - Gauss's law}. \quad (3.7)$$

3.1.3 The Logarithmic Singularity

Now, the function G_ζ (the subscript marks a parameter dependence on source location ζ) with $\partial G_\zeta = F_\zeta$ is introduced. It has the form:

$$G_\zeta := \frac{1}{2\pi} \log(z - \zeta), \quad (3.8)$$

where the main branch of the logarithm is taken to determine the function uniquely.² Letting the operator $\bar{\partial}$ act on F_ζ , leads to the fundamental result ($\delta_\zeta := \delta(|\zeta - z|)$ is the complex version of the Dirac-delta-distribution):

$$\bar{\partial}F_\zeta = \bar{\partial}\partial G_\zeta = \frac{1}{2\pi}\bar{\partial}\frac{1}{z-\zeta} = \frac{\delta_\zeta}{2} \quad (3.9)$$

For a proof of $\bar{\partial}\frac{1}{\pi(z-\zeta)} = \delta_\zeta$ We refer to [81], p.119. Employing now the right hand side of eq. (3.2), where $\zeta \in M$, yields:

$$\int_M \bar{\partial}F_\zeta \, d\bar{z} \wedge dz = \int_M \bar{\partial}\partial G_\zeta \, d\bar{z} \wedge dz = i \int_M \delta_\zeta \, dx dy = i. \quad (3.10)$$

On the other hand, from the left side of eq. (3.1), one sees:

$$\int_{\partial M} F_\zeta \, dz = \frac{1}{2\pi} \int_{\partial M} \frac{1}{z-\zeta} \, dz = i. \quad (3.11)$$

Taking the imaginary part shows that G_ζ generates a pure source field (eq. (3.5)). If the real and imaginary part of eq. (3.8) are separated, it can be seen that half of the field is generated by the real and half is generated by the imaginary part of the logarithmic singularity:

$$F_\zeta = \partial G_\zeta = \frac{1}{2\pi} \partial \left(\frac{1}{2} \log(z-\zeta)(z^*-\zeta^*) + i \arg(z-\zeta) \right) \\ = \frac{1}{2\pi} \left(\underbrace{\frac{1}{2} \frac{1}{z-\zeta}}_{\text{half source}} + \underbrace{\frac{i}{2} \frac{1}{i(z-\zeta)}}_{\text{half source}} \right) = \frac{1}{2\pi} \frac{1}{z-\zeta}. \quad (3.12)$$

Providing either the real or imaginary part is sufficient to generate a field and with this motivation we introduce:

Definition 3.1. The quantity $\frac{1}{4\pi} \log zz^*$ is the fundamental solution Γ of the Laplacian:

$$\Gamma(z) = \frac{1}{4\pi} \log zz^* = \frac{1}{2\pi} \log |z|. \quad (3.13)$$

The fields can be generated using eq. (3.13):

- $\bar{\partial}\Gamma(z) = \frac{1}{4\pi} \frac{1}{z^*} \xrightarrow{\text{eq. (3.5)}} \text{pure source field,}$
- $i\bar{\partial}\Gamma(z) = \frac{i}{4\pi} \frac{1}{z^*} \xrightarrow{\text{eq. (3.4)}} \text{pure circulation field.}$

The generated fields are depicted in figs. 3.3a and 3.3b. The equipotential lines of $\Gamma(z)$ (in green) are perpendicular to the field lines $F = \bar{\partial}\Gamma(z)$ (in blue) in the case

² Although the disambiguity cancels with the differentiation in eq. (3.9), hence it has no physical relevance.

of a source field, which is the case for electrostatic fields (fig. 3.3a). While for a circulation field the field lines are tangential to the equipotential lines, which is the case for magnetostatic fields $F = i\bar{\partial}\Gamma(z)$ (in blue) (fig. 3.3b). This can be summarized as: The logarithmic singularity is the generator of the physical fields.

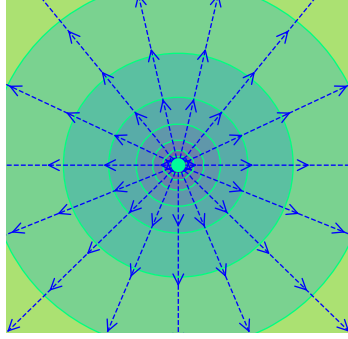


Fig. 3.3a The equi-potential lines of $\Gamma(z)$ (green) and the field direction $F = \bar{\partial}\Gamma(z)$ (blue).

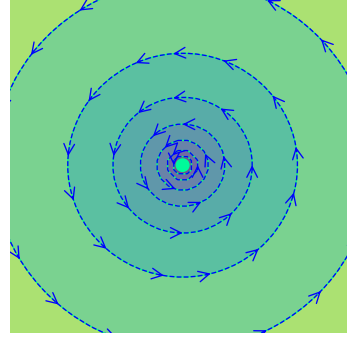


Fig. 3.3b The equi-potential lines of $\Gamma(z)$ (green) and the field direction $F = i\bar{\partial}\Gamma(z)$ (blue).

3.1.4 The Complex Green Function

The field is generated by the complex logarithmic singularity. To take the interaction with the environment into account (illustrated in fig. 3.1), we add a holomorphic function $\text{Rbf}(z, \zeta)$ in M for z and ζ (source and circulation free in M), the response boundary function, to G_ζ :

$$G_\zeta = \underbrace{\frac{1}{2\pi} \log(z - \zeta)}_{\text{elementary excitation}} + \underbrace{\text{Rbf}(z, \zeta)}_{\substack{\text{environmental} \\ \text{response to field}}} . \quad (3.14)$$

This is the general form of the complex Green function. Now we discuss some important properties of it.

3.1.4.1 Conformal Invariance

A fundamental property of the logarithmic singularity is its conformal invariance. A mapping φ is called conformal if it is meromorphic and $\varphi(z)' \neq 0$, $z \in M$ (details are given in section 3.2.2). As a consequence we can develop φ into a Taylor series around a point ζ as $\varphi = \varphi(\zeta) + (z - \zeta)\psi(z)$ with $\psi(z) \neq 0$, $z \in M$ and $\psi(z)$ holomorphic.

Proof. Transforming $\tilde{G}(z, \zeta) := G(\varphi(z), \varphi(\zeta))$ shows:

$$\begin{aligned}\tilde{G}_\zeta &= \frac{1}{2\pi} \log(\varphi(z) - \varphi(\zeta)) + \text{Rbf}(\varphi(z), \varphi(\zeta)) \\ &= \frac{1}{2\pi} \log(\varphi(\zeta) + (z - \zeta)\psi(z) - \varphi(\zeta)) + \text{Rbf}(\varphi(z), \varphi(\zeta)) \\ &= \frac{1}{2\pi} \log(z - \zeta) + \tilde{\text{Rbf}}(z, \zeta),\end{aligned}\tag{3.15}$$

with the holomorphic function $\tilde{\text{Rbf}}$. □

This can be formulated as: The logarithmic singularity preserves its physical properties under conformal mappings.

A remark: G_{z_0} on $\bar{\mathbb{C}}$ ("free space") reveals two singularities, one at ζ and the other with the opposite sign at $z = \infty$, the north-pole. The first we call source singularity and the second, sink singularity which is shown in the right plot of fig. 3.10.

3.1.5 The Connection to Harmonic Functions

The following can be observed: $\text{Rbf} := \phi + i\psi$ was defined in the previous paragraph and since Rbf , and consequently ∂Rbf , are holomorphic it holds that $\bar{\partial} \partial \text{Rbf} = 0$. In the classical notation this can be written as:

$$\bar{\partial} \partial \text{Rbf} = \frac{1}{4} (\partial_x + i\partial_y)(\partial_x - i\partial_y)(\phi + i\psi) = \frac{1}{4} (\partial_x^2 + \partial_y^2)(\phi + i\psi) = 0. \tag{3.16}$$

The real ϕ and the imaginary ψ part of the response boundary function are harmonic. In the following, $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial n}$ denote the mathematical positively oriented tangential and outwards pointing normal vector of ∂M , respectively. Note that ϕ and ψ are not in-dependent, which follows from eq. (3.3):

$$\begin{aligned}0 &= 2 \int_{\partial M} \omega = 2 \int_{\partial M} \partial \text{Rbf} \, dz \\ &= \int_{\partial M} (\partial_x - i\partial_y)(\phi + i\psi)(dx + i dy) \\ &= \int_{\partial M} (\partial_x \phi dx + \partial_y \phi dy) + (\partial_y \psi dx - \partial_x \psi dy)\end{aligned}$$

$$\begin{aligned}
& + i \int_{\partial M} (-\partial_y \phi dx + \partial_x \phi dy) + (\partial_x \psi dx + \partial_y \psi dy) \\
& = \int_{\partial M} |dz| \left(\frac{\partial \phi}{\partial s} - \frac{\partial \psi}{\partial n} \right) + i \int_{\partial M} |dz| \left(\frac{\partial \phi}{\partial n} + \frac{\partial \psi}{\partial s} \right).
\end{aligned}$$

Since the real and the imaginary parts are zero, it follows:

$$\frac{\partial \phi}{\partial s} = \frac{\partial \psi}{\partial n} \qquad \frac{\partial \phi}{\partial n} = -\frac{\partial \psi}{\partial s}. \quad (3.17)$$

These equations are called Cauchy-Riemann equations. We see that they are not completely symmetric. Another property is implied by these equations: the equipotential lines of the real and the imaginary part are orthogonal. For example if $\phi = \text{const.}$ this implies $\frac{\partial \phi}{\partial s} = 0 = \frac{\partial \psi}{\partial n}$ along the level curve ϕ . As another consequence we see:

$$dRbf = \frac{\partial \phi}{\partial s} + i \frac{\partial \psi}{\partial s} \stackrel{\text{eq. (3.17)}}{=} \frac{\partial \phi}{\partial s} - i \frac{\partial \phi}{\partial n} \stackrel{\text{eq. (3.17)}}{=} \frac{\partial \psi}{\partial n} + i \frac{\partial \psi}{\partial s} \quad (3.18)$$

Rbf is fully (up to a physically irrelevant constant) determined by its real or imaginary part within a simply-connected domain. If we prescribe the Rbf along a closed path $\gamma \subset \bar{M}$ it is sufficient to define the real values along γ to fix Rbf. The real or imaginary part of the complex Green function eq. (3.14) fully determines the physics. We note this by ($\text{rbf} := \text{Re} \{ \text{Rbf} \}$):

$$g(z, z_0) = \text{Re } G = \Gamma(z - \zeta) + \text{rbf}(z, \zeta). \quad (3.19)$$

As shown in eq. (3.15), the logarithmic singularity is invariant under conformal mappings. To find solutions for different problems, we extensively use this invariance property. The focus is the search for such suitable conformal mappings, which transform a domain M into a canonical domain, where the solution of the problem is known.

In general, domains are treated, where the mapping cannot be extended analytically to the boundary, since the boundary contains singularities. In this case, the solution, of course, is not holomorphic along the boundary, but still in the domain enclosed by the boundary. For large classes, namely Jordan domains, we can extend the function continuously and one to one to the boundary. In some cases, the domain is not a Jordan domain, and the consequences of this boundary behaviour are the subject of the following section.

3.2 Topology, Conformal Mapping and Boundary Behaviour

3.2.1 Topology and the Riemann-sphere

We now set up the topological landscape. The plane is equipped with the standard topology induced by the neighbourhoods $a \in \mathbb{C}$ and $r > 0$:

$$\begin{aligned}\mathbb{D}_r(a) &:= \{z \in \mathbb{C} : |z - a| < r\}, \\ \bar{\mathbb{D}}_r(a) &:= \{z \in \mathbb{C} : |z - a| \leq r\}, \\ \partial\mathbb{D}_r(a) &:= \{z \in \mathbb{C} : |z - a| = r\}.\end{aligned}\tag{3.20}$$

$\mathbb{D}_r(a)$, $\bar{\mathbb{D}}_r(a)$ and $\partial\mathbb{D}_r(a)$ are an open and closed disc centred at a with radius r and its boundary respectively. If $a = 0$ and $r = 1$ we write $\mathbb{D}$ and $\bar{\mathbb{D}}$ for these discs and $\mathbb{T}$ for the unit circle $\partial\mathbb{D}$.

We add the point ∞ to the complex plane $\mathbb{C}$. This yields the so-called Riemann-sphere³ $\bar{\mathbb{C}} := \mathbb{C} \cup \infty$. This is a very comfortable environment to work in since $\bar{\mathbb{C}}$ is compact and the procedure of adding the point ∞ to $\mathbb{C}$ is called one-point-compactification. $\bar{\mathbb{C}}$ is a one dimensional complex manifold with two charts (define discs on ∞).

A domain M symbolizes a proper connected open subset of $\bar{\mathbb{C}}$, ∂M the topological boundary of M and the closure of $\bar{M} := M \cup \partial M$. We only address simply-connected submanifolds of $\bar{\mathbb{C}}$, whereas they can be topologically characterized as follows:

A submanifold M is simply-connected if its complement $\bar{\mathbb{C}} \setminus M$ is connected.

We distinguish now three classes:

- $\infty \notin \bar{M}$, which is the standard case of a simply-connected domain on $\mathbb{C}$,
- $\infty \in M$, which is conformal equivalent to the former class on $\bar{\mathbb{C}}$ exhibiting new properties and is called exterior domain,
- $\infty \in \partial M$ is called unbounded domain.

The different classes are important to characterize the solutions of the magnetostatic problems, which are discussed in the dedicated section 3.6.

3.2.2 Conformal Mapping

We now introduce one of the most important theorems of analysis the fundamental theorem of conformal mapping. Let us interpret a complex valued function $F(x, y) = u(x, y) + iv(x, y)$ on the plane $\mathbb{R}^2$ as $(x, y) \mapsto (u, v)$. In the following $\mathcal{J}(z_0)$ denotes the Jacobi matrix. Using the Cauchy-Riemann operators eq. (3.1) and demanding $\partial F \neq 0$, the total differential dF at a point z_0 of a holomorphic function F ($\bar{\partial}F = 0$) can be written as:

³ Riemann has shown that there exists a homeomorphism of a sphere onto $\bar{\mathbb{C}}$, this is the origin of the name.

$$\begin{aligned}
dF(z_0) &= \partial F dz = |\partial F| \frac{\partial F}{|\partial F|} dz = |\partial F| e^{i \arg \partial F} dz = \alpha e^{i\varphi} dz, \\
\Rightarrow \vec{dF}(z_0) &= \mathcal{J}(z_0) \cdot \begin{pmatrix} dx \\ dy \end{pmatrix} = \alpha R_\varphi \cdot \begin{pmatrix} dx \\ dy \end{pmatrix}, \\
\det |\mathcal{J}(z_0)| &= \alpha \det |R_\varphi| = \alpha.
\end{aligned}$$

$\alpha > 0$ and R_φ is a rotation matrix with $\varphi = \arg \partial F$. The total differential of F is composition of a rotation of the angle φ with a scaling α . It means that the infinitesimally transformation dF preserves angles and the orientation up to a rotation φ . Such a function is called conformal at z_0 . Let $F : M \rightarrow \mathbb{C}$ be an $\mathbb{R}$ -differentiable function in an open set $M \subset \mathbb{C}$, then F is conformal if and only if it is holomorphic and $F' \neq 0$ [82].

Definition 3.2. A F maps the domain $M_1 \subset \bar{\mathbb{C}}$ conformally onto $M_2 \subset \bar{\mathbb{C}}$ if

1. F is meromorphic in M_1 ;
2. F injective (one-to-one), that is $z, z' \in M_1, z \neq z' \Rightarrow F(z) \neq F(z')$;
3. $F(M_1) = M_2$.

The principle of conformal maps is illustrated in fig. 3.4. The curves $z(t)$ and $\zeta(t)$ have in z_0 the unit tangential vectors $\frac{\dot{z}(t)}{|\dot{z}(t)|}$ and $\frac{\dot{\zeta}(t)}{|\dot{\zeta}(t)|}$, respectively. The function F transforms the curves to $F(z(t))$ and $F(\zeta(t))$ and their unit tangential vectors at $F(z_0)$ onto $F'(z_0) \frac{\dot{z}(t)}{|\dot{z}(t)|}$ and $F'(z_0) \frac{\dot{\zeta}(t)}{|\dot{\zeta}(t)|}$, respectively. This is a multiplication by a complex number $\neq 0$, a simple rescaling and rotation. In the plot the transformed tangential vectors were normalized and the angle is chosen to be $\frac{\pi}{2}$ or a multiplication by i . We conclude the $\text{Re}\{F\} = \text{const.}$ and $\text{Im}\{F\} = \text{const.}$ are orthogonal, since x and iy are orthogonal. Note that the inverse function $F(z) = w$, $dF(z) = \alpha e^{i \arg \partial F} dz = dw$

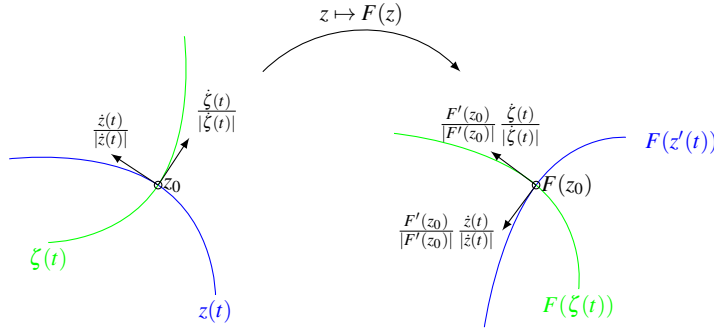


Fig. 3.4: The principle of a conformal map.

implies $dz(w) = \frac{1}{\alpha} e^{-i \arg \partial F} dw$ is also conformal. In addition, we see, if the Jacobi determinant $\det |\mathcal{J}(z_0)| = \alpha > 0$ does not vanish in for $z \in M$, the mapping is one-to-one and onto $U := F(M)$. U is again a domain (proof see [83]). So the function

F transplanted M onto U . These properties are essential for this work.

Now the Fundamental theorem of conformal mapping can be stated as:

Theorem 3.1 (Riemann mapping theorem - RMT). *Let $M \subsetneq \mathbb{C}$ be a simply-connected domain. There exists a conformal map of M onto the unit disc $\mathbb{D}$.*

Since the RMT theorem is regarded as the most important theorem of the 19th century by many mathematicians we provide at this occasion some historical remarks. Historically this theorem was given by Riemann in his doctoral thesis of 1851 [84], although his proof, which relies on the existence of the solution of the Dirichlet problem, was not completely right. This issue was solved by Hilbert (1905) 50 years later still demanding some restrictions on the boundary of the domain (details see [85] or [86]). Carathéodory in 1912 was the first who gave a complete proof for the theorem. Different modern versions of a proof can be found in [41], [83] or [37]. The RMT states that every proper sub-domain of $\mathbb{C}$ is conformal equivalent (there exists a conformal mapping onto to the unit disc). Conformal equivalence implies topological equivalence for proper sub-domains of $\mathbb{C}$ ([82], p. 282).

On $\bar{\mathbb{C}}$ we can reformulate the RMT:

Theorem 3.2 (Uniformization theorem). *Any simply-connected domain M on $\bar{\mathbb{C}}$ is conformally equivalent either to $\bar{\mathbb{C}}$, $\mathbb{C}$ or to $\mathbb{D}$*

The proof can be found in e.g. [82] or [28].

Some additional comments: The automorphisms of $\bar{\mathbb{C}}$ are the linear rational functions so-called Moebius transformations. Three complex parameters uniquely define these (as discussed in section 4.1.1). They allow sending specific points on $\bar{\mathbb{C}}$ to ∞ . Hence by a Moebius transformation we always can conformally transform an exterior simply-connected domain M ($\infty \in M$) onto a domain $\infty \notin M$ by putting a point $a \in \bar{\mathbb{C}} \setminus M$ onto ∞ . simply-connected domains $M \subsetneq \mathbb{C}$ on $\bar{\mathbb{C}}$, either interior ($\infty \notin M$) or exterior ($\infty \in M$), are conformally equivalent.

Consequently, it is sufficient to study the properties of M , which are conformal invariant on $\mathbb{D}$ and, the properties of conformal mappings. The most interesting properties are properties, which reflect the symmetries of a problem (its invariances). One conformal invariant we already saw in the previous chapter, namely the fundamental singularity. Also, holomorphicity, as well as complex integration, are conformal invariants. How this behaviour extends to the boundary is the subject of the following section.

3.2.3 Boundary Behaviour

So far, we only discussed the properties of the functions on M . Here we take a closer look at the behaviour when we approach the boundary of M , ∂M . A large class of domains, the Jordan domains show a relatively generous behaviour, and we now define the term Jordan domain by starting with the definition of a Jordan curve on $\bar{\mathbb{C}}$:

Definition 3.3. A curve K is a mapping $t \mapsto z(t)$, $a \leq t \leq b$, $z \in \bar{\mathbb{C}}$.

K is called a Jordan curve J if $t \mapsto z(t)$ is a homeomorphism between $\mathbb{T}$ and $[a, b]$,

as it can be found in [87]. The following theorem is of fundamental importance, since it allows us to classify domains on $\bar{\mathbb{C}}$:

Theorem 3.3 (Jordan curve theorem). *Let J be a Jordan curve on $\bar{\mathbb{C}}$, then $\bar{\mathbb{C}} \setminus J$ has two components M_1 and M_2 satisfying $\partial M_1 = \partial M_2 = J$.*

M_1 and M_2 are called Jordan domains, which are simply-connected. If $\infty \in M_2$, we call M_1 interior and M_2 exterior domain. We want to understand the behaviour if we approach the boundary. Which properties of conformal mappings of the interior of M are preserved when getting closer and closer to the boundary ∂M ? Mathematically written (F denotes a conformal mapping of $\mathbb{D}$ onto $M \subseteq \bar{\mathbb{C}}$):

$$F(\zeta) = \lim_{z \rightarrow \zeta} F(z) \in \bar{\mathbb{C}} \text{ whereas: } z \in \mathbb{D}, \zeta \in \mathbb{T} \quad (3.21)$$

If this limit exists for all $\zeta \in \mathbb{T}$, F is continuous on $\bar{\mathbb{D}}$ (In the case $\infty \in \partial M$ the continuity is meant in the spherical metric.). Fortunately, the question concerning the continuity of F approaching the boundary has an entirely topological answer ([35], p.18). A set shall be called locally connected if a connected subset of small diameter can join nearby points, then we can state:

Theorem 3.4 (Continuity theorem). *The function F has a continuous extension to $\bar{\mathbb{D}}$ if and only if M is locally connected.*

We now want to know if this extension is a homeomorphism. This is answered by a theorem of Carathéodory ([82]):

Theorem 3.5 (Carathéodory-Osgood). *If M is a Jordan domain, the conformal mapping $F : M \rightarrow \mathbb{D}$ can be extended to a homeomorphism between $\bar{M} \rightarrow \bar{\mathbb{D}}$.*

A solution for the boundary problem can be found if there exists a homeomorphism between the boundaries. This is a very impressive result due to the fact that Jordan domains can be very pathologic like the Koch snowflake [82] or the comb domain [88], p.110. The approximative solution for the Koch snowflake is shown in figs. 4.3a and 4.3b for the Green function of the first and the second kind.

Although Jordan domains form a large class covering many cases, very common applications of practical interest, like in the case of a strip-domain, do not belong to this class. The strip, denoted as S_d of width d , is defined as $S_d := \{z : |\operatorname{Im} z| < d/2\}$ where ∂S corresponds to two circles on $\bar{\mathbb{C}}$ (lines on $\mathbb{C}$ are circles on $\bar{\mathbb{C}}$) see fig. 3.5a, which touch ∞ (the north-pole on $\bar{\mathbb{C}}$) as depicted in fig. 3.5b: the boundary on the plane as lines correspond to circles on $\bar{\mathbb{C}}$. $\mathbf{N}$ denotes the north-pole corresponding to the point ∞ . The blue and the red line show the trace of two points ζ_- and ζ_+ running to ∞ . In fig. 3.5b we see how they meet at $\mathbf{N}$. The light shaded region indicates the projected strip domain. The boundary of it touches itself at the north-pole. S_d is an unbounded star-like domain (definition 3.4). As a consequence, the boundary touches ∞ from two sides and so there exists no homeomorphism between ∂S_d and $\mathbb{T}$ on $\bar{\mathbb{C}}$.

This domain is a so-called *regulated* domain, a term introduced by Ostrowski [35]. Originally the term regulated domain allows finitely many discontinuities of the tangential vectors of the boundary curve and finitely many self-intersections. Since only simply-connected domains are discussed in this thesis, self-intersections are neglected in all following discussions about a regulated domain which is bounded⁴. As mentioned before a subclass of the regulated domains, is the class of the star-like domains on $\bar{\mathbb{C}}$ (where $[\zeta, w]$ is the linear connection between the points ζ and w):

Definition 3.4. A star-like domain M with respect to ζ is a domain, where $w \in M$ and $[\zeta, w] \subset M$.

This means there is a central point from which all rays cross or touch the boundary. A conformal function F is star-like if $F(0) = 0$ and $F : \mathbb{D} \rightarrow M$, with M star-like. For later use we provide the following theorem ($\lim_{r \rightarrow 1^-}$ denotes the left limit to one) :

Theorem 3.6. If F is star-like, then $\lim_{r \rightarrow 1^-} F(re^{i\varphi}) \in \bar{\mathbb{C}}$ exists for every φ .

A proof can be found in [35]. This insures that we can approach the boundary of M as a radial limit function of F of conformal functions $F_r(z) := F(rz)$, $0 < r < 1$, in other words F is continuous in the spherical sense up to the boundary. This signals

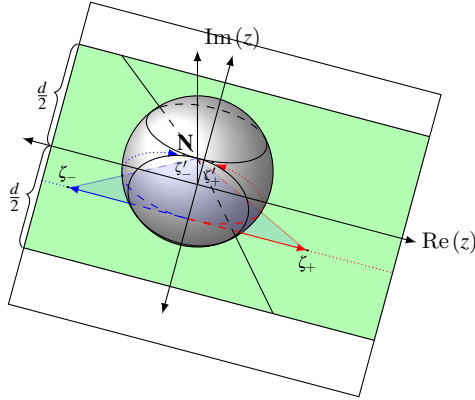


Fig. 3.5a The strip domain S_d (green) on the Riemann-sphere $\bar{\mathbb{C}}$.

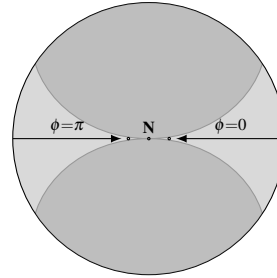


Fig. 3.5b The bird's eye view onto $\bar{\mathbb{C}}$ of the strip as drawn in fig. 3.5a.

the end of the considerations on the boundary behaviour. The basic terminology for a concise discussion on the treated problems in this work was introduced and we address now the physical aspects.

⁴ In this context the term bounded regulated is used equivalent to bounded piecewise regular simply-connected and the term unbounded regulated is synonymous to unbounded star-like.

3.3 The Physical Aspects

In the following section, we introduce the physical setting for the cases of perfectly conducting boundaries in the cases of electric and perfect permeable boundaries in the magnetic case. We move from the classical technical notation to the complex notation. Due to the fact that transcendental functions act upon and deliver unitless dimensionless numerical values [89], the previously discussed mathematical quantities have to be modified when moving to the physical picture. In the following, the quantities in transcendental functions are always normalised to a meaningful reference radius $|z_{\text{ref}}|$, as given for example as the minimal distance to the boundary at the centre of the shape. In also cures the arising difficulty when obtaining the two dimensional potentials of infinitely elongated structures out of the Biot-Savart Law and Gauss's law leading to non-regular fields, as discussed in [90], p. 198. The Green functions, modelling the physical potentials have to be understood in the same manner: their arguments used in the physical context are always in the normalised system, where the reference normalisations vanish as the physical fields are derived from the Green functions.

3.3.1 The Magnetostatic Case in the Limit of Perfect Permeability

We start with the empirical law called *Ampère's law* (see [91], p.107 or [36], p.14). Assuming that the magnetic intensity H has only an azimuthal component around a wire located at the origin, and since - from symmetry arguments - its modulus should only depend on the distance r , to the wire. Where, μ_0 be the permeability of free space, so we have $\vec{s}$ is oriented tangentially to the integration path, using the technical notation):

$$\oint \vec{H} \cdot d\vec{s} = I. \quad (3.22)$$

This law states that we always obtain the enclosed current I , when integrating along a closed path around the wire. From the previous chapter, it follows that this equation shows the circulation is caused by a fundamental source of strength I at the origin eq. (3.7). The current I is produced by moving charged particles, which can be expressed by the mean velocity $c\beta_0$, with c being the speed of light of the particles and the charge line density λ (Coulomb/m) via: $I = c\beta_0\lambda$. The complex B-field $B = \mu_0 H$ ($B = (\text{Re}\{B\}, \text{Im}\{B\})$) in free space can be formulated as (z_{ref} is a normalisation constant as discussed at the beginning of this chapter):

$$B(z) = 2i\mu_0 c\beta_0 \lambda \bar{\partial} \Gamma \left(\frac{z}{z_{\text{ref}}} \right) = i \frac{\mu_0 c\beta_0 \lambda}{2\pi} \frac{1}{z^*}. \quad (3.23)$$

If we assume that the material is a so called linear material the magnetic intensity H with a relative permeability μ and the magnetic field B are related linearly via:

$$B(z) = \mu H(z). \quad (3.24)$$

Linear materials are homogeneous and isotropic. The change of the field from a material, M_1 with μ_1 to another material, M_2 , with μ_2 , can be obtained by using a Gaussian pillbox [74] by assuming for our purposes no surface current. From this we derive the continuity conditions at the boundary of the two materials, where the subscript 1 stands for the material 1 (domain M_1) and the subscript 2 for material 2 (domain M_2). The subscripts t and n denote the tangential and the normal component respectively (see fig. 3.6):

$$B_{1n} = B_{2n} \quad (3.25a)$$

$$H_{1t} = H_{2t}. \quad (3.25b)$$

This means that the normal component of the B field (eq. (3.25a)) and the tangential component of the intensity H (eq. (3.25b)) is continuous through the boundary. Our interest is now the limiting case when material 2 reaches an extremely high relative permeability. Since the materials are linear it follows for eq. (3.25a) from eq. (3.24) taking the limit $\mu_2 \rightarrow \infty$:

$$\begin{aligned} B_{1n} = B_{2n} \text{ implies } \mu_1 H_{1n} &= \mu_2 H_{2n}, \\ \lim_{\mu_2 \rightarrow \infty} \frac{\mu_2}{\mu_1} H_{2n} &= H_{1n} \text{ bounded implies } H_{2n} \rightarrow 0. \end{aligned}$$

The physical observable $H_{2n} \rightarrow 0$ since H_1 is bounded. It follows that the magnetic intensity is purely tangential approaching the boundary from within M_2 and is continuous through the boundary.

Let us assume now, that material 1, with permeability μ_1 , is simply connected and enclosed by M_2 , with permeability μ_2 ($\mu_2 \gg \mu_1$). In the enclosed M_1 be a wire which is carrying a current $c\beta_0\lambda$. From eq. (3.22) we observe: $\oint_\gamma \vec{H}_2 \cdot d\vec{s} = c\beta_0\lambda$, hence the field is tangential and the integration along the boundary coming from the exterior to M_1 , meaning the path γ lies completely in M_2 , yields $c\beta_0\lambda$. In the equilibrium there will be a stationary flow in M_2 . With the path length $|\partial M| = \oint_\gamma |ds|$ the flow is $\frac{I}{|\partial M|} \vec{t}$. Through the continuity this also is valid for the interior limit and hence $\vec{H}_1 = \frac{\beta_0\lambda}{|\partial M_1|} \vec{t}$ along the boundary of M_1 , where $|\partial M_1|$ is the length of ∂M_1 . In fig. 3.6 this is illustrated. The material M_1 (blue) with a permeability of μ_1 has the boundary γ to the material M_2 (green), which has a permeability of $\mu_2 \gg \mu_1$. Terms $\mathbf{t}$ and $\mathbf{n}$ show the unit tangential and unit normal vector of $\gamma(s_0)$ at the point s_0 . As described, the tangential component of the magnetic intensity H is continuous through γ , whereas the normal component in M_2 vanishes. H in M_2 is purely tangential as indicated by the green line. In the equilibrium H_t is constant in M_2 with a value $H_t = \frac{I}{|\partial M_1|}$, where $I = c\beta_0\lambda$ is the enclosed current.

Defining a vector $\mathbf{t} = t_x + it_y$ and its conjugate as $\mathbf{t}^* = t_x - it_y$, where $\vec{t} = (t_x, t_y)$ is the positive oriented tangential unit vector to γ . Defining another vector $\mathbf{n}$ and its conjugate as $\mathbf{n}^* = n_x - in_y = (n_x, n_y) = i\mathbf{t}^*$, with $\vec{n} = (n_x, n_y)$, the exterior normal unit vector of γ and expressing H in terms of a function g , which satisfies the relation $B = 2ic\beta_0\lambda\mu_0\bar{\partial}g$, we see:

$$H \stackrel{\text{eq. (3.24)}}{=} \frac{1}{\mu} B = 2ic\beta_0\lambda\bar{\partial}g \quad (3.26)$$

$$H_t = \text{Re} \{H\mathbf{t}^*\} = 2\text{Re} \{ic\beta_0\lambda\bar{\partial}g\mathbf{t}^*\} = 2c\beta_0\lambda\text{Re} \{\bar{\partial}g\mathbf{n}^*\} = 2c\beta_0\lambda\frac{\partial g}{\partial n}. \quad (3.27)$$

If now a current is located inside M_1 with strength $I = c\beta_0\lambda$, g has to obey along the boundary γ (separating M_1 and M_2) with $\mu_2 \gg \mu_1$ (see fig. 3.6):

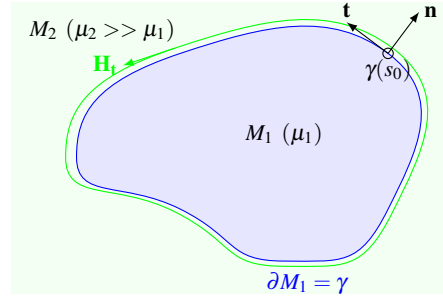
$$\text{Re} \{\bar{\partial}g\mathbf{n}^*\} \Big|_{\gamma} = \text{Re} \{\partial g\mathbf{n}\} \Big|_{\gamma} = \frac{1}{|\partial M_1|}, \quad |\partial M_1| := \oint_{\partial M_1(s)} |ds|, \quad (3.28)$$

and additionally (γ encloses the wire):

$$2\text{Im} \left\{ \int_{\gamma} \partial g dz \right\} = 1. \quad (3.29)$$

The function g is the Neumann function and in section 3.5 we give a unique definition for g and study the function in detail.

Fig. 3.6 Perfect magnetic boundary conditions.



3.3.1.1 The problem of infinite energy density

As a last remark, we discuss the problem of infinite energy density caused in the limit of $\mu \rightarrow \infty$. Here we follow the arguments of [31]. It must be clear that we address a limit configuration with increasing permeability. In fig. 3.12, a material with very high permeability is shown in an experimental setting. As the comparison with the calculated result shows, the experimental result is very close to the theoretically predicted value. In the limit, a finite H would lead to an infinite energy density inside the material, and surface currents would prevent the penetration of a field into such a material, meaning a steady-state solution does not exist within a finite time span. In real materials nevertheless, the time is always finite, and hence the steady-state solution will be reached within finite time followed by high energy storage. An experimental verification of the modelling in this manner can be found in [31, 92, 93].

3.3.2 The Electrostatic Case in the Limit of Perfect Conductivity

Electrostatic phenomenons are described by *Gauss's law*. The electric field $\vec{E}$ is caused by the line charge density λ . It can be found in, e.g. [74, 94] and reads as ($\vec{n}$ is the normal vector to the integration path):

$$\oint \varepsilon_0 \vec{E} \cdot d\vec{n} = \lambda. \quad (3.30)$$

The electric field is a source field. Analogously to the B field, it can be expressed in free space by a fundamental source with a strength λ at the origin, where $\vec{E} = (\operatorname{Re}\{E\}, \operatorname{Im}\{E\})$ holds. $z_{\text{ref.}}$ is a normalisation constant as discussed at the beginning of this chapter:

$$E(z) = \frac{2\lambda}{\varepsilon_0} \bar{\partial} \Gamma \left(\frac{z}{z_{\text{ref.}}} \right) = \frac{\lambda}{2\pi \varepsilon_0} \frac{1}{z^*}. \quad (3.31)$$

Now, analogous to the previous technique employed at the B field, we assume two linear materials. A linear material can be written as:

$$D = \varepsilon E, \quad (3.32)$$

where the dielectric constant is denoted by ε and the flux density by D . A material M_1 with a dielectric constant ε_1 is enclosed by another material M_2 having a dielectric constant ε_2 . From the same Gaussian pillbox argument we derive the following conditions at the boundary γ between M_1 and M_2 (the subscripts mark the corresponding quantities of material M_1 and M_2):

$$E_{1t} = E_{2t} \quad (3.33a) \quad D_{1n} = D_{2n}, \quad (3.33b)$$

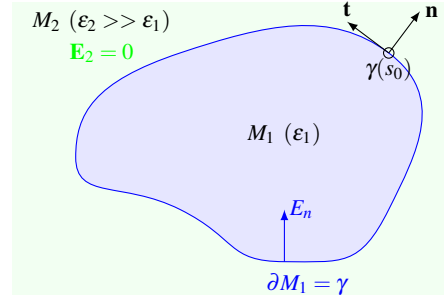
since the charges can distribute nearly free in a good conductor as metals, they compensate external fields and in an equilibrium state the field vanishes inside the conductor. Now by letting M_2 be metallic ($\varepsilon_2 \gg \varepsilon_1$). It follows:

$$\begin{aligned} D_{1n} = D_{2n} & \stackrel{\text{eq. (3.32)}}{\text{implies}} \underbrace{\varepsilon_1 E_{1n}}_{\text{bounded}} = \varepsilon_2 E_{2n} \\ \lim_{\varepsilon_2 \rightarrow \infty} \varepsilon_2 E_{2n} \text{ (bounded)} & \Rightarrow E_{2n} \rightarrow 0 \\ E_{1t} = E_{2t} & \stackrel{\text{eq. (3.32)}}{\text{implies}} \frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2} \\ \lim_{\varepsilon_2 \rightarrow \infty} \frac{D_{2t}}{\varepsilon_2} = 0 & = \frac{D_{1t}}{\varepsilon_1} \text{ implies } \frac{D_{1t}}{\varepsilon_1} = E_{1t} = E_{2t} \rightarrow 0. \end{aligned}$$

The electric field $\vec{E}_2$ vanishes inside the metal. Due to the continuity of the tangential component of the electric field $E_{2t} = 0 = E_{1t}$, the field $\vec{E}_1$ is perpendicular to the

boundary γ . Figure 3.7 illustrates this result. The material M_1 (blue) with a dielectric constant of ε_1 has the boundary γ to the material M_2 (green), which has a dielectric constant of $\varepsilon_2 \gg \varepsilon_1$. $\mathbf{t}$ and $\mathbf{n}$ show the unit tangential and unit normal vector of $\gamma(s_0)$ at the point γ . As described in the text, the tangential component of the electric field $\vec{E}_2 = 0$ is continuous through γ . $\vec{E}_1$ is normal onto the boundary γ as indicated by E_{1n} (blue). Expressing the electric field E now with help of a function g , similar

Fig. 3.7 Perfect electric boundary conditions.



to the magneto-static case, we can write (in air $\varepsilon_r \approx \varepsilon_0$):

$$E = \frac{2\lambda}{\varepsilon_0} \bar{\partial} g, \quad (3.34)$$

$$E_t = \text{Re} \{ E \mathbf{t}^* \} = \frac{2\lambda}{\varepsilon_0} \text{Re} \{ \bar{\partial} g \mathbf{t}^* \} := \frac{\lambda}{\varepsilon_0} \frac{\bar{\partial} g}{\partial t} = 0, \quad (3.35)$$

$$g|_{\gamma} = \text{const.} \quad (3.36)$$

Additionally, it follows that (γ encloses the wire):

$$2\text{Im} \left\{ \int_{\gamma} \partial g dz \right\} = 1. \quad (3.37)$$

The function g and its properties is uniquely defined and is called Green function. It is discussed in detail in section 3.4.

Summary: The first function we investigated was the *Green function of magnetostatic problems* and the corresponding boundary problem is known as the *Neumann problem* (defined rigorously in definition 3.6). The source is located in the domain M ($|\partial M| := \oint_{\partial M} |ds|$):

$$H = 2ic\beta_0\lambda\bar{\partial}g, \quad (3.38)$$

$$\frac{\partial g}{\partial n} \Big|_{\partial M} = \frac{1}{|\partial M|}. \quad (3.39)$$

The second investigated function was the *Green function of electrostatic problems* and the corresponding boundary problem is known as the *Dirichlet problem* (defined rigorously in definition 3.5). The source be located in the domain M :

$$E = \frac{2\lambda}{\varepsilon_0} \bar{\partial} g, \quad (3.40)$$

$$g|_{\partial M} = 0. \quad (3.41)$$

3.4 The Green Function of the Classical Dirichlet Boundary Problem

Now we define the Green function of the electro and magnetostatic type. In both cases, we derive the concrete solution using absolute convergent harmonic series for the circular case. As we have seen in the previous chapters, the physics is caused by the source singularity, which in turn interacts with the boundary due to singularities (these are the images of the singularity generated at the boundaries). From eqs. (3.40) and (3.41), the electrostatic problem is defined and we show now, that the solution can be expressed through the Green function.

We define the for a domain M uniquely existing⁵ Green function as:

Definition 3.5. The classical Green function g :

Let $M \subseteq \bar{\mathbb{C}}$, a Green function for M is a map $g_M : M \times M \mapsto (-\infty, \infty]$, so that for each $\zeta \in M$:

- $g_M(\cdot, \zeta)$ is harmonic on $M \setminus \zeta$ and bounded for $z \notin \mathbb{D}_r(\zeta)$ for every r ,
- $g_M(z, \zeta) = \infty$ and as $z \rightarrow \zeta$:

$$g_M(z, \zeta) = \begin{cases} \log |z| + O(1), & \zeta = \infty \\ \log |z - \zeta| + O(1), & \zeta \neq \infty \end{cases}$$

- $g_M(z, \zeta) \rightarrow 0$ as $z \rightarrow \tau$, for $\tau \in \partial M$.

Γ denotes the fundamental singularity and rbf, harmonic in M , be the response boundary function, then the function g_M can be written as:

$$g_M(z, \zeta) = \Gamma(z, \zeta) + \text{rbf}_M(z, \zeta). \quad (3.42)$$

⁵ It is true in the non-classical case for arbitrary non-polar domains [20] and in the classical case for admissible domains [18, 20, 26].

With this notation the connection to the classical Dirichlet boundary problem [20] becomes evident. The response boundary function rbf satisfies the Dirichlet problem:

- rbf is harmonic,
- $\lim_{z \rightarrow \tau} \text{rbf}(z, \zeta) = -\Gamma(z, \zeta)$ for $\tau \in \partial M$.

The classical Dirichlet problem can be solved on any regular domain [20]. Within this study we are mainly interested in simply connected domains. If M is a simply connected domain and $M \setminus \bar{\mathbb{C}}$ contains more than two points, then the domain is regular [20].

This insures that the rbf, and consequently the Green function, exists for a large class of domains M . On the other hand if we find a conformal mapping of M onto $\mathbb{D}$, we can solve the Dirichlet problem there. The existence of such a mapping is guaranteed by the RMT theorem 3.1. As we have seen, the fundamental singularity is invariant under conformal mappings. As rbf as a real harmonic function always has a harmonic conjugate function, in a simply connected domain, we find the following result:

Theorem 3.7. *g is a conformal invariant.*

Let F map conformally $M_1 \rightarrow M_2$. , then: $g_{F(M_1)}(F(z), F(\zeta)) = g_{M_2}(z, \zeta)$

Proof. The conformal invariance of the singularity Γ , was already demonstrated in section 3.1.2, eq. (3.15). $g_{M_1}(z, \zeta)|_{z \in \partial M_1} = 0$ by definition 3.5.

Since $z \in \partial M_2 \rightarrow F(z) \in \partial F(M_1) = \partial M_2$

$\Rightarrow g_{M_2}|_{z \in \partial M_2} = g_{M_1}(F(z), F(\zeta))|_{F(z) \in \partial F(M_1)} = 0.$ □

3.4.1 A Novel Approximation Method for Simply-connected Domains including an Error-bound

Providing the mapping of M onto a solvable domain (e.g. $\mathbb{D}$) solves the Dirichlet problem for M . The following theorems are essential for the approximative solution:

Theorem 3.8. *If $M_1 \subset M_2$, then: $g_{M_1}(z, \zeta) \leq g_{M_2}(z, \zeta)$, with $z, \zeta \in M_1$.*

Theorem 3.9. *Let M such that $(M_n)_{n \geq 1}$ are sub-domains such that $M_1 \subseteq M_2 \subseteq M_3 \dots$ and $\bigcup_n M_n = M$. Then: $\lim_{n \rightarrow \infty} g_{M_n}(z, \zeta) = g_M(z, \zeta)$, with $z, \zeta \in M$.*

The proofs can be found in [18, 20, 35].

Now, we prove that the Green function g_M of a domain M , which is between two domains $M_1 \supseteq M \supseteq M_2$, satisfies: $g_{M_1} \leq g_M \leq g_{M_2}$:

Theorem 3.10. *Let M such that $(M_n)_{n \geq 1}$ are subdomains such that $M_1 \supseteq M_2 \supseteq M_3 \dots$ and $\bigcap_n M_n = M$. Then: $\lim_{n \rightarrow \infty} g_{M_n}(z, \zeta) = g_M(z, \zeta)$, with $z, \zeta \in M$.*

Proof. We fix $\zeta \in M$. For $n \geq 1$ define:

$$h_n = g_{M_n}(z, \zeta) - g_M(z, \zeta) \quad (z \in M \setminus \{w\}).$$

Then h_n is harmonic on $M \setminus \{w\}$, and by the removable singularity theorem [18], h_n extends to be harmonic on M . Theorem 3.8 implies that $h_n \leq h_{n+1}$, for each n , so $u := \lim_{n \rightarrow \infty} h_n$ is sub-harmonic on M . Since $h_n \leq -g_M(\cdot, w)$ on M , for each n , it follows that $u \leq -g_M(\cdot, w)$ on M . Hence u is bounded above on M and also $\limsup_{z \rightarrow \zeta} u(z) \leq 0$ for $\zeta \in \partial M$. Therefore by the maximum principle ([95], p.7) $u \leq 0$ on M . From this we see:

$$\limsup_{n \rightarrow \infty} g_{M_n}(z, \zeta) \leq g_M(z, \zeta) \quad (z \in M).$$

From theorem 3.8 we also have:

$$\liminf_{n \rightarrow \infty} g_{M_n}(z, \zeta) \geq g_M(z, \zeta) \quad (z \in M).$$

Combining these two inequalities yields the result. $\square$

We arrived at a key outcome of this work:

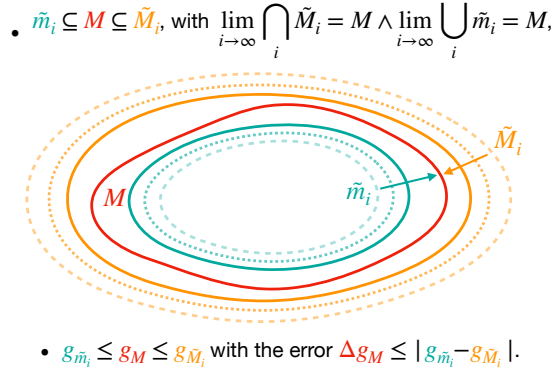
Lemma 3.1. *If we look for the Green function of the first kind for a domain (simply connected) M , which is not known a priori, and we can solve the problem for the domains $\tilde{m}$ and $\tilde{M}$, with $\tilde{m} \subseteq M \subseteq \tilde{M}$, the solution g_M is bounded by $g_{\tilde{m}} \leq g_M \leq g_{\tilde{M}}$, which provides an error bound. If approaching the domain from the in- and out-side via $\tilde{m}_1 \subseteq \tilde{m}_2 \subseteq \tilde{m}_3 \dots$ with $\bigcup_n \tilde{m}_n = M$, and $\tilde{M}_1 \supset \tilde{M}_2 \supset \tilde{M}_3 \dots$ with $\bigcap_n \tilde{M}_n = M$, we are able to formulate the solution of g_M approximative either through $g_{\tilde{m}}$ or $g_{\tilde{M}}$ with the maximal error $|g_{\tilde{m}} - g_{\tilde{M}}|$.*

Figure 3.8 illustrates the concept as proved in lemma 3.1. $\tilde{m}_i$ is enclosed by M and $\tilde{M}_i$ is enclosing M . The solutions of $\tilde{m}_i$ and $\tilde{M}_i$ can be calculated. We reduce the known upper error bound by increasing the accuracy of the approximated domains $\tilde{M}_i$ and $\tilde{m}_i$ with respect to M , as indicated by the lines-style (increasing density and decreasing opacity) and arrows and stop if the error bound is satisfying (e.g. when the error bound is lower than the measurement error).

The approximation of the domain through polygons allows for good command over the convergence. In our application the approximative solution is given in terms of in/out scribed polygonal domains, which can be obtained with help of the Schwarz-Christoffel-transformations (section 4.1) and is applied to the vacuum chamber of the LHC (section 7.2.1).

In the next section the solution for the Green function explicitly for $\mathbb{D}$ will be derived.

Fig. 3.8 The concept of the approximation of a domain M , where no exact solution is known. $\tilde{m}_i$ is enclosed by M and $\tilde{M}_i$ is enclosing M . The solutions of $\tilde{m}_i$ and $\tilde{M}_i$ are given. An upper bound for the error, as proved in lemma 3.1, is known.



3.4.2 How to find the Explicit Green Function

If one solution of the problem of a canonical domain exists, along with a good command over conformal mappings onto this domain, many solutions can be obtained. The domain in the plane with the highest symmetry - the unit disc $\mathbb{D}$ - is the entry point.

The strategy is to use the assumption that rbf can be represented as an absolute convergent harmonic series. Deriving a solution under this assumption allows for more flexibility afterwards. The function of interest rbf, has to satisfy $\text{rbf}(z, \zeta) = -\Gamma(z, \zeta) = \frac{1}{2\pi} \log |z - \zeta|$, for $z \in \partial\mathbb{T}$ ($z = re^{i\varphi}$):

$$\begin{aligned}
 \text{rbf}_{\mathbb{D}}(re^{i\sigma}, \zeta) &= \text{Re} \left\{ \sum_{n=0}^{\infty} a_n(\zeta) \rho^n e^{2\pi i n \sigma} \right\} \\
 &= -\text{Re} \left\{ \sum_{n=0}^{\infty} 2\rho^n e^{2\pi i n \sigma} \int_0^1 d\sigma' \rho^n e^{-2\pi i n \sigma'} \log(e^{2\pi i \sigma'} - \zeta) \right\} \\
 &= -\text{Re} \left\{ \sum_{n=0}^{\infty} \int_0^1 d\sigma' \rho^n e^{2\pi i n (\sigma - \sigma')} (\log(e^{2\pi i \sigma'} - \zeta) + \log(e^{-2\pi i \sigma'} - \zeta^*)) \right\} \\
 &= -\text{Re} \left\{ \int_0^1 d\sigma' \sum_{n=0}^{\infty} \rho^n e^{2\pi i n (\sigma - \sigma')} \left(\sum_{k=1}^{\infty} \frac{\zeta^k e^{-2\pi i k \sigma'}}{k} + \sum_{k=1}^{\infty} \frac{\zeta^{*k} e^{2\pi i k \sigma'}}{k} \right) \right\} \\
 &= -\text{Re} \left\{ \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} \frac{z^n \zeta^{*k}}{k} \underbrace{\int_0^1 d\sigma' e^{2\pi i \sigma' (k-n)}}_{\delta_{n,k}} \right\} = -\text{Re} \{ \log(1 - z\zeta^*) \}.
 \end{aligned} \tag{3.43}$$

The function $\text{rbf}_{\mathbb{D}}(z, \zeta) = -\text{Re} \{ \log(1 - z\zeta^*) \}$ is harmonic on $\bar{\mathbb{D}}$, so it can be holomorphic extended onto the boundary.

We derived the Green function g of the first kind of the unit disc:

$$g_{\mathbb{D}}(z, \zeta) = \frac{1}{2\pi} \log \left| \frac{z - \zeta}{1 - z\zeta^*} \right|, \quad (3.44)$$

The general solution of the Green function g of the first kind of a domain M , with $F_{\zeta}^M(M) \mapsto \mathbb{D}$ and $F_{\zeta}^M(\zeta) = 0$, F_{ζ}^M being conformal, can be given as:

$$g_M(z, \zeta) = \frac{1}{2\pi} \log |F_{\zeta}^M(z)|. \quad (3.45)$$

Fig. 3.9 The behaviour of the so-called Blaschke factor.

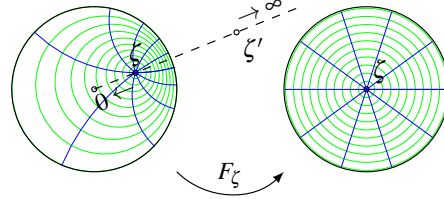
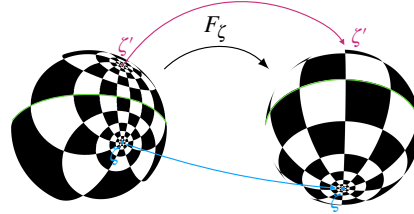


Fig. 3.10 The Blaschke factor on $\mathbb{C}$ (same ζ as in fig. 3.9 is taken).



The term $F_{\zeta}(z) := \frac{z - \zeta}{1 - z\zeta^*}$ is called Blaschke factor, which is unimodular on $\mathbb{D}$ transporting $\zeta \rightarrow 0$. The mapping is illustrated in fig. 3.9 below, where $\zeta = \frac{1}{2} + \frac{2}{10}i$. Additionally in fig. 3.9, the contour lines are also plotted, corresponding to the electric potential and the field lines. As displayed in fig. 3.9 the solution is a mapping onto the high symmetric case of a source at the origin. The Blaschke factor maps the unit disk onto itself. It appears (naturally) as conformal mapping in the (complex) Green function of the unit circle and is a special case of the Moebius class [96].

Interestingly the mapping can be interpreted in terms of mirror functions: $F_{\zeta}(z)$ maps $\zeta \rightarrow 0$ and hence its mirror on the unit circle $\zeta' = \frac{1}{\zeta^*} \rightarrow \infty$ along the symmetry axis

(as presented as dashed line in fig. 3.9.). This minimizes the influence of the image source. It reflects the minimizing character of the solution of the Dirichlet problem. Since for simply connected domains we can conformally transform the invariant Green function onto the unit disk (RMT - theorem 3.1) and solve the problem there with help of the Blaschke factor F_ζ . The mirror image ζ' is transported to ∞ , which easily can be seen on $\tilde{\mathbb{C}}$ in fig. 3.10.

In the next section, we discuss the solution of the magnetostatic problem.

All solutions found for the Neumann function (the Green function of the second kind), which are inherently harder to find than for the Green function of the first kind, naturally extend to the latter as well.

3.5 Novel solutions of the Green Function of the Generalized Neumann Problem of Simply Connected Domains

We discuss in the following the classification of the Green function of the magnetostatic problem. As mentioned earlier, the situation is more involved than in the electrostatic case, since here the Green function is not conformally invariant.

We provide the necessary proofs to the full extent.

The magnetostatic boundary behaviour is explained in section 3.3.1, and the Green function has a constant normal derivative, namely the constant circulation of the enclosed current along the boundary.

We start with a general definition, and later on we see that our interest is mainly in a particular variant of the Neumann function. For the classical Neumann function the normal derivative has to be defined everywhere along the boundary. To cover possible discontinuities of the normal derivative along the boundary, we formulate the generalized Neumann function instead of the classical Neumann function. The problem is solved for bounded piecewise regular domains and afterwards is extended onto more general domains.

Definition 3.6. The generalized Neumann function n :

Let $M \subseteq \mathbb{C}$ bounded regulated, a Neumann function for M is a map $n_M : M \times M \mapsto (-\infty, \infty]$, so that for each $\zeta \in M$:

- $n_M(\cdot, \zeta)$ is harmonic on $M \setminus \zeta$ and bounded for $z \notin \mathbb{D}_r(\zeta)$ for every r ,
- $n_M(z, z) = \infty$ and as $z \rightarrow \zeta$:

$$n_M(z, \zeta) = \log |z - \zeta| + O(1)$$

- $\frac{\partial n_M}{\partial n}(z, \zeta) = \frac{1}{|\partial M|}$ as $z \rightarrow \tau$, for not everywhere $\tau \in \partial M$.

To uniquely define n_M one can impose a so-called normalizing condition ($|dz|$ denotes the line integral):

$$\int_{\partial M} n_m(z, \cdot) |dz| = 0. \quad (3.46)$$

Since the normalization has no influence on the physical fields, we omit it in the following. In order to find the Neumann function, a special form of the Neumann problem has to be solved:

Definition 3.7. The generalized Neumann problem: Let M be a bounded regulated domain. Where $h(z)$ be a real-valued piecewise continuous function on ∂M .

- n_M is continuous and piecewise differentiable on ∂M ,
- n_M is harmonic in M ,
- Additionally we demand: $\lim_{z \rightarrow \tau} \operatorname{Re} \{ \partial n_M \mathbf{n} \} = h(\tau)$ for not everywhere $\tau \in \partial M$.

A necessary condition for h is [33]:

$$\int_{\partial M} h(z) |dz| = 0. \quad (3.47)$$

Definition 3.7 has a unique solution ([33], p.266). The regulated domain was introduced in section 3.2.3 and the restriction that M has to be bounded and regular ensures the existence of the normal derivative along the boundary with a finite number of discontinuities (jumps). Such a domain is illustrated in fig. 3.11. We see there that at the vertices ($a, b, c, \dots, f$) of the boundary the normal $\mathbf{n}$ green reveals discontinuities of the amount φ_i as we approach the vertices from the right and left side, respectively. The red arrow shows the mean of the right and left side limit. To see that the generalized Neumann function can be found through the generalized

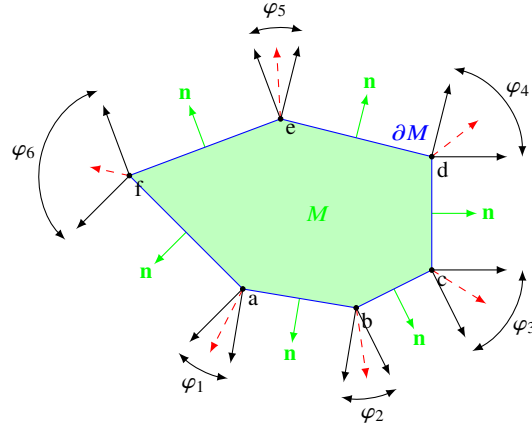


Fig. 3.11: An example of a piecewise regular domain M , a polygonal domain.

Neumann problem, n_M is formulated as the sum of the fundamental singularity and

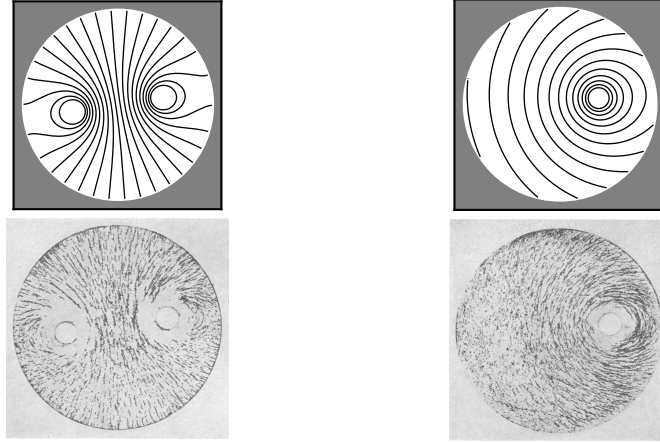


Fig. 3.12: The Neumann function vs. the experiment. (Lower pictures (photographs) taken from [31]).

the in M harmonic boundary response function rbf_M :

$$n_M(z, \zeta) = \Gamma(z, \zeta) + \text{rbf}_M(z, \zeta) \quad (3.48)$$

Along ∂M n_M has to satisfy the condition:

$$2\text{Re} \left\{ \partial n_M(z, \zeta) \mathbf{n} \right\} \Big|_{z \in \partial M} = \text{Re} \left\{ \mathbf{n} \left(\frac{1}{2\pi(z - \zeta)} + 2\partial \text{rbf}_M(z, \zeta) \right) \right\} \Big|_{z \in \partial M} = \frac{1}{|\partial M|}. \quad (3.49)$$

It is a Neumann problem for $\text{Re} \{ \text{rbf}_M \}$ with the boundary value ($z \in \partial M$):

$$2\text{Re} \{ \mathbf{n} \partial \text{rbf}_M(z, \zeta) \} = \frac{1}{|\partial M|} - \text{Re} \left\{ \frac{\mathbf{n}}{2\pi(z - \zeta)} \right\}. \quad (3.50)$$

Indeed this obeys the necessary condition eq. (3.47):

$$\int_{\partial M} 2\text{Re} \{ \mathbf{n} \partial \text{rbf}_M(z, \zeta) \} |dz| = \int_{\partial M} \frac{1}{|\partial M|} - \text{Re} \left\{ \frac{\mathbf{n}}{2\pi(z - \zeta)} \right\} |dz| = 0 \quad (3.51)$$

Now, we develop a strategy to find rbf_M and consequently the Neumann function n_M . In section 3.4, we assumed rbf to be representable as an absolute convergent series and here again we use this approach.

3.5.1 Dini's Formula

To find a function rbf_M which solves the Neumann problem eq. (3.49) on a unit disc $\mathbb{D}$, $\text{rbf}_\mathbb{D}$ can be represented as an absolute convergent harmonic series⁶ ($\frac{\partial}{\partial n} = \frac{\partial}{\partial \rho}$):

$$\begin{aligned}
 \text{rbf}_\mathbb{D}(z; \zeta) &= \text{rbf}_\mathbb{D}(\rho, \sigma; \zeta) = \text{Re} \left\{ \sum_{n=1}^{\infty} a_n(\zeta) \rho^n e^{2\pi i n \sigma} \right\} \\
 \frac{\partial \text{rbf}_\mathbb{D}(\rho, \sigma; \zeta)}{\partial \rho} \Big|_{\rho=1} &= \text{Re} \left\{ \sum_{n=1}^{\infty} a_n(\zeta) n \rho^{n-1} e^{2\pi i n \sigma} \right\} \Big|_{\rho=1} \\
 &\Rightarrow a_n(\zeta) = 2 \int_0^1 d\sigma \frac{e^{-2\pi i n \sigma}}{n} \frac{\partial \text{rbf}_\mathbb{D}(\rho, \sigma; \zeta)}{\partial \rho} \Big|_{\rho=1} \\
 \text{rbf}_\mathbb{D}(\rho, \sigma; \zeta) &= \text{Re} \left\{ \sum_{n=1}^{\infty} 2 \int_0^1 d\sigma' \frac{\rho^n e^{2\pi i n (\sigma - \sigma')}}{n} \frac{\partial \text{rbf}_\mathbb{D}(\rho', \sigma'; \zeta)}{\partial \rho'} \right\} \Big|_{\rho'=1} \\
 &= 2 \text{Re} \left\{ \int_0^1 d\sigma \underbrace{\sum_{n=1}^{\infty} \frac{\rho^n e^{2\pi i n (\sigma - \sigma')}}{n}}_{-\log(1 - \rho e^{2\pi i (\sigma - \sigma')})} \frac{\partial \text{rbf}_\mathbb{D}(\rho', \sigma'; \zeta)}{\partial \rho'} \right\} \Big|_{\rho'=1} \\
 &= -2 \text{Re} \left\{ \int_0^1 d\sigma' \log(1 - \rho e^{2\pi i (\sigma - \sigma')}) \frac{\partial \text{rbf}_\mathbb{D}(\rho', \sigma'; \zeta)}{\partial \rho'} \right\} \Big|_{\rho'=1}. \tag{3.52}
 \end{aligned}$$

The function on the disc can be expressed in terms of it's normal derivative on the unit circle $\mathbb{T}$. Equation (3.52) is known as Dini's formula (see [32] p.277). The form of $\text{rbf}_\mathbb{D}$ to be the image function of $\mathbb{T}$ ($|\mathbb{T}| = 2\pi$) is:

$$\frac{\partial \text{rbf}_\mathbb{D}(\rho, \sigma; \zeta)}{\partial \rho} \Big|_{\rho=1} = -\frac{\partial \log r}{\partial \rho} \Big|_{\rho=1} + 1. \tag{3.53}$$

Employing eq. (3.52) yields:

$$\begin{aligned}
 &-2 \text{Re} \left\{ \int_0^1 d\sigma' \log(1 - \rho e^{2\pi i (\sigma - \sigma')}) \left(-\frac{\partial \log r}{\partial \rho} \Big|_{\rho=1} + 1 \right) \right\} \\
 &= 2 \text{Re} \left\{ \int_0^1 d\sigma' \log(1 - \rho e^{2\pi i (\sigma - \sigma')}) \left(\frac{\partial \log |\rho e^{2\pi i \sigma'} - \zeta|}{\partial \rho} \Big|_{\rho=1} - 1 \right) \right\} \\
 &= \text{Re} \left\{ \int_0^1 d\sigma' \log(1 - \rho e^{2\pi i (\sigma - \sigma')}) \left(\frac{\zeta e^{-2\pi i \sigma'}}{1 - \zeta e^{-2\pi i \sigma'}} + \frac{\bar{\zeta} e^{2\pi i \sigma'}}{1 - \bar{\zeta} e^{2\pi i \sigma'}} \right) \right\} \\
 &= \text{Re} \left\{ \int_0^1 d\sigma' \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\rho^n e^{2\pi i n (\sigma - \sigma')}}{n} \sum_{k=1}^{\infty} \left(\bar{\zeta}^k e^{2\pi i k \sigma'} + \zeta^k e^{-2\pi i k \sigma'} \right) \right\}
 \end{aligned}$$

⁶ The series starts at $n = 1$ because a_0 has to be zero for the boundary problem to be solvable. This can be seen directly from the Green formula ([33], p.265).

$$\begin{aligned}
&= \operatorname{Re} \left\{ \sum_{n=1}^{\infty} (-1)^{n+1} \sum_{k=1}^{\infty} \frac{z^n \bar{\zeta}^k}{n} \underbrace{\int_0^1 d\sigma' e^{2\pi i(k-n)\sigma'}}_{\delta_{n,k}} + \frac{z^n \bar{\zeta}^k}{n} \underbrace{\int_0^1 d\sigma' e^{-2\pi i(k+n)\sigma'}}_{\delta_{n,-k}} \right\} \\
&= \operatorname{Re} \left\{ \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(z\bar{\zeta})^n}{n} \right\} = \operatorname{Re} \{ \log(1 - z\bar{\zeta}) \}.
\end{aligned} \tag{3.54}$$

Hence the analytic form of the $\operatorname{rbf}_{\mathbb{D}} = \operatorname{Re} \{ \log(1 - z\bar{\zeta}) \}$ is found.

The Neumann function of the circle has the form:

$$n_{\mathbb{D}}(z; \zeta) = \frac{1}{2\pi} \log |(z - \zeta)(1 - z\bar{\zeta})|. \tag{3.55}$$

The experimental verification of eq. (3.55) is shown in fig. 3.12. The left plot shows a source while the right plot shows a source and a sink. Due to Ampère's Law, integration along the boundary yields the enclosed current. The latter is invariant under conformal mappings as proven in section 3.5.3 and the integral vanishes.

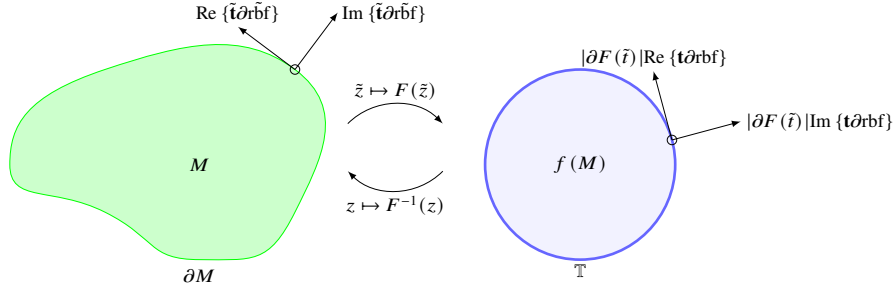


Fig. 3.13: The conformal behaviour of the tangential $\frac{\partial \operatorname{rbf}(z)}{\partial t} = 2\operatorname{Re} \{ \partial \operatorname{rbf}(z) \mathbf{t} \}$ and normal derivative $\frac{\partial \operatorname{rbf}(z)}{\partial n} = 2\operatorname{Im} \{ \partial \operatorname{rbf}(z) \mathbf{t} \}$.

Now we generalize the solution of the circle $\mathbb{D}$. Let us look at the normal derivative under conformal mappings. An illustration can be found in fig. 3.13. The normal and tangential vectors onto the ∂M are visualized and their orientation is kept under the conformal mapping on $\mathbb{T}$, only a scaling factor comes into play, which was also shown in fig. 3.4 for the principle of a conformal map:

Lemma 3.2. *Let F map M , which is a smooth bounded simply-connected domain, conformally onto $\mathbb{D}$. If we transform $\operatorname{rbf}(z) = \tilde{\operatorname{rbf}}(\tilde{z})$ with $F(\tilde{z}) = z$, then $\tilde{\mathbf{t}} \partial \tilde{\operatorname{rbf}}(F(z)) = \frac{\mathbf{t}}{|\partial F^{-1}(z)|} \partial \operatorname{rbf}(z)$.*

Proof. Assume $\tilde{z} \mapsto F(\tilde{z}) = z$ and $\tilde{\operatorname{rbf}}(\tilde{z}) = \operatorname{rbf}(F(\tilde{z}))$,
 $\tilde{\mathbf{t}} = \frac{\dot{\tilde{z}}}{|\dot{\tilde{z}}|} = \frac{\partial F^{-1}(z) \dot{\tilde{z}}}{|\partial F^{-1}(z) \dot{\tilde{z}}|}$ on ∂M and $\mathbf{t} = \frac{\dot{z}}{|\dot{z}|}$ on $\mathbb{T}$, hence:

$$\begin{aligned} \partial \tilde{\text{rbf}}(\tilde{z}) \tilde{\mathbf{t}} &= \partial \tilde{\text{rbf}}(\tilde{z}) \frac{\partial F^{-1}(z) \dot{z}}{|\partial F^{-1}(z) \dot{z}|} = \partial \text{rbf}(F(\tilde{z})) \partial F(\tilde{z}) \partial F^{-1}(z) \frac{\dot{z}}{|\dot{z}|} \frac{1}{|\partial F^{-1}(z)|} \\ &= \frac{\partial \text{rbf}(z)}{|\partial F^{-1}(z)|} \mathbf{t}. \end{aligned} \quad \square$$

Equation (3.52) can now be written as ($\mathbf{t}$ is the positively oriented complex tangential unit vector of $\mathbb{T}$):

$$\text{rbf}_{\mathbb{D}}(z, \zeta) = -\frac{1}{\pi} \oint_{|\tau|=1} |d\tau| \log |1 - z\tau^*| 2\text{Im} \{ \mathbf{t} \partial \text{rbf}_{\mathbb{D}}(\tau; \zeta) \}. \quad (3.56)$$

Equation (3.56) is by construction harmonic for $|z| < 1$.

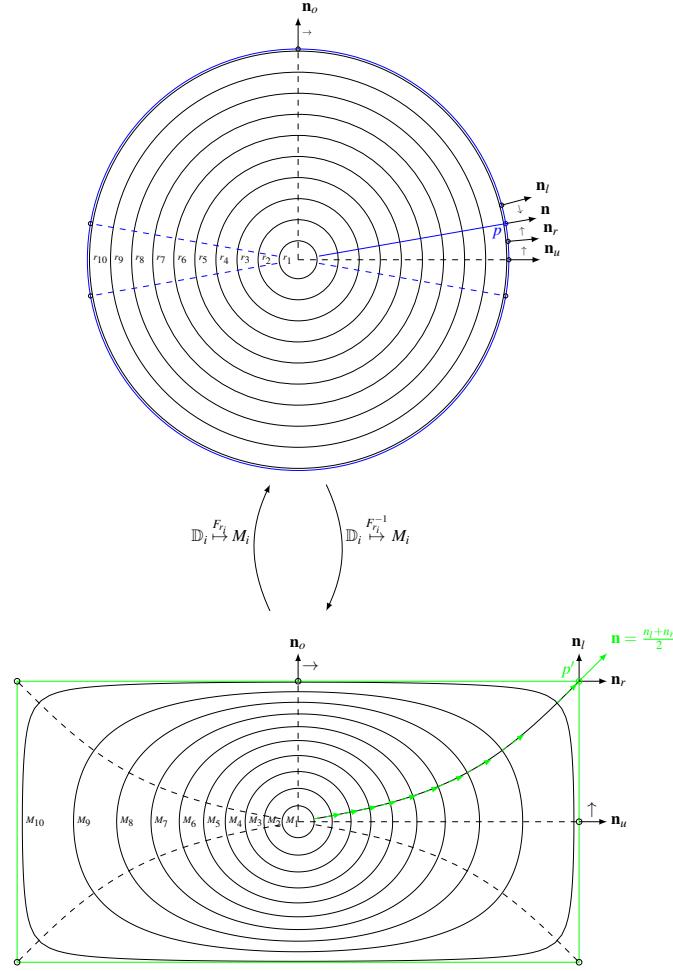


Fig. 3.14: The concept of the radial limiting angle. A linear homotopy from the circle onto a rectangle is indicated.

3.5.2 A Novel Integral Representation of the Neumann Function

Now we give a novel integral representation of the Neumann function of a smoothly bounded simply-connected domain M with the map F mapping M conformally onto $\mathbb{D}$:

Theorem 3.11. *The Neumann function of a smoothly bounded (no corners) simply-connected domain M , with the conformal map $F : M \rightarrow \mathbb{D}$, can be given as:*

$$n_M(z, \zeta) = \frac{1}{2\pi} \left(\log |(F(z) - F(\zeta))(1 - F(z)F^*(\zeta))| - \frac{2}{\pi} \oint_{|\tau|=1} |d\tau| \log |1 - F(z)\tau^*| \left(1 - \frac{|\partial F^{-1}(\tau)|}{|\partial M|}\right) \right). \quad (3.57)$$

Remark: The boundary integral does not depend on the source point ζ !

To prove this statement we first introduce the radial exhaustion of a domain M . Let F map a domain M conformally on $\mathbb{D}$. The radial exhaustion is a sequence of domains M_i , which are the images M_i under F of concentric circles $\mathbb{D}_i$ with increasing radii $r_i < r_{i+1} \leq 1$, which is depicted in fig. 3.14 for F mapping a rectangle onto $\mathbb{D}$. Each domain M_i is smooth and hence there is a well defined tangential and normal direction along its boundary ∂M_i . Jumps of the exterior normal and their average at the edges of the rectangle are shown.

Let us define the mapping $F_r^{-1}(z) := F^{-1}(rz)$, so $\mathbb{D} \xrightarrow{F_r^{-1}(z)} M_r$ and $\tilde{z} = F_r^{-1}(z)$. From lemma 3.2 it follows for a harmonic function u : $\tilde{t}_r \partial \tilde{u}(\tilde{z}) = \partial u(z) \frac{\tilde{z}}{|\partial F_r^{-1}(z)| |\tilde{z}|} =$

$$\frac{\partial u(z) \mathbf{t}}{r |\partial F_r^{-1}(z)|}.$$

Now we prove eq. (3.57):

Proof. From definition 3.6 follows that the Neumann function has to be 1. harmonic on $M \setminus \zeta$, 2. reveals a logarithmic singularity at $z = \zeta$ and 3. the normal derivative of the Neumann function has to be constant with the value $\frac{1}{|\partial M|}$ along ∂M . To show this we separate eq. (3.57) (omitting the normalization):

$$n_M(z, \zeta) = \underbrace{\log |(F(z) - F(\zeta))(1 - F(z)F^*(\zeta))|}_{\textcircled{1}} - \underbrace{\frac{2}{\pi} \oint_{|\tau|=1} |d\tau| \log |1 - F(z)\tau^*| \left(1 - \frac{|\partial F^{-1}(\tau)|}{|\partial M|}\right)}_{\textcircled{2}}.$$

Start with condition 1. and 2. and part $\textcircled{1}$: The term $\log |(1 - F(z)F^*(\zeta))|$ is harmonic for $z \notin \partial M$ ($F(z) \in \mathbb{D}$). And as $F'(z) \neq 0$, we can develop F into a Taylor series

around a point ζ as $F = F(\zeta) + (z - \zeta)\psi(z)$ with $\psi(z) \neq 0$, $z \in M$ and $\psi(z)$ holomorphic. The term $\log |(F(z) - F(\zeta))|$ can be written as:

$$\begin{aligned} \operatorname{Re} \{\log(F(z) - F(\zeta))\} &= \operatorname{Re} \{\log(F(\zeta) + (z - \zeta)\psi(z) - F(\zeta))\} \\ &= \log |z - \zeta| + \underbrace{\log |\psi(z)|}_{\text{harmonic}}. \end{aligned} \quad (3.58)$$

② is of the form eq. (3.56) and hence is harmonic for $z \in M$ ($|F(z)| < 1$). This means the function is harmonic for z inside M and reveals a logarithmic singularity at $z = \zeta$.

Now we show 3. and start with ①:

We use the previously stated fact about functions of the form $u(F(\tilde{z}))$ and we define $F_r(\zeta) = w$:

$$\begin{aligned} &2\tilde{\mathbf{t}}_r \partial \log |(F_r(z) - F_r(\zeta))(1 - F_r(z)F_r^*(\zeta))| \Big|_{z \in \partial M_r} \\ &= \frac{\mathbf{i}e^{it}}{r|\partial F_r^{-1}(e^{it})|} \left[\frac{1}{e^{it} - w} - \frac{w^*}{1 - e^{it}w^*} \right] \\ &= \frac{\mathbf{i}}{r|\partial F_r^{-1}(e^{it})|} \left[\frac{e^{it}}{e^{it} - w} - \frac{e^{it}w^*}{1 - e^{it}w^*} \right] \\ &= \frac{\mathbf{i}}{r|\partial F_r^{-1}(e^{it})|} \left[\frac{1}{1 - e^{-it}w} - \frac{1}{1 - e^{it}w^*} + 1 \right] \\ &= \frac{\mathbf{i}}{r|\partial F_r^{-1}(e^{it})|} \left[1 + 2\mathbf{i}\operatorname{Im} \frac{1}{1 - e^{-it}w} \right] = \frac{\mathbf{i}}{r|\partial F_r^{-1}(e^{it})|} + \text{real residual}. \end{aligned}$$

Hence, $2\operatorname{Im} \tilde{\mathbf{t}} \partial ①|_{z \in \partial M} = |\partial F_r^{-1}(e^{it})|^{-1}$. Now we look at ②:

In this case we approximate the solution with the method of radial exhaustion starting with a domain $M_r \subset M$:

$$\begin{aligned} &2\tilde{\mathbf{t}}_r \frac{2}{\pi} \oint_{|\tau|=1} |\mathrm{d}\tau| \log |1 - F_r(z)\tau^*| \left(1 - \frac{|\partial F_r^{-1}(\tau)|}{|\partial M|}\right) \Big|_{z \in \partial M} \\ &= \frac{2\mathbf{i}e^{it}}{\pi r|\partial F_r^{-1}(e^{it})|} \oint_{|\tau|=1} |\mathrm{d}\tau| \log |1 - F_r(z)\tau^*| \overbrace{\Psi_r(\tau)}^{:=\Psi_r(\tau)} \\ &= \frac{2\mathbf{i}}{\pi r|\partial F_r^{-1}(e^{it})|} \oint_{|\tau|=1} |\mathrm{d}\tau| \frac{e^{it}\tau^*}{1 - e^{it}\tau^*} \Psi_r(\tau) \\ &= \frac{\mathbf{i}}{\pi r|\partial F_r^{-1}(e^{it})|} \int_0^{2\pi} \mathrm{d}\sigma \frac{e^{\mathbf{i}(t-\sigma)}}{1 - e^{\mathbf{i}(t-\sigma)}} \Psi_r(\tau) \\ &= \frac{\mathbf{i}}{\pi r|\partial F_r^{-1}(e^{it})|} \int_0^{2\pi} \mathrm{d}\sigma \frac{1}{2} \left[\frac{1 + e^{\mathbf{i}(t-\sigma)}}{1 - e^{\mathbf{i}(t-\sigma)}} - 1 \right] \Psi_r(e^{\mathbf{i}\sigma}) \end{aligned}$$

$$= \frac{i}{2\pi r |\partial F_r^{-1}(e^{it})|} \left[\int_0^{2\pi} d\sigma \frac{1 + e^{i(t-\sigma)}}{1 - e^{i(t-\sigma)}} \Psi_r(e^{i\sigma}) - \underbrace{\int_0^{2\pi} d\sigma \Psi_r(e^{i\sigma})}_{=0} \right]$$

As long as $\Psi_r(z)$ has a continuous extension to ∂M_r , it holds (as can be deduced from the Schwarz kernel [97] or [35], p. 43):

$$\frac{i}{2\pi r |\partial F_r^{-1}(e^{it})|} \int_0^{2\pi} d\sigma \underbrace{\frac{e^{i\sigma} + e^{it}}{e^{i\sigma} - e^{it}}}_{\text{Schwarz kernel}} \Psi_r(e^{i\sigma}) = \frac{i}{r |\partial F_r^{-1}(e^{it})|} \Psi_r(e^{it}).$$

In the limit $r \rightarrow 1$ for a smooth boundary $\partial M_{r \rightarrow 1} = \partial M$, we obtain:

$$\lim_{r \rightarrow 1} \frac{i}{r |\partial F_r^{-1}(e^{it})|} \Psi_r(e^{it}) = \frac{i}{|\partial F^{-1}(e^{it})|} \Psi(e^{it}). \quad (3.59)$$

Putting in $\Psi(\tau)$ yields: $\frac{i}{|\partial F^{-1}(e^{it})|} (1 - \frac{|\partial F^{-1}(e^{it})|}{|\partial M|}) = i(|\partial F^{-1}(e^{it})|^{-1} - |\partial M|^{-1})$. So we conclude: $2\text{Im} \tilde{t}\partial(\textcircled{1} + \textcircled{2}) = |\partial M|^{-1}$. $\square$

If the function $\Psi_r(z)$ can be extended continuously to ∂M_r , we can use the integral representation. If ∂M contains corners $\Psi(z)$ jumps at the corners and is piecewise continuous only. Using the concept of a radial limiting angle the solution of the generalized Neumann functions exists (definition 3.7). The concept is shown in fig. 3.14 for a rectangle. In the limit the normal derivative jumps at the corners and the limiting radial angle is the average direction of left and right limit.

Nonetheless, an approximative solution of the classical Neumann function can be given employing the integral representation. It can be obtained by taking a smooth domain close to the true boundary, so $M_r \approx M$. In fig. 3.15 the solution is drawn for a square, approximated by a domain near the true boundary ($r \approx 1$).

The solution of a smooth domain, an ellipse, is shown in fig. 3.16. Equation (3.57) is used to estimate the impact of combined-function magnets in the CERN PS accelerator on the beam in section 6.3.2.

Some comments on theorem 3.11: the main idea of eq. (3.57) is to follow the tactics used for the Green function. One maps the problem of the domain of interest M onto the unit disk. The Neumann function is not a conformal invariant, because the normal derivative along the boundary ∂M onto the circle picks up a scaling factor $|\partial F|$ as shown in fig. 3.13. This is corrected by the second term in eq. (3.57), the boundary integral (derived via eq. (3.56) - Dini's formula) and therefore the normal derivative satisfies the third condition in definition 3.6. One advantage of this form is that the boundary integral is independent of the source point. The approach has several consequences; for example, the first term can be calculated easily if the position of the source is varied, while the second more complicated term is not changed and could be stored in a look-up table. Other more important consequences as a form conformal invariance are presented in the following.

Fig. 3.15 Example of the novel integral representation of a limiting angle on a bounded domain.

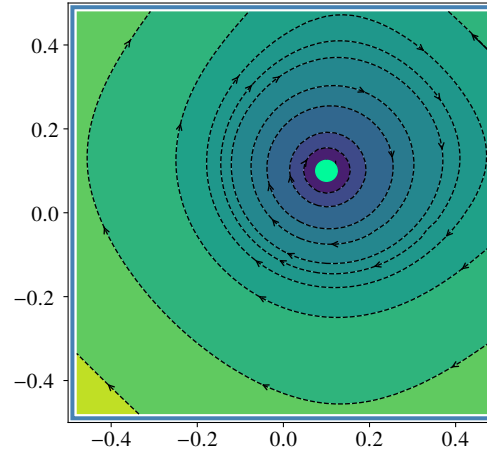
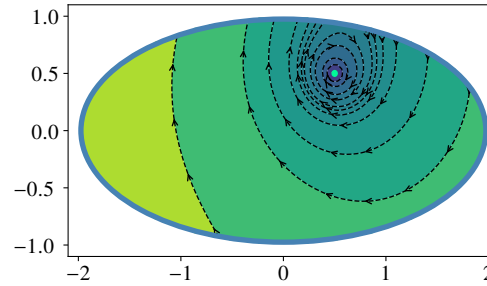


Fig. 3.16 Example of the novel integral representation of a smooth domain, non circular - using eq. (3.57).



3.5.3 First Consequence of eq. (3.57): the Neumann Function as a Conformal Invariant

Lemma 3.3. *The difference of two Neumann functions is a conformal invariant.*

Proof. Starting from theorem 3.11, we subtract two Neumann functions having two sources at different locations:

$$\begin{aligned}
 n_M(z, \zeta_1, \zeta_2) &:= n_M(z, \zeta_1) - n_M(z, \zeta_2) = \\
 &\frac{1}{2\pi} \left(\log |(F(z) - F(\zeta_1))(1 - F(z) F^*(\zeta_1))| \right. \\
 &\quad \left. - \frac{2}{\pi} \oint_{|\tau|=1} |d\tau| \log |1 - F(z)\tau^*| \left(1 - \frac{|\partial F^{-1}(\tau)|}{|\partial M|} \right) - \right.
 \end{aligned}$$

$$\begin{aligned}
& \log |(F(z) - F(\zeta_2))(1 - F(z)F^*(\zeta_2))| \\
& \quad - \frac{2}{\pi} \oint_{|\tau|=1} |\mathrm{d}\tau| \log \left| 1 - F(z)\tau^* \left(1 - \frac{|\partial F^{-1}(\tau)|}{|\partial M|} \right) \right| \\
& = \frac{1}{2\pi} \log \left| \frac{(F(z) - F(\zeta_1))(1 - F(z)F^*(\zeta_1))}{(F(z) - F(\zeta_2))(1 - F(z)F^*(\zeta_2))} \right|. \tag{3.60}
\end{aligned}$$

□

We found a conformal invariant version of the Neumann function. The boundary integrals vanish and hence it also follows: $\mathrm{Im} \, \mathbf{t} \partial n_M(z, \zeta_1, \zeta_2) = 0$. We know from section 3.3.1, that the tangential component of the field is continuous. As a consequence, the field vanishes in the M surrounding material. The situation of the circular shape is shown in fig. 3.12 for an experimental setting.

Let F map M , which is a simply-connected domain, conformally onto $\mathbb{D}$. ζ_1 and ζ_2 are different points in M (The case if a point touches the boundary is discussed in section 3.5.5.).

$$n_M(z, \zeta_1, \zeta_2) = \frac{1}{2\pi} \log \left| \frac{(F(z) - F(\zeta_1))(1 - F(z)F^*(\zeta_1))}{(F(z) - F(\zeta_2))(1 - F(z)F^*(\zeta_2))} \right| \tag{3.61}$$

is the pendant of the Green function of the first kind. The real part of the Green function of the first kind vanishes along the boundary of a domain (or is constant), the imaginary part of the conformal invariant Neumann function vanishes along a boundary (or is constant). For such cases we find always the closed-form if the conformal mapping can be expressed through closed-forms as eg. the circle fig. 3.12, the ellipse fig. 3.17b and the rectangle fig. 3.17a.

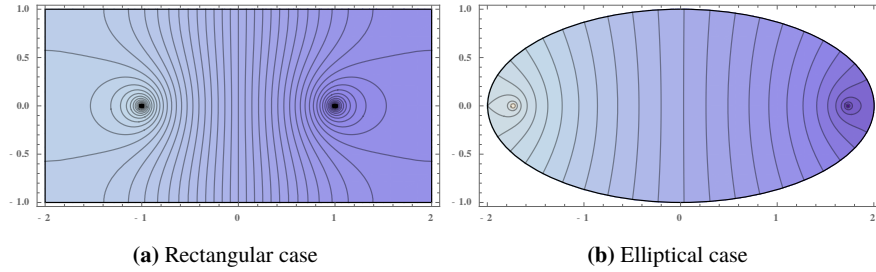


Fig. 3.17: The invariant representation of the Neumann function eq. (3.60) in the case of a rectangular and elliptical boundary.

3.5.4 Second Consequence of eq. (3.57): the Neumann Function for Unbounded Regulated Simply-connected Domains

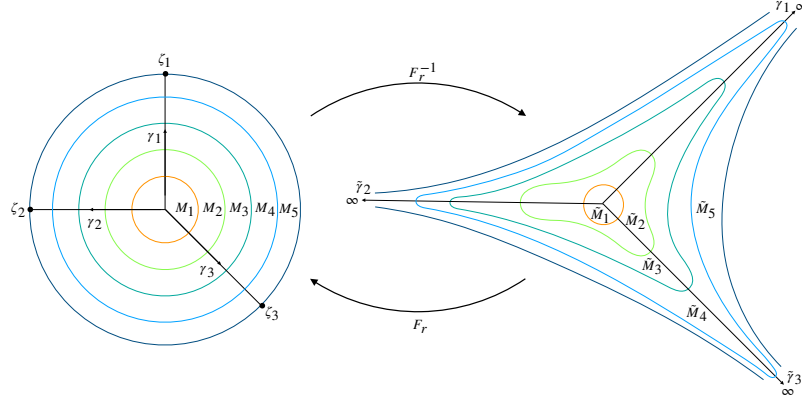


Fig. 3.18: An unbounded star-like domain M with three openings to the north-pole.

Let M be a *star-like regulated unbounded domain* on $\bar{\mathbb{C}}$. M touches the north-pole from N different directions, which is sketched in fig. 3.24 for $N = 2, 4, 6$. We term this domains regulated domain with a degeneracy N . ϕ_k denotes the argument of the rays, along which the north-pole can be reached from within M , where $k \in \{0, 1, 2, \dots, N\}$. Another example is drawn in fig. 3.18 for three openings to the north-pole, where also the concept of radial-exhaustion is shown: The mapping $F_r^{-1}(z)$ maps circles with different radii r ($M_1 \dots M_4$, with different colours) onto different domains ($\tilde{M}_1 \dots \tilde{M}_4$, with different colours). The boundary $\partial \tilde{M}_n$ increases with r , and in the limit $r \rightarrow 1$ the boundary touches the north-pole from three different directions, which is shown by the different paths ($\tilde{\gamma}_1, \tilde{\gamma}_2$ and $\tilde{\gamma}_3$). The corresponding paths (γ_1, γ_2 and γ_3) touch the unit disc $\mathbb{T}$ along the rays with the argument ϕ_1, ϕ_2 and ϕ_3 . Hence, $\lim_{r \rightarrow 1} \frac{1}{F_r^{-1}(re^{i\phi_k})} = 0$.

Conjecture 3.1. Let F map M_N , which is an unbounded star-like domain with a degeneration of degree N , conformally onto $\mathbb{D}$. Additionally it holds: $\lim_{r \rightarrow 1} \frac{1}{F_r^{-1}(re^{i\phi_k})} = 0$ (the north-pole can be reached along N rays with the argument ϕ_k). The Neumann function reveals the form:

$$n_{M_N}(z, \zeta) = \frac{1}{2\pi} \log \frac{|(F(z) - F(\zeta))(1 - F(z)F^*(\zeta))|}{\prod_k^N |1 - F(z)e^{-i\phi_k}|^{\frac{2}{N}}}. \quad (3.62)$$

Proof (Sketch). Let us start form the integral representation for bounded domain theorem 3.11 (omitting the normalization):

$$\begin{aligned} n_{M_r}(z, \zeta) = & \underbrace{\log |(F_r(z) - F_r(\zeta))(1 - F_r(z)F_r^*(\zeta))|}_{\textcircled{1}} \\ & - \underbrace{\frac{2}{\pi} \oint_{|\tau|=1} |d\tau| \log |1 - F_r(z)\tau^*|}_{\textcircled{2}} + \underbrace{\frac{2}{\pi} \oint_{|\tau|=1} |d\tau| \log |1 - F_r(z)\tau^*| \frac{|\partial F_r^{-1}(\tau)|}{|\partial M_r|}}_{\textcircled{3}} \end{aligned}$$

Integral ②: $\oint_{|\tau|=1} |d\tau| \log |1 - F_r(z)\tau^*| = 0$ for $|F_r(z)| < 1$ for each r ([35]). Integral ③ has the form:

$$\textcircled{3}_r = \frac{2}{\pi} \oint_{|\tau|=1} |d\tau| \underbrace{\log |1 - F_r(z)\tau^*|}_{:=\phi(\tau)} \underbrace{\frac{|\partial F_r^{-1}(\tau)|}{|\partial M_r|}}_{:=K_r(\tau)}$$

- $\phi(\tau)$ is a continuous bounded function in M_r for $|F_r(z)| \leq 1$.
- $K_r(\tau)$ can be brought to the form of Dirac sequences appendix B.

In the limit $\lim_{r \rightarrow 1} K_r(\tau) = \frac{1}{N} \sum_k^N \delta(\tau - e^{i\phi_k})$, which means the points touching the north-pole on $\bar{\mathbb{C}}$ produce a sink like singularity:

$$\begin{aligned} \lim_{r \rightarrow 1} \textcircled{3}_r &= \frac{2}{\pi} \oint_{|\tau|=1} |d\tau| \log |1 - F(z)\tau^*| \frac{1}{N} \sum_k^N \delta(\tau - e^{i\phi_k}) \\ &= \frac{2}{N} \sum_k^N \log |1 - F(z)e^{-i\phi_k}| \end{aligned}$$

□

This form of the Neumann function allows us to estimate the influence of boundaries of specific shapes, by opening a bounded domain to infinity at specific locations, yielding a *closed-form* (eq. (3.62)).

This representation makes it possible to give closed-form solutions for the n -pole family (as shown in the next chapter in fig. 4.7), as given in section 4.3.2 and is used to derive closed-forms for the combined-function magnets in the CERN PS accelerator, section 6.3.2.

3.5.5 Some Comments on the Source Boundary Behaviour

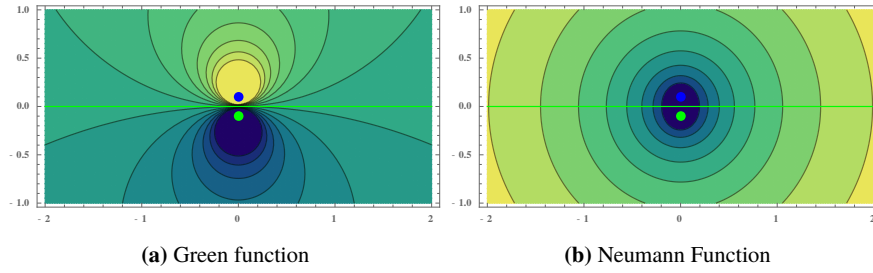


Fig. 3.19: The reflection of the fundamental singularity near the boundary of the Green and the Neumann function.

The logarithmic singularities generate the fields and in appendix D.1, it was shown that the indirect field can be interpreted as a reflection of the source singularity eq. (3.13). A look at the solutions of the Green functions eqs. (3.44) and (3.57) tells us:

$$g_{\mathbb{D}}(z, \zeta) = \frac{1}{2\pi} \log \left| \frac{z - \zeta}{1 - z\zeta^*} \right|, \quad (3.63)$$

and:

$$n_{\mathbb{D}}(z, \zeta) = \frac{1}{2\pi} \log |(z - \zeta)(1 - z\zeta^*)|. \quad (3.64)$$

If the source singularity approaches the boundary $\zeta \rightarrow \mathbb{T}$, the first equation vanishes while the second yields a logarithmic singularity of order two when $z \rightarrow \zeta$. It is also true for general domains since the fundamental singularity is invariant under conformal mappings. For g , the image is negative, and in the case of n , it is positive. In section 5.2 this terminology is used to formulate the image tensors.

An illustration of a source near a boundary provided in fig. 3.19. A green line indicates the boundary and the green point marks the source and the blue point, its reflection at the boundary. An extended definition of the conformal invariant version of the Neumann function allowing the source location be on the boundary can be found in [26] and is not further discussed here for the limited scope of this work. A last remark: a boundary point can be mapped conformally onto an arbitrary other boundary point. This is also true for every inner point.

3.6 A New Classification of the Neumann function on the Riemann-sphere

We classify the Neumann function for some instances of regulated unbounded domains and exterior domains. As the *north-pole* is contained in these domains, it naturally generates a sink, the equipotential lines and consequently the field lines are perpendicular to the boundaries. Two cases are distinguished if the domain is unbounded. The boundary touches the north-pole and is either degenerated or non-degenerated. If the domain is degenerated, the mapping of the north-pole, which has a logarithmic singularity of order two at the boundary (section 3.5.5), onto $\mathbb{T}$, is not univalent and belongs to a special subclass of regulated domains on the Riemann-sphere. The discussion in this work is restricted to unbounded star-like domains due to the limited scope, but could be extended to more general domains. Examples for every class are presented in the following with the focus on cross-section appearing in accelerator elements.

Various new closed-form solutions of the Neumann problem are offered throughout the text. Primarily, the explicit formulas are provided to act as a reference for more practical purposes, e.g. to implement them into a numerical tracking code.

Three different cases are distinguished:

- Domains M : $\infty \in \partial M$
 - ∞ non-degenerated: unbounded domain.
 - ∞ degenerated: unbounded star-like domain.
- Domains M : $\infty \in M$ - exterior domains - conformally equivalent to bounded domains.

In the following, examples for these cases are provided and the corresponding Neumann functions and fields are given explicitly.

3.6.1 Unbounded Domains: ∞ Non-degenerated

In this case, the boundary extends to infinity and is univalently defined there. The sink-singularity at the north-pole is "picked" up and consequently the function re-

veals a logarithmic singularity of order two at the north-pole. Nevertheless, a sink-singularity at infinity has no impact on the local field. As examples the half-plane, the half-strip and a square depression are provided in the following. The half-strip and the square-depression can be used to model the iron yoke of a dipole magnet.

3.6.1.1 The Upper Half-Plane

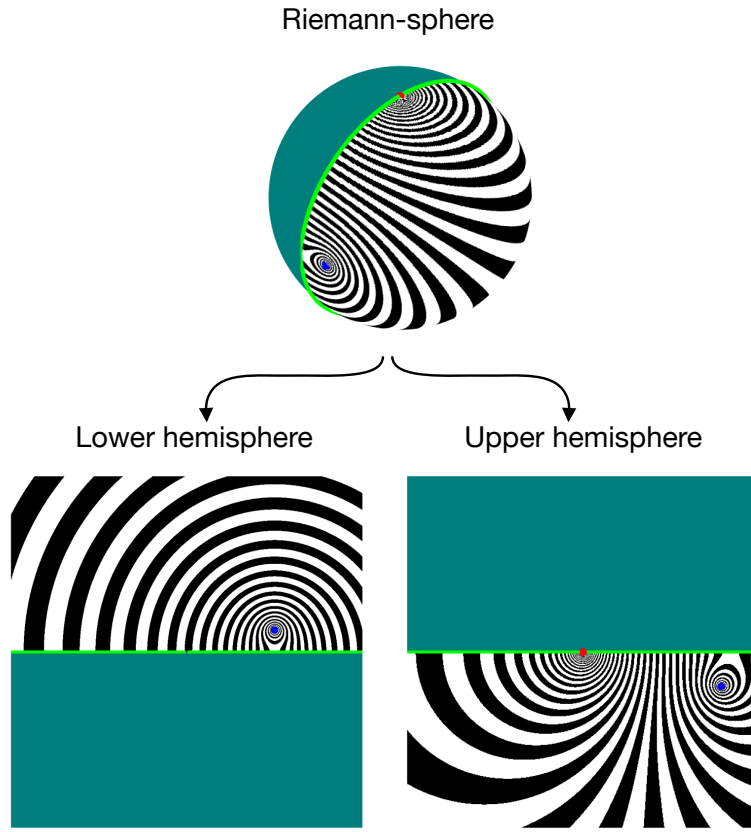


Fig. 3.20: The half-plane on Riemann-sphere and the projections on the two hemispheres.

The simplest example of an unbounded non degenerated regulated domain is the upper half plane $\mathbb{H}^+ := \{z | \text{Im}\{z\} \geq 0\}$. The Neumann function $N_{\mathbb{H}^+}$ of the upper half-plane can be calculated employing eq. (3.62) and the inverse Cayley mapping

(see section 4.1.1):

$$N_{\mathbb{H}^+}(z, \zeta) := \frac{1}{2\pi} \log(z - \zeta)(z - \zeta^*). \quad (3.65)$$

A visualisation on the Riemann-sphere is provided in fig. 3.20. The blue point indicates the location of the source. The sink like logarithmic singularity at the north-pole is shown in red. The boundary of $\mathbb{H}^+$ is green and the complementary domain $\mathbb{H}^-$ is dark green. One sees the projections on the lower hemisphere, which is the well known physical plane. The projection on the upper hemisphere (lower right plot) shows the north-pole touching the boundary. Using this interpretation makes it evident why the field lines, indicated as alternating black and white regions, are perpendicular to the boundary in unbounded domains.

3.6.1.2 The Half-Strip

If approximating, for example, the solution of a C-shaped dipole magnet it might be useful to use a closed-form expression - the half-strip - to model the cross-section of the iron yoke. The mapping of the half-strip $S_{\frac{1}{2}, 2b} := \{z | |\operatorname{Im} z| \leq b \wedge \operatorname{Re} \{z\} \geq 0\}$ with the width $2b$ parallel to the x -axis (vertices of the polygon at $(ib, -ib, \infty)$) onto $\mathbb{T}$ is (see eq. (4.9)): $z \mapsto 1 - \frac{2}{1 + \sinh \frac{\pi z}{2b}}$, hence the Neumann function using eq. (3.62) is:

$$N_{S_{\frac{1}{2}, 2b}}(z, \zeta) = \frac{1}{2\pi} \log \left[\left(\sinh \frac{\pi z}{2b} - \sinh \frac{\pi \zeta}{2b} \right) \left(\sinh \frac{\pi z}{2b} + \sinh \frac{\pi \zeta^*}{2b} \right) \right]. \quad (3.66)$$

A plot of this function is shown in fig. 3.21 on the Riemann-sphere. The graphics is similar to fig. 3.20 which was already explained in section 3.6.1.1. One sees that the two parallel lines of the boundary meet at the north-pole, where a sink like logarithmic singularity is located. The complex magnetic field is according to eq. (3.38) (cgs - units):

$$H_{S_{\frac{1}{2}, 2b}}(z, \zeta) = -i \frac{\beta_0 \lambda}{b} \left(\frac{\cosh \frac{\pi z}{2b}}{\sinh \frac{\pi z}{2b} + \sinh \frac{\pi \zeta^*}{2b}} + \frac{\cosh \frac{\pi \zeta}{2b}}{\sinh \frac{\pi z}{2b} - \sinh \frac{\pi \zeta}{2b}} \right)^*. \quad (3.67)$$

3.6.1.3 A Square Depression

The next step in complexity of approaching the example of a C-shaped magnet could be done by a so-called depression. To approximate the C-shape C_a (fig. 3.22b) with help of special functions, the points A ($-A$) and B ($-B$) are shifted to ∞ giving a polygon with $n = 4$ points (fig. 3.22a). The elliptic function $z \mapsto \frac{E(m^2)}{a'} E(\sin^{-1} z | m^2)$ maps the upper half-plane onto this shape ([98] p.17, (eq.119.02)). The parameter m has to be found numerically via: $\frac{E'(m^2)}{a'} = \frac{K'(m^2) - E'(m^2)}{b'}$. $E(z | m^2)$ denotes the

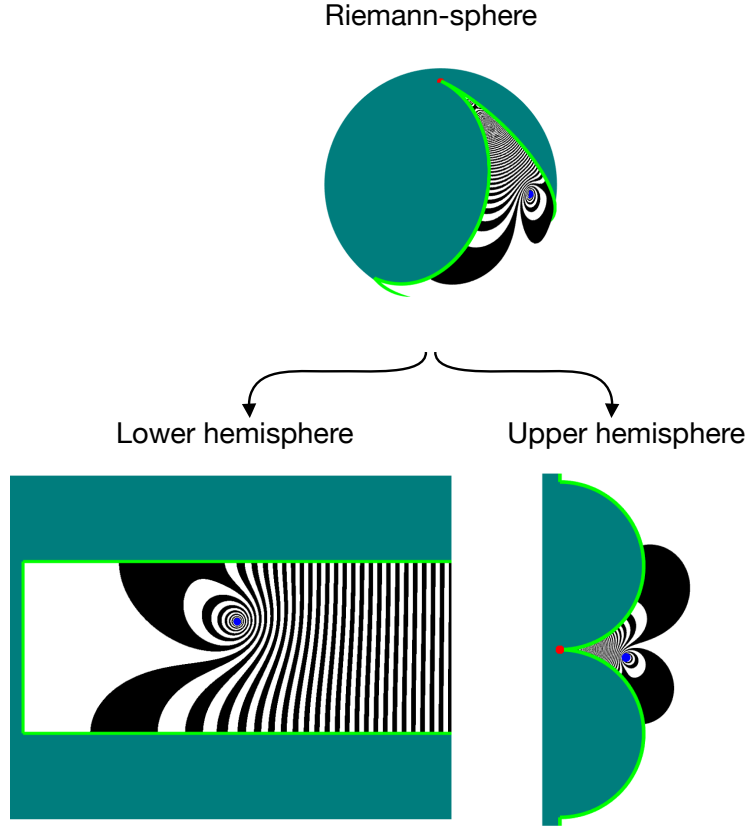


Fig. 3.21: The half-strip on Riemann-sphere and the projections on the two hemispheres.

elliptic integral of the second kind. $K(m^2)$ and $E(m^2)$ denote the complete elliptic integrals of the first and second kind, respectively. The term $E'(m^2) := E(1 - m^2)$ and $K'(m^2) := K(1 - m^2)$. Definitions of the used elliptic functions are given in appendices C.1.1 and C.1.2. The Neumann function has the form:

$$N_{C_a}(z, \zeta) := \frac{1}{2\pi} \log(\tilde{z} - \tilde{\zeta})(\tilde{z} - \tilde{\zeta}^*), \quad (3.68)$$

with the abbreviations $\tilde{z} = \sin E^{-1}(\frac{a}{E}z|m^2)$ and $\tilde{\zeta} = \sin E^{-1}(\frac{a}{E}\zeta|m^2)$. A visualisation can be seen in fig. 3.23 (explanations in section 3.6.1.1). The corresponding field is eq. (3.38) (cgs - units):

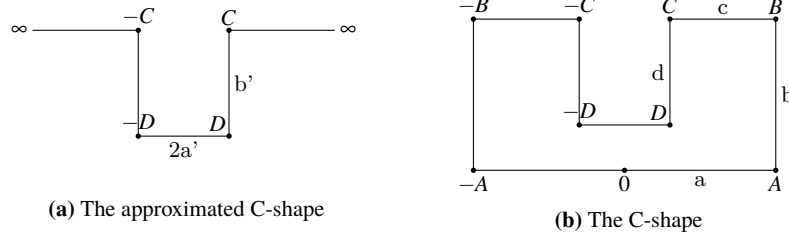


Fig. 3.22: The C-shaped polygon to model the cross-section of a dipole.

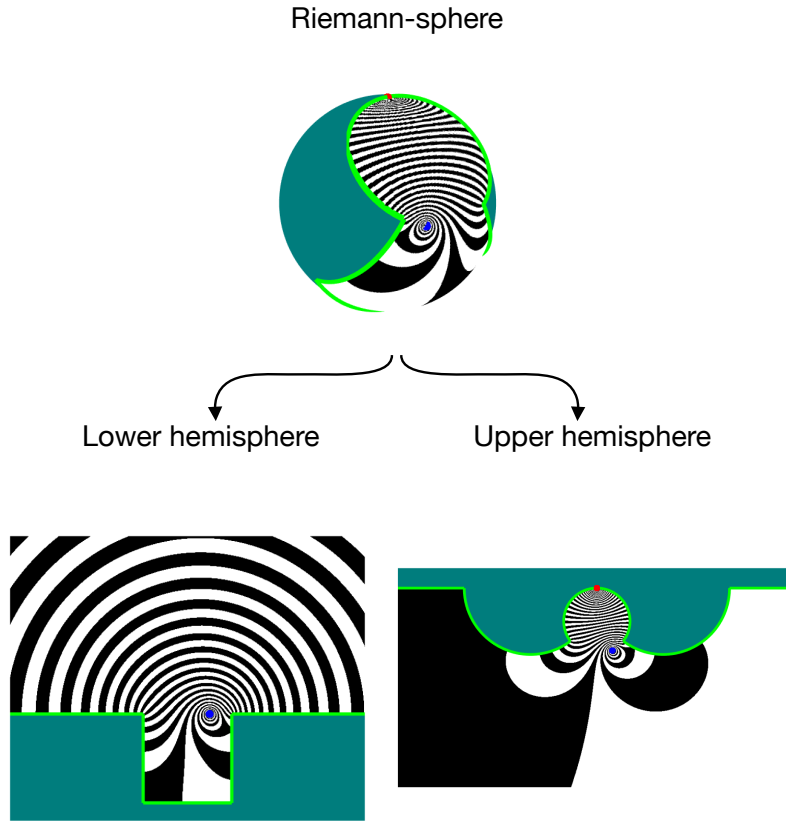


Fig. 3.23: A depression on Riemann-sphere and the projections on the two hemispheres.

$$H_{C_a}(z, \zeta) := -2i\beta_0\lambda \left(\frac{2\sqrt{\tilde{z}^2 + 1}(\tilde{z} - \operatorname{Re}\{\tilde{\zeta}\})}{\sqrt{1 - m^2\tilde{z}^2}[\tilde{\zeta}^2 + \tilde{z}(\tilde{z} - 2\operatorname{Re}\{\tilde{\zeta}\})]} \right)^*. \quad (3.69)$$

3.6.2 Regulated Domains: ∞ Degenerated

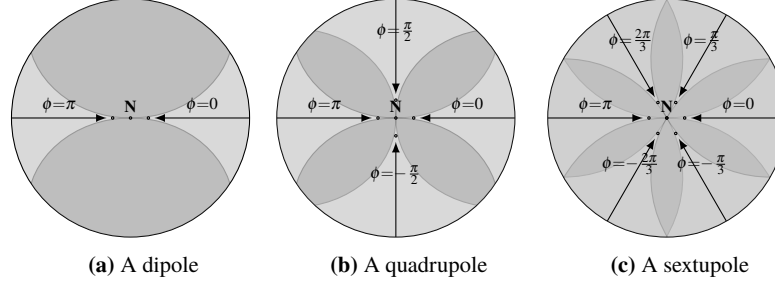


Fig. 3.24: The Riemann-sphere seen from above.

It is possible that the star-like domain M of interest is degenerated at ∞ , which means that on the compactified complex plane $\bar{\mathbb{C}}$ the complement of M , $\bar{\mathbb{C}} \setminus M$ is not simply connected. In such cases, it is necessary to investigate the behaviour at the point $\infty \in \partial M$. Of course, if such a domain is mapped onto $\mathbb{D}$, the mapping is not unique at the boundary at ∞ , which are termed regulated (section 3.2.3).

After deriving the solution eq. (3.62) based on the integral equation eq. (3.57), an alternative interpretation of the solution is now formulated:

We demand, following *Curie's principle* [99] stating that *symmetric premises* lead to *symmetric conclusions*, that the solution (the fields) of a physical symmetric configuration preserves this symmetry. Infinity can be reached along N rays from one centred point, the centre of symmetry (set to 0) as shown in fig. 3.24. Three different bird eyes views show a degeneracy of two, four and six, as occurring in the n -poles of increasing order. The domains are labelled as M_N , N indicating the degeneracy at infinity. A singularity is placed at the end (in the limit to infinity) of each ray approaching infinity as depicted in fig. 3.24 (k^{th} -ray with argument ϕ_k). We now construct a function F_k of the domain of interest where every "infinity"-ray k is mapped onto one on the unit circle, which can be always achieved by an interposition of a rotation (which is obviously conformal). Finally, after averaging the Neumann reveals the form:

$$N_{M_N}(z, \zeta) = \frac{1}{N} \sum_k \frac{1}{2\pi} \log \frac{(F_k(z) - F_k(\zeta))(1 - F_k(z)\bar{F}_k(\zeta))}{(1 - F_k(z)e^{-ik\phi_k})^2}. \quad (3.70)$$

Equation (3.70), as mentioned, was previously derived in a formal way. Nevertheless, following the argumentation above, the construction of the Neumann function for the class of the n -pole problem is fairly straightforward.

Next, this method is demonstrated for the strip, where the degeneration $k = 2$:

3.6.2.1 The Strip

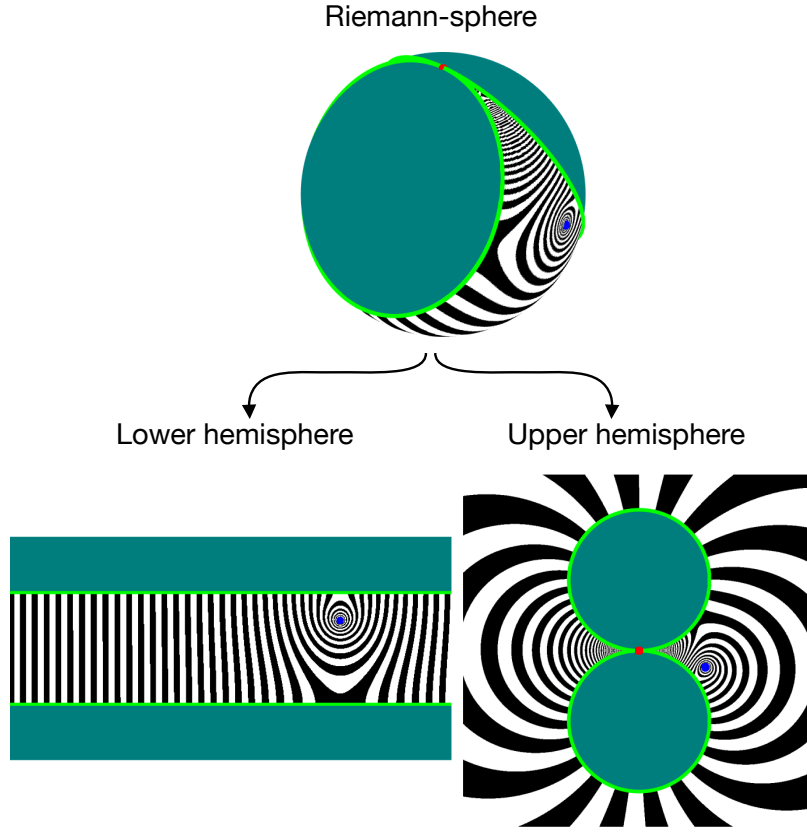


Fig. 3.25: The strip on Riemann-sphere and the projections on the two hemispheres.

The transformation $z \mapsto F_1(z) = \tanh \frac{\pi z}{4d}$ (eq. (4.7)) maps a strip S_d of width $2d$, parallel to the real axes (symmetry axis equals real axis) onto the unit disk. One observes $\infty \mapsto 1$. ∞ can be reached along the rays $z(t) = e^{i\phi_1 t}$ with $\phi_1 = 0$ and $\phi_2 = \pi$. This is depicted in fig. 3.24 projected on the Riemann-sphere. Rotation the circle by $\phi_2 = \pi$ yields: $z \mapsto F_2(z) = -\tanh \frac{\pi z}{4d}$, hence $-\infty \mapsto 1$. After averaging (using eq. (3.70)) and some algebra we arrive at:

$$N_{S_d}(z, \zeta) = \frac{1}{2\pi} \log \left[\sinh \frac{\pi(z - \zeta)}{4d} \cosh \frac{\pi(z - \zeta^*)}{4d} \right]. \quad (3.71)$$

The field according to eq. (3.38) is (cgs - units):

$$H_{S_d}(z, \zeta) = -\frac{i\beta\lambda}{2d} \left[\tanh \frac{\pi(z - \zeta^*)}{4d} + \coth \frac{\pi(z - \zeta)}{4d} \right]^* . \quad (3.72)$$

This is in agreement with the result obtained by the method of images (eq. (D.11)). We continue with the next member of the n -pole class, the quadrupole.

A visualization of this Neumann function can be found in fig. 3.25. One sees the invariant nature of circles under Moebius transformations on the Riemann-sphere in the projection onto the upper hemisphere (lower right plot). Two circles touch each other at the north-pole (red point). The complementary domain of the strip is not simply-connected and eq. (3.62) yields the correct result.

3.6.2.2 The Quadrupolar Structure

The quadrupole structure S^2_d has four directions ($k = 4$) pointing to infinity. $z \mapsto F_1(z) = \sqrt{\tanh \frac{\pi z^2}{4d}} \operatorname{sgn} \operatorname{Re} \{z\}$ (see eq. (4.44)) maps this structure onto the unit disk and $\infty \mapsto 1$. We have to rotate by i three times yielding: $F_2(z) = iF_1(z)$, $F_3(z) = -F_1(z)$ and $F_4(z) = -iF_1(z)$. Employing eq. (3.70) and again executing some algebra yields ($\operatorname{Re} \{\zeta\} > 0$):

$$N_{S^2_d}(z, \zeta) = \frac{1}{2\pi} \log \cosh \frac{\pi z^2}{4d} \left[\sqrt{\tanh \frac{\pi z^2}{4d}} \operatorname{sgn}(\operatorname{Re} \{z\}) - \sqrt{\tanh \frac{\pi \zeta^2}{4d}} \right] \left[\operatorname{sgn}(\operatorname{Re} \{z\}) \sqrt{\tanh \frac{\pi z^2}{4d} \tanh \frac{\pi (\zeta^*)^2}{4d}} - 1 \right] . \quad (3.73)$$

A visualization of this Neumann function is given in fig. 3.26 on the Riemann-sphere. The degeneracy of the complementary domain at the north-pole (red) can be seen in the projection onto the upper hemisphere (lower left plot). Using eq. (3.62) yields the correct result. To emphasize the difference to the Green function of the first kind, the Green function for the same configuration is shown in fig. 3.27. Here the north-pole has no special characteristics. The chess board styled representation shows the fact that the field lines are perpendicular to the boundary and to the potential lines.

3.6.2.3 The n -Pole Structures

In this case depending on the pole, the boundary reaches ∞ from several directions. For $k = 1, 2, 3$ as depicted in fig. 3.24. The general formula to map an n -pole S^n_d onto the circle is given in eq. (4.45).

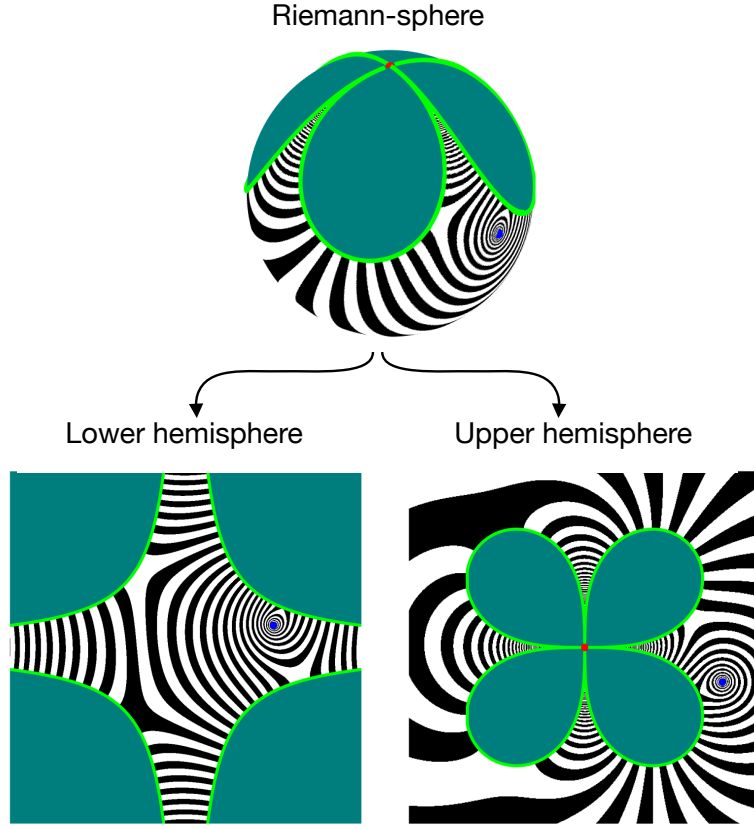


Fig. 3.26: The quadrupole on Riemann-sphere and the projections on the two hemispheres for the magnetostatic problem.

However it is possible to map the n -pole structure via the n^{th} power onto a strip, neglecting the *Riemann sheet* for symmetric charge configurations, hence the solution reflects the order of the pole. The symmetrized classical Neumann function of the n -pole structure is:

$$N_{S^n_d}(z, \zeta) = \frac{1}{2n\pi} \log \left[\sinh \frac{\pi(z^n - \zeta^n)}{4d} \cosh \frac{\pi(z^n - \zeta^{n*})}{4d} \right]. \quad (3.74)$$

This formula can be used for radial symmetric current distributions only and if the current is located at the origin, we get:

$$N_{S^n_d}(z, 0) = \frac{1}{2n\pi} \log \left[\sinh \frac{\pi z^n}{4d} \cosh \frac{\pi z^n}{4d} \right]. \quad (3.75)$$

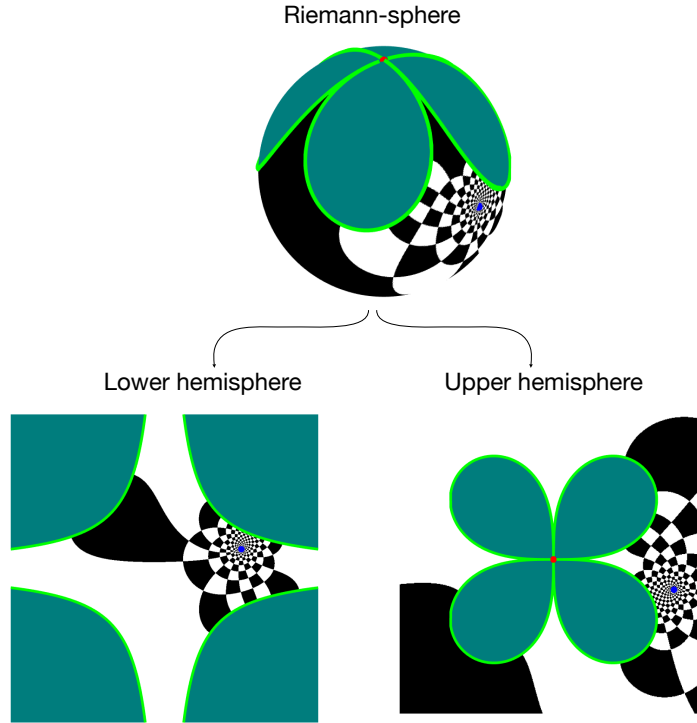


Fig. 3.27: The quadrupole on Riemann-sphere and the projections on the two hemispheres for the electrostatic problem.

The corresponding field is:

$$H_{S^n_d}(z) = -i \frac{\beta_0 \lambda n}{d} \left(z^{n-1} \coth \frac{\pi z^n}{d} \right)^* . \quad (3.76)$$

For off-axis source points the solutions become extremely tedious and are not further discussed due to the limited scope of this thesis. They can be handled using computer algebra systems. Notwithstanding, the full solution in the case of the quadrupole image coefficients are presented in section 5.3.7.

Figure 4.7 visualizes the Green function of the first and the second kind of the n -pole family for a non-symmetric charge distribution up to the 4th order, where all solutions are expressed as closed-forms.

3.6.3 Exterior Domains $M : \infty \in M$

When a domain contains the north-pole (the point ∞), an additional singularity originated in the compactification of the complex plane appears. For the case of the fundamental solution, as remarked in section 3.1.2, the "free" space case includes a sink at the north-pole and in fig. 3.10 (left) the solution of the "free" space can be seen. This property of the fundamental solution is true for all domains containing the north-pole. Interestingly, it naturally leads to the conformal invariant form of the Neumann function eq. (3.60), meaning the imaginary part of the function vanishes along the boundary, or differently said the field is perpendicular to exterior domains. Assume $F : M \mapsto \mathbb{D}$ ($M : \infty \in M$). It is to possible to find the solution via the inverse Cayley-transformation ([100] p.85, eq.(6.3.7.1)): $z \mapsto -\frac{i(z-1)}{z+1}$:

$$N_M(z, \zeta, z_1) = \frac{1}{2\pi} \log \frac{(F(z) - F(\zeta)) (F(z)F(\zeta)^* - 1)}{(F(z) - F(z_1)) (F(z)F(z_1)^* - 1)}. \quad (3.77)$$

If, as assumed, $\bar{\mathbb{C}} \setminus M$ is bounded, one always finds an $F(\infty) \stackrel{!}{=} 0$, hence:

Lemma 3.4. *The Neumann function of exterior domains M can written in the form:*

$$N_M(z, \zeta) = \frac{1}{2\pi} \log \frac{(F_\infty(z) - F_\infty(\zeta)) (F_\infty(z)F_\infty(\zeta)^* - 1)}{F_\infty(z)}, \quad (3.78)$$

with F_∞ mapping the north-pole to the origin, $\lim_{z \rightarrow \infty} F_\infty(z) = 0$.

The additional sink-singularity at the north-pole has no impact on the local field. Now, we apply this form of the Neumann function on the case of the C-shaped magnet. It is not possible to express the function in terms of closed-forms any-more, since a polygon with five vertices is involved, which is discussed in chapter 4.

3.6.3.1 The Full C-Shape Model

The transformation of $\mathbb{D}$ onto the outside of 'C-shaped'-polygon C , as depicted in fig. 3.22, can be found with help of the Schwarz-Christoffel-Transformation section 4.1:

$$F(z) = \alpha + \gamma \int^z \frac{\sqrt{z'^2 - 1} \sqrt{z'^2 - \zeta_2^2} \sqrt{z'^2 - \zeta_3^2}}{z'^2 \sqrt{z'^2 - \zeta_1^2}} dz'. \quad (3.79)$$

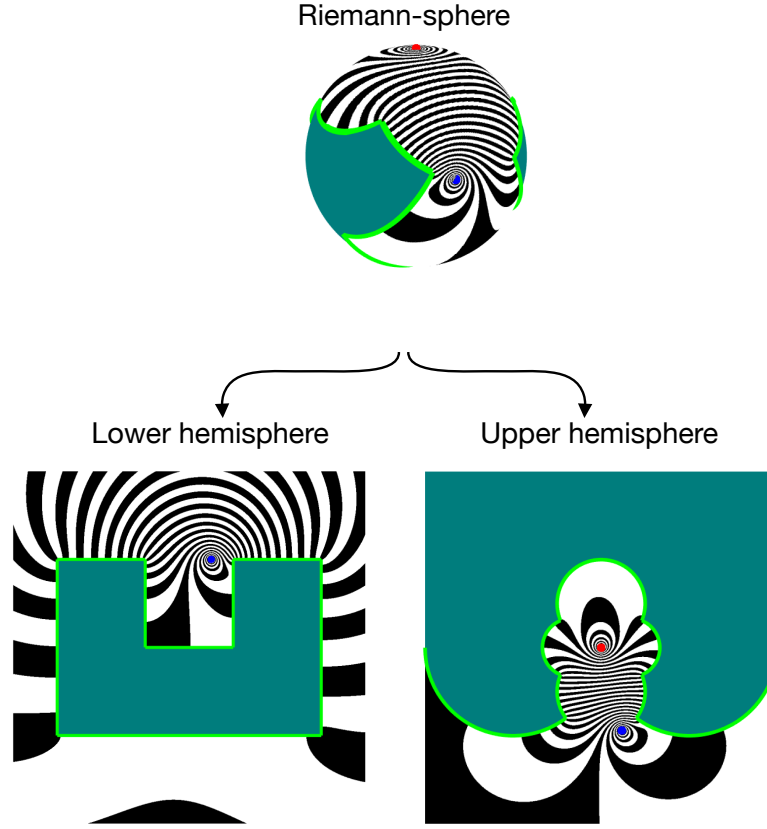


Fig. 3.28: The C-shaped magnet on Riemann-sphere and the projections on the two hemispheres as an example of an exterior domain for the Neumann function.

The parameters α, γ and $\zeta_1, \zeta_2, \zeta_3 \in \partial\mathbb{D}$ are determined numerically via (for simplicity w.l.o.g. a is set to 1.):

$$\frac{1}{2\zeta_1} = \frac{1}{2\zeta_2} + \frac{1}{2\zeta_3}, \quad F(\pm) = \pm A, \quad (3.80)$$

$$\frac{\left| \int_1^{\zeta_1} F'(z) dz \right|}{\left| \int_0^1 F'(z) dz \right|} = b, \quad \frac{\left| \int_{\zeta_1}^{\zeta_2} F'(z) dz \right|}{\left| \int_0^1 F'(z) dz \right|} = c, \quad \frac{\left| \int_{\zeta_2}^{\zeta_3} F'(z) dz \right|}{\left| \int_0^1 F'(z) dz \right|} = d. \quad (3.81)$$

The Neumann function can now be formally expressed as:

$$N_C(z, \zeta) = \frac{1}{2\pi} \log \frac{(F^{(-1)}(z) - F^{(-1)}(\zeta)) (F^{(-1)}(z) F^{(-1)}(\zeta)^* - 1)}{F^{(-1)}(z)}. \quad (3.82)$$

This function is plotted in fig. 3.28. The north-pole (red point) generates the sink-like logarithmic singularity and does not influence the local field (lower right plot). The sink can be moved to an arbitrary point in the domain. The corresponding field, again expressed formally, is:

$$H_C(z, \zeta) = -2i\beta_0\lambda \frac{(F^{(-1)}(z)^2 F^{(-1)}(\zeta)^* - F^{(-1)}(\zeta))}{F^{(-1)}(z) (F^{(-1)}(z) - F^{(-1)}(\zeta)) F' (F^{(-1)}(z)) (F^{(-1)}(z) F^{(-1)}(\zeta)^* - 1)}. \quad (3.83)$$

The method of approximating an exterior domain with the help of polygons is used in section 6.3.2 to give an approximative solution for the combined-function magnets in the CERN PS. In the next chapter, the explicit construction of conformal transformation of polygons onto $\mathbb{D}$ is discussed.

The difference to the Green function of the first kind is shown in fig. 3.27 for the same configuration. As already mentioned, here the north-pole has no special characteristics. The chess board styled representation indicates the field lines and potential lines of the Green function, which are perpendicular. The field is perpendicular to the boundary.

Summary: The *unbounded domains* concerning the *Neumann function* were classified into three categories:

- Domains M : $\infty \in \partial M$
 - ∞ *non-degenerated*: unbounded domain. The north-pole is mapped univalently onto $\mathbb{T}$. The standard mapping technique can be used.
 - ∞ *degenerated*: unbounded star-like domain. The north-pole is not mapped univalently onto $\mathbb{T}$. The new formulae eq. (3.62) has to be used.
- Domains M : $\infty \in M$ - *exterior domain* - naturally conformally invariant. The north-pole can be mapped univalently onto 0. Equation (3.78) can be used.

For all domains, an example was given focusing on the approximation of a *C-shaped magnet* and higher-order magnetic structures, which can be found in accelerators. The visualization of higher order degenerated domains is provided in fig. 4.7 for the n -pole family for the Green functions of the first and the second kind.

In all other cases the methods as described in sections 3.5.2 and 3.5.3 can be utilized.

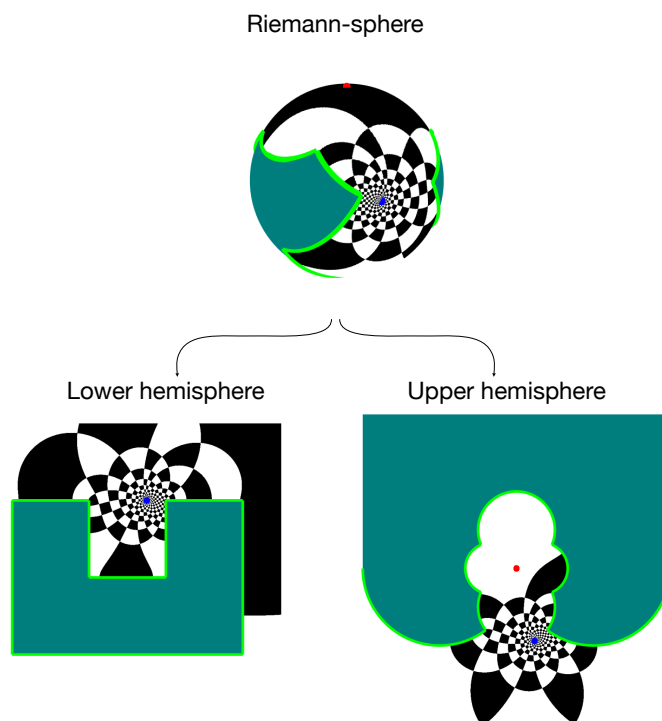


Fig. 3.29: The C-shaped magnet on Riemann-sphere and the projections on the two hemispheres as an example of an exterior domain for the Green function of the first kind.

Chapter 4

Conformal Mappings

*The theory of elliptic functions is the fairyland of mathematics.
The mathematician who once gazes upon this enchanting and
wondrous domain crowded with the most beautiful relations and
concepts is forever captivated.*

— Richard E. Bellman

Abstract In this chapter, we bestow some basic *conformal mappings*. Importantly, if found, conformal mappings allow for the boundary problem to be solved, and the indirect field problem for a specific geometry, in a simple form. In order to construct new mappings, it is possible to composite known mappings, since the composition is again conformal.

We use this property to construct more complicated shapes as the mapping of an ellipse onto $\mathbb{D}$. Although, as already mentioned, we know about the existence of mappings of simply-connected domains onto $\mathbb{D}$, through to the RMT, the theorem is not constructive. However methods exist, which allow us to obtain approximative solutions of the true domain if we cannot formulate the exact mapping. For polygonal shapes there exists a method termed *Schwarz-Christoffel-transformation* which is studied in this chapter in detail. We also attempt to find symmetric representations, which commute with the conjugation $F^*(z^*) = F(z)$, in order to simplify the analysis later on.

As specified in the introduction, these calculations are given in full detail since some discrepancies were found in the literature for simple cross-sections when formulating the effect of the indirect field through operators of the obtained mappings. For the ellipse, the mapping onto the upper-half plane is new and described explicitly. The n -pole mappings are used to reveal the novel n -pole Green functions.

Alternative approaches to find conformal mappings using numerical methods can be found, e.g. in [38–40, 101].

4.1 The Schwarz-Christoffel-Transformation

The method of Schwarz-Christoffel-transformation allows mapping polygons onto $\mathbb{D}$. The focus on Schwarz-Christoffel-transformation is based on the concept that the approximation of domains via polygons is relatively simple. Some basic mappings for simple cases, for which the so-called *parameter problem* (section 4.1.3) can be solved analytically, are demonstrated at the beginning. If the number of vertices of

the polygon is smaller than five, we can find closed-forms for the transformation. In all other cases, except for high symmetrical cases, no closed-form exists.

The mapping of a rectangle onto the unit circle involves special functions, namely *Jacobi elliptic functions*. Based on this result the mapping of the ellipse is formulated in turn with the Jacobi elliptic functions (section 4.3.1). The high symmetric case of the regular polygon leads to hypergeometric functions. One use-case of regular polygons is the *beam-screen* of the HL-LHC, which will consist partly of such geometries (section 7.5). Finally, the arbitrary polygon is addressed which cannot be expressed through closed-forms any more, is applied to approximate the LHC beam-screen (chapter 7).

4.1.1 Elementary Operations

Before we provide the details of the Schwarz-Christoffel-transformation, we introduce some basic transformations. The elementary operations as translation $z \mapsto z + d$, $d \in \mathbb{C}$, dilatation $z \mapsto az$, $a \in \mathbb{R}^+$, rotation $z \mapsto az$, $a \in \mathbb{T}$ and inversion $z \mapsto \frac{1}{z}$ are conformal on $\bar{\mathbb{C}}$. Using these operations, we are able to construct the *Moebius transformation* (a bilinear transformation) which is again conformal:

$$F(z) = \frac{az + b}{z + c} \quad a, b, c \in \mathbb{C} \quad ac \neq b. \quad (4.1)$$

The condition $ac \neq b$ ensures that the function is not constant. F maps "generalized" circles, as, e.g. the real axis, onto circles on $\bar{\mathbb{C}}$. Finding such a mapping involves three ordered points along the circle, which are mapped onto three points in the same order on the mapped circle. The inner of a circle is transported to the inner of the mapped circle and it is always on the left side of the boundary, following the three ordered points along this circle.

In the following, we extensively use these properties to symmetrize our transformations. Some examples are listed here.⁷:

- The mapping of the unit circle onto itself - an automorphism, with the point z_0 mapped onto the origin, has the form (proof [100] p.81 sec.6.2.2):

$$F_{z_0}(z) = \frac{z - z_0}{1 - z_0^* z}. \quad (4.2)$$

This has already been discussed as the Blaschke factor as depicted in figs. 3.9 and 3.10.

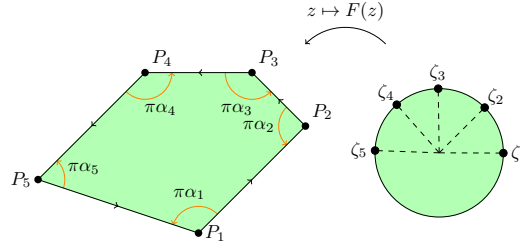
- The linear fractional transformation $z \mapsto \frac{i-z}{i+z}$ maps the upper half-plane $\{z : \text{Im } z > 0\}$ conformally onto $\mathbb{D}$. This function is called *Cayley transformation*.
- The exponential function maps strips parallel to the y -axis with width 2π periodically onto the complex plane ([103] ch2.6 pp.109-122). The strip S_d (where

⁷ For an exhaustive treatment consult [102].

d is width) is mapped onto the upper half-plane $\{z : \text{Im } z > 0\}$ via: $z \mapsto ie^{\frac{\pi z}{2d}}$. A composition with the Cayley transform yields: $z \mapsto \tanh \frac{\pi z}{4d}$, mapping S_d onto $\mathbb{D}$. $0 \mapsto 0$ and it commutes with the conjugation $F^*(z^*) = F(z)$. This result is also be derived employing the Schwarz-Christoffel-transformation (eq. (4.7)).

4.1.2 The Construction of the Schwarz-Christoffel-transformation

Fig. 4.1 The SCT maps a polygon onto the circle.



The Schwarz-Christoffel-transformation is a method to transform a circle (and here the real axis, which is a degenerated circle on $\mathbb{C}$) onto any polygon of the complex plane, conformally. The transformation is given as a first-order differential equation with a complex variable, where the *prevertices* (the image of these points are the vertices of the regarded polygon) are unknown. As previously discussed, a bilinear transformation - a Moebius transformation has three arbitrary parameters. There is the possibility to choose three prevertices freely and one needs to use numerical methods to find $n - 3$ prevertices if the number of vertices exceeds three. The general equation to map the upper plane onto the given polygon is (see [104], Ch.13, p.41):

$$dF^{-1} = \gamma \prod_{k=1}^n (\zeta - z_k)^{\alpha_k - 1} d\zeta. \quad (4.3)$$

Where γ is an arbitrary complex constant. α_k is the interior angle in multiples of π of the polygon at a vertex, where it is conventional to go along the polygon sides with the inner region of the polygon always to the left. It holds: $\sum_k (1 - \alpha_k) = 2\pi$ (the polygon is closed). As depicted in fig. 4.1, the vertices P_i $i \in \{1, 2, 3, 4, 5\}$ are mapped onto the prevertices ζ_i $i \in \{1, 2, 3, 4, 5\}$. The corresponding angles α_i $i \in \{1, 2, 3, 4, 5\}$ are also drawn. The transformation F^{-1} is unique if the angles α_k and the prevertices z_k are given, up to the integration constants (rotation and translation). The Schwarz-Christoffel formula of mapping the inner of the polygon onto the unit disc is (δ, γ denote the integration constants):

$$F^{-1}(z) = \delta + \gamma \int^z \prod_{k=1}^n \left(1 - \zeta e^{-i\varphi_k}\right)^{\alpha_k - 1} d\zeta. \quad (4.4)$$

φ_k are the angles of the prevertices on $\mathbb{D}$ (eq. (4.4) can be easily obtained employing a Moebius transformation of eq. (4.3)). For a given polygon, a set of vertices P_i and angles α_k , it is not straight forward to find the corresponding prevertices φ_k . It is called the *parameter problem* and how to solve it is subject of the following section.

4.1.3 The Parameter Problem

In [105], p.11, eq.(2.4) a method is given to solve the parameter problem. ζ_k are the pre-vertices (the unknown parameters) on the unit disc, belonging to the vertices P_k of the polygon. Following ([105], pp.24-25), one can pin down three points, $z_{n-2} = -1$, $z_{n-1} = -i$, $z_n = 1$. The remaining $n - 3$ parameters are given by:

$$\frac{\left| \int_{z_j}^{z_{j+1}} F^{-1'}(\zeta) d\zeta \right|}{\left| \int_{z_1}^{z_2} F^{-1'}(\zeta) d\zeta \right|} = \frac{|P_{j+1} - P_j|}{|P_2 - P_1|}, \quad j = 2, 3, \dots, n-2, \quad (4.5)$$

where $F^{-1'}$ is given by eq. (4.4). If we map onto the upper plane (eq. (4.3)), of course the three arbitrary preimages must to be chosen to lie on the real axis. In the following the parameter problem is solved exactly for several simple domains, while the general problem involves solving the non-linear system of equations eq. (4.5).

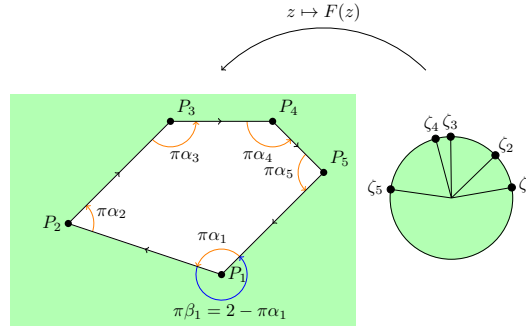
4.1.4 A Variant of the Schwarz-Christoffel-transformation

It is also possible to map the unit disc onto the *exterior* of a polygon. The formula is (see e.g.: [30] p.353, eq.(5.6.3c)):

$$F^{-1}(z) = \delta + \gamma \int \frac{1}{(1+z^2)^2} \prod_{k=1}^n (\zeta - z_k)^{1-\alpha_k} d\zeta. \quad (4.6)$$

which is sketched in fig. 4.2. The direction of the vertices P_i is clockwise, since the inner domain - per convention - is on the left side of the edge. β_i (in orange) indicates the inner angle, which is replaced by $2 - \alpha_i$ (in blue) in eq. (4.6). Note that for an exterior domain of a polygon also the exterior angles $\beta_k = 2 - \alpha_k$ have to be taken $\alpha_k - 1 \rightarrow 1 - \alpha_k$. The used symbols are defined in section 4.1.

Fig. 4.2 The exterior Schwarz-Christoffel-transformation maps an exterior polygon onto the circle.



4.2 Polygonal solutions

The first addressed shapes are the strip, having two vertices on $\mathbb{C}$ and one on $\bar{\mathbb{C}}$ at the north-pole and the half-strip with three vertices. These domains are regulated domains as classified in section 3.2.3.

4.2.1 The Strip and the Half-Strip

The Strip

In view of eq. (4.4) we obtain for the strip ($n = 2$) of width $2b$, mapping the two vertices at infinity ($\alpha = 0!$) onto ± 1 on the unit circle:

$$F^{-1}(z) = \delta + \gamma \int^z \frac{1}{(1-\zeta)(1+\zeta)} d\zeta = \delta + \gamma \tanh^{-1} z. \quad (4.7)$$

Demanding $F(ib) = i$ ($\gamma = \frac{4b}{\pi}$) and $F(-ib) = -i$ ($\gamma = \frac{4b}{\pi}$), (which fixes $\gamma = \frac{\pi}{4b}$ and $\delta = 0$) we get the result of section 4.1.1: $z \mapsto \tanh \frac{\pi z}{4b}$, which is symmetric w.r.t. the imaginary axis as already stated.

The Half-Strip

For the half-strip ($n = 3$) of width $2b$ the vertices are: $\{-ib, ib, \infty\}$. Two angles are $\alpha = 1/2$ and one at infinity is 0. Using now eq. (4.3) and mapping the two vertices to $\pm i$:

$$F^{-1}(z) = \delta + \gamma \int^z \frac{1}{\sqrt{(1-i\zeta)(1+i\zeta)}} d\zeta = \delta + \gamma \sinh^{-1} z. \quad (4.8)$$

Demanding $F(0) = 0$ ($\delta = 0$) and $F(ib) = i$ ($\gamma = \frac{2b}{\pi}$) we get $z \mapsto \sinh \frac{\pi z}{2b}$.

Symmetrizing

With help of the Cayley transform and a specific rotation the mapping onto the unit circle is:

$$F(z) = 1 - \frac{2}{1 + \sinh \frac{\pi z}{2b}}. \quad (4.9)$$

The function commutes with the conjugation: $F^*(z^*) = F(z)$.

4.2.2 The Rectangle

In the rectangular case, we obtain solutions in terms of elliptic functions. The derivation is presented in more detail since it is also used when mapping an ellipse onto the circle. Additionally, the calculation of the image tensors of the rectangle is delivered in section 5.3.1.

Equation (4.3) is applied in the case of a rectangle with the side lengths a and b . Assuming the lower side of the rectangle with length a coincidences with the real axes, whereas the short side (side length b) lie symmetric to the imaginary axis. The first vertex has the coordinate $p_1 = a/2$, the second $p_2 = a/2 + ib$ and the third p_3 and the forth p_4 are at $-a/2 + ib$ and $-a/2$, respectively. Since we can choose three points arbitrarily, we set $F^{-1}(0) = 0$, $F^{-1}(1) = p_1$ and $F^{-1}(1/m) = p_2$, $m \in (0, 1]$. Due to symmetry reasons we know $F^{-1}(-1) = p_4$ and $F^{-1}(-1/m) = p_3$. The angles $\alpha_1, \dots, \alpha_4$ are 0.5. Putting this into eq. (4.3) yields:

$$\begin{aligned} F^{-1} &= \frac{\gamma}{\sqrt{(\zeta^2 - 1)(\zeta^2 - \frac{1}{m^2})}} d\zeta = \frac{\tilde{\gamma}}{\sqrt{(1 - \zeta^2)(1 - (m\zeta)^2)}} d\zeta \\ F^{-1} &= \frac{\tilde{\gamma}}{\sqrt{(1 - (m \sin(\phi))^2)}} d\phi \quad (\text{using: } \zeta := \sin(\phi)) \end{aligned} \quad (4.10)$$

$\tilde{\gamma}$ denotes a negligible rescaling in the integration constant γ . Integrating eq. (4.10) leads to an elliptic integral of the first kind and involves the parameter m appendix C.1.1. It can be found if we use eq. (4.5):

$$\frac{|p_2 - p_1|}{|p_1 - p_4|} = \frac{b}{a} = \frac{\left| \int_{\frac{\pi}{2}}^{\arcsin(1/m)} \frac{1}{\sqrt{(1 - (m \sin(\phi))^2)}} d\phi \right|}{\left| \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\sqrt{(1 - (m \sin(\phi))^2)}} d\phi \right|} \quad (4.11)$$

$$= \frac{\left| \int_{\frac{\pi}{2}}^{\arcsin(1/m)} \frac{1}{\sqrt{(1-(m \sin(\phi))^2)}} d\phi \right|}{2 \left| \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{(1-(m \sin(\phi))^2)}} d\phi \right|} \quad (4.12)$$

To simplify the numerator we substitute ϕ by:

$$\sin \phi = \frac{1}{\sqrt{1-m'^2 \sin^2 \phi'}} \implies d\phi = d\phi' \frac{-im' \cos \phi}{\sqrt{1-m'^2 \sin^2 \phi'}^3} \quad (4.13)$$

With $m' = \sqrt{1-m^2}$. The limits of the transformed integral are:

$$\phi'(1/m) = \arcsin \frac{\sqrt{1-m^2}}{m'} = \arcsin 1 = \frac{\pi}{2}, \quad \phi'(0) = 0 \quad (4.14)$$

To insure the unique inverse transformation, we restricted arcsin to its principal value. The numerator becomes:

$$\int_{\frac{\pi}{2}}^{\arcsin(1/m)} \frac{1}{\sqrt{(1-(m \sin(\phi))^2)}} d\phi = \quad (4.15)$$

$$\int_0^{\frac{\pi}{2}} d\phi' \frac{-im' \cos \phi}{\sqrt{1-m'^2 \sin^2 \phi'}^3} \frac{\sqrt{1-m'^2 \sin^2 \phi'}}{\sqrt{(m' \sin(\phi))^2 - m^2}} = -i \int_0^{\frac{\pi}{2}} d\phi' \frac{1}{\sqrt{1-m'^2 \sin^2 \phi'}}. \quad (4.16)$$

Hence we obtain the equation, which implicitly gives the parameter m :

$$\frac{2b}{a} = \frac{\left| \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{(1-(m' \sin(\phi))^2)}} d\phi \right|}{\left| \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{(1-(m \sin(\phi))^2)}} d\phi \right|} = \frac{K(1-m^2)}{K(m^2)}. \quad (4.17)$$

$K(m^2)$ is the *complete elliptic integral* of the first kind. The parameter m has to be found numerically (the dependency on a and b is plotted in appendix C.1.3). We see immediately if $2a = b \implies m = 1/\sqrt{2}$, $\lim_{a \rightarrow \infty} \implies m \rightarrow 1$ and $\lim_{b \rightarrow \infty} \implies m \rightarrow 0$. Using eqs. (4.3), (4.10) and (C.1) we obtain the mapping:

$$w := F^{-1}(z) = \delta + \gamma F(\arcsin z | m^2) \quad (4.18)$$

We demanded $F^{-1}(0) = 0$ and $F^{-1}(1) = \frac{a}{2}$. Inserting yields $\gamma = \frac{a}{2K(m^2)}$ and $F_{\mathbb{H}^+}$ is $\text{sn}(z|m)$ - a *Jacobi elliptic* function as defined in appendix C.1.2 :

$$\frac{2K(m^2)}{a} w = F(\arcsin z | m^2) \quad (4.19)$$

$$z = F(w) = \sin(\text{am}(w \frac{2K(m^2)}{a} | m^2)) = \text{sn}(\frac{2K(m^2)}{a} w | m^2) \quad (4.20)$$

Finally we found the function mapping a rectangle with corners: $\{-a/2, a/2, a/2 + ib, -a/2 + ib\}$ onto the upper half-plane. Due to symmetry reasons this function maps the conjugate rectangle $\{-a/2, a/2, a/2 - ib, -a/2 - ib\}$ onto the lower half plane. The symmetry axis $z \in \{-ib, ib\}$ is mapped onto the imaginary axes. These properties are used when mapping the ellipse onto the half-plane.

Symmetrizing

Since we mapped the rectangle onto the upper half-plane, we can get the mapping onto the circle with help of the Cayley transform and a specific rotation. The composition with eq. (4.2), mapping the ordered four points crossing the two symmetry axis of the rectangle onto $\{1, i, -1, -i\}$, yields:

$$F_m(z) = i + \frac{2}{i + \sqrt{m} \operatorname{sn}\left(\frac{(\frac{ib}{2} + z)K(1-m^2)}{b}, m^2\right)}. \quad (4.21)$$

This function has the properties: $F(0) \mapsto 0$ and $F^*(z^*) = F(z)$.

The limiting Case

Since $\lim_{a \rightarrow \infty} \Rightarrow m \rightarrow 1$ as directly seen from eq. (4.17) with $\operatorname{sn}(z|1) = \tanh(z)$ ([98] p.21, eq.(122 .09)) and $K(0) = \pi/2$ ([98] p.10, eq.(111.02)):

$$\lim_{m \rightarrow 1} F_m(z) = \tanh \frac{\pi z}{2b}, \quad (4.22)$$

the mapping of a strip of a width b onto the circle.

Now, we take the limits $b \rightarrow \infty$ leading to $m \rightarrow 0$. Starting from eq. (4.20) and using the fact that $\operatorname{sn}(z|0) = \sin(z)$ ([98] p.21, eq.(122 .08)), one obtains:

$$F(z) = \sin \frac{\pi z}{a}, \quad (4.23)$$

the map of the half-strip $\{z | \operatorname{Re}(z) \geq 0 \wedge \operatorname{Im}(z) \leq a\}$ onto $\mathbb{D}$.

4.2.3 The Regular Polygon

The vertices of a regular n -polygon are mapped onto roots of one (see e.g. [106], p.329, Ch.5, Example 5.9, eq.(69)). Employing eq. (4.4) we obtain:

$$dF^{-1} = \frac{\gamma}{\prod_{k=1}^n (\zeta - z_k)^{1-\alpha_k}} d\zeta = \frac{\gamma}{(\zeta^n - 1)^{\frac{2}{n}}} d\zeta = \frac{\tilde{\gamma}}{(1 - \zeta^n)^{\frac{2}{n}}} d\zeta, \quad (4.24)$$

$$\alpha_k = 1 - \frac{2}{n}, \prod_{k=1}^n (\zeta - z_k) = \zeta^n - 1. \quad (4.25)$$

We can use the symmetry to simplify the integral, and express the solution as '2-1' *Hypergeometric functions*. The identity as used in eq. (4.31):

$$\int_0^z \frac{d\zeta}{(1 - \zeta^n)^{\frac{2}{n}}} = z {}_2F_1\left(\frac{2}{n}, \frac{1}{n}, \frac{n+1}{n}; z^n\right), \quad (4.26)$$

can be shown with help of the Euler representation of the Hypergeometric Function ${}_2F_1(a, b, c; z)$ (see e.g. [107], p.11, eq.(1.40)):

$${}_2F_1(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a} dt. \quad (4.27)$$

The Γ function and the Hypergeometric Function ${}_2F_1(a, b, c; z)$ are defined in appendix C.1.5. One obtains:

$${}_2F_1\left(\frac{2}{n}, b, c; z^n\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1 - \underbrace{tz^n}_{:=\zeta^n(t)})^{\frac{2}{n}}} dt \quad (4.28)$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} n \int_0^z \frac{\left(\frac{\zeta}{z}\right)^{n(b-1)} \left(1 - \left(\frac{\zeta}{z}\right)^n\right)^{c-b-1}}{(1 - \zeta^n)^{\frac{2}{n}}} \frac{\zeta^{n-1}}{z^n} d\zeta. \quad (4.29)$$

Setting $c - b - 1 = 0$ and $nb - 1 = 0$ and using $\Gamma(z+1) = z\Gamma(z)$ we see:

$$\int_0^z \frac{d\zeta}{(1 - \zeta^n)^{\frac{2}{n}}} = z {}_2F_1\left(\frac{2}{n}, \frac{1}{n}, \frac{n+1}{n}; z^n\right). \quad (4.30)$$

Integration and the use of eq. (4.26) and eq. (4.4) gives the transformation of the unit disc onto a regular n-polygon with a distance r from the origin to a vertex:

$$F^{-1}(z) = \frac{2^{2/n} \sqrt{\pi} r}{\Gamma\left(\frac{1}{2} - \frac{1}{n}\right) \Gamma\left(1 + \frac{1}{n}\right)} z {}_2F_1\left(\frac{2}{n}, \frac{1}{n}, \frac{n+1}{n}; z^n\right). \quad (4.31)$$

The limiting Case

In the limit $n \rightarrow \infty$, F^{-1} becomes a rescaling onto the circle with the radius r :

$$\lim_{n \rightarrow \infty} F^{-1}(z) = \lim_{n \rightarrow \infty} \frac{2^{2/n} \sqrt{\pi} r z {}_2F_1\left(\frac{1}{n}, \frac{2}{n}; 1 + \frac{1}{n}; z^n\right)}{\Gamma\left(\frac{1}{2} - \frac{1}{n}\right) \Gamma\left(1 + \frac{1}{n}\right)} \quad (4.32)$$

$$= \lim_{n \rightarrow \infty} \frac{2^{2/n} \sqrt{\pi} r z \sum_{k=0}^{\infty} \frac{(\frac{2}{n})_k (\frac{1}{n})_k}{(\frac{1+n}{n})_k} \frac{z^{n^k}}{k!}}{\Gamma\left(\frac{1}{2} - \frac{1}{n}\right) \Gamma\left(1 + \frac{1}{n}\right)} \quad (4.33)$$

$$\begin{aligned} & \xrightarrow{\text{eq. (4.35)}} \frac{\lim_{n \rightarrow \infty} 2^{2/n} \sqrt{\pi} r z \left(1 + \sum_{k=1}^{\infty} \frac{(\frac{2}{n})_k (\frac{1}{n})_k}{(\frac{1+n}{n})_k} \frac{z^{n^k}}{k!}\right)}{\lim_{n \rightarrow \infty} \underbrace{\Gamma\left(\frac{1}{2} - \frac{1}{n}\right) \Gamma\left(1 + \frac{1}{n}\right)}_{\rightarrow \sqrt{\pi}}} = rz \quad (4.34) \end{aligned}$$

In the limit we obtain the mapping F^{-1} of a circle of the radius r . Since $|z| \leq 1$:

$$\begin{aligned} \left| \sum_{k=1}^{\infty} \frac{(\frac{2}{n})_k (\frac{1}{n})_k}{(\frac{1+n}{n})_k} \frac{z^{n^k}}{k!} \right| & \leq \sum_{k=1}^{\infty} \frac{(\frac{2}{n})_k (\frac{1}{n})_k}{(\frac{1+n}{n})_k} \frac{|z|^{n^k}}{k!} \\ & \leq \sum_{k=1}^{\infty} \frac{(\frac{2}{n})_k (\frac{1}{n})_k}{(\frac{1+n}{n})_k} \frac{1}{k!} = \frac{2^{-2/n} \Gamma\left(\frac{1}{2} - \frac{1}{n}\right) \Gamma\left(1 + \frac{1}{n}\right) - \sqrt{\pi}}{\sqrt{\pi}} \\ \lim_{n \rightarrow \infty} \frac{2^{-2/n} \Gamma\left(\frac{1}{2} - \frac{1}{n}\right) \Gamma\left(1 + \frac{1}{n}\right) - \sqrt{\pi}}{\sqrt{\pi}} & = 0 \Rightarrow \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \frac{(\frac{2}{n})_k (\frac{1}{n})_k}{(\frac{1+n}{n})_k} \frac{z^{n^k}}{k!} = 0. \quad (4.35) \end{aligned}$$

4.2.4 Arbitrary Polygons

If the number of vertices $n > 3$, the parameters have to be determined with the aid of hypergeometric functions. In the case of quadrilaterals ($n = 4$), the parameter is given in terms of elliptic functions. In all other cases, the parameter problem has to be solved numerically. The solution has to be inverted to get the final mapping of the polygon onto the desired domain. In this work, all domains are mapped onto the unit circle.

Additionally, the conjugate of the mapping is needed. It is expressed in terms of the complex conjugate of the argument, which can always be done in the following way: If we have found the mapping F (assuming that there is no rotation of the x-axis) we can formulate the mapping F_c of the conjugated domain (mirroring all points of the domain w.r.t. the x-axis). The mapping F_c maps all conjugate points onto the conjugate unit circle, which is, of course, the needed function $F_c(z^*) = F^*(z)$.

Written in terms of the SC-transform eq. (4.4) it becomes:

$$Fc(z) = \delta^* + \gamma^* \int^z \prod_{k=1}^n (1 - \zeta e^{i\varphi_k})^{\alpha_k - 1} d\zeta. \quad (4.36)$$

4.2.4.1 The Method of Driscoll and Vavasis

General mappings can be computed numerically using the MATLAB-based *SC Toolbox* by Driscoll and Trefethen [108]. Numerical algorithms for Schwarz-Christoffel mappings became feasible by the work of Lloyd Trefethen [109]. Trefethen invented a reliable algorithm to solve the parameter problem numerically which additionally addresses the crowding phenomenon, which describes convergence problems of elongated domains since the prevertices of such structures lie very closely. Driscoll and Vavasis [105] developed a novel approach based on cross-ratios and *Delaunay triangulation*. The method is so fast and reliable that polygons can be used to approximate arbitrary domains⁸. The graphics of the approximation of the *Koch-fractal* with around 700 vertices is shown in figs. 4.3a and 4.3b.

4.3 Composed Mappings

Here, more complicated geometries are studied. Let us start with the ellipse, which is mapped onto a screwed annulus then onto a rectangle and finally onto a circle. The second case is the n -pole, which is mapped onto a strip and afterwards onto a sheeted circle. The mapping of the ellipse are newly formulated, and the Green/Neumann functions of the n -poles are new. Therefore both mappings are presented in detail.

4.3.1 The Ellipse - Including a New Mapping

As depicted in fig. 4.4 we construct the mapping of an ellipse onto the imaginary axis, where the inner is mapped onto $\{z | \operatorname{Re}(z) > 0\}$.

We start with a function based on the inverse of the *Joukowski transform* $z \mapsto w + \sqrt{w^2 - 1}$ ([110] pp.66-68). We generalize the function to:

$$\mathcal{J}(z) = \left(-i \frac{z + \sqrt{z^2 - k^2}}{b + \sqrt{b^2 + k^2}} \right)^{k+1} \quad z \in \mathbb{C}, \quad b, k \in \mathbb{R}_0^+. \quad (4.37)$$

⁸ Remark: Once the prevertices were calculated, they can be stored and reused.

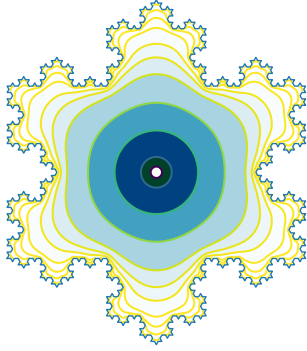


Fig. 4.3a Green function of a fractal namely the Koch snowflake - an electrostatic potential of a beam in a fractal beam pipe.

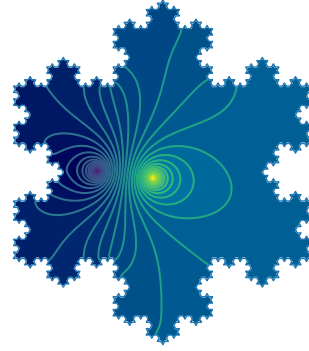


Fig. 4.3b The Neumann function of a fractal - the magnetostatic potential of two beams of equal intensity running in opposite directions.

The ellipse is characterized by two parameters: b , the short semi-axis and $\pm k$, the position of the foci. The long semi-axis is given by: $a^2 = b^2 + k^2$. As seen in fig. 4.4, the function $\mathcal{J}(z)$ maps the ellipse onto a sliced and screwed annulus. In dependence of k the argument winds around the origin $\frac{k+1}{2\pi}$ -times, so it is necessary to leave the complex plane and construct a temporary *Riemann surface*.

The ellipse is mapped onto the outer boundary of the annulus (blue) with an absolute value of 1. The line connecting the two foci (green) is mapped onto the inner boundary (green). The ellipse is cut along the real axis, starting at $-k$ in a positive direction. If the logarithm is taken, which of course now acts on the Riemann surface, the annulus is mapped onto a rectangle back on the plane again. The side lengths of the rectangle are determined e.g. through (symbols as used in fig. 4.4): $|b'' - 0''| = (1 + k) \left| \log\left(\frac{k}{b + \sqrt{b^2 + k^2}}\right) \right|$ and $4|0'' - k''| = 2\pi(k + 1)$.

The ellipse itself is mapped onto the imaginary axis onto $\frac{i\pi(k+1)}{2}[-1, 3]$ (blue in fig. 4.4). Now we employ an elliptic function, mapping the rectangle onto the imaginary axis and the inner onto $\{z|\operatorname{Re}(z) > 0\}$. As depicted first we translate the system (periodic! in the imaginary direction modulus $2\pi(k+1)$) to symmetrize the problem and rescale to obtain the side length K (real direction). Afterwards we map the upper half of the rectangle onto $\{z|\operatorname{Re}(z) = 0 \wedge \operatorname{Im}(z) \geq 0\}$ and its inner onto $\{z|\operatorname{Re}(z) > 0 \wedge \operatorname{Im}(z) > 0\}$ (1. quadrant) and the lower half onto $\{z|\operatorname{Re}(z) = 0 \wedge \operatorname{Im}(z) \leq 0\}$ and its inner onto $\{z|\operatorname{Re}(z) > 0 \wedge \operatorname{Im}(z) < 0\}$ (4. quadrant).

As explained in section 4.2.2, the sine-amplitude $\operatorname{sn}(z|m)$ maps a rectangle with the origin at the symmetry centre and the side lengths $2K'$ (imaginary direction) and $2K$ (real direction) onto the complex plane, where the symmetry line along the imaginary direction is stretched due to the symmetry onto the imaginary axis (the functions are defined in appendices C.1.1 and C.1.2). The parameter m is found, using eq. (4.17):

$$\frac{K(m^2)}{K'(m^2)} = \frac{|\log(\frac{k}{b+\sqrt{b^2+k^2}})|}{\pi} = \frac{|\sinh^{-1} \frac{b}{k}|}{\pi}. \quad (4.38)$$

The mapping has the form:

$$\operatorname{sn}_z := \operatorname{sn} \left(\frac{K'}{\pi} \log \frac{\sqrt{z^2 - k^2} + z}{\sqrt{b^2 + k^2} + b} \middle| m^2 \right). \quad (4.39)$$

It is possible to algebraically manipulate eq. (4.39):

$$\begin{aligned} \operatorname{sn}(z) &= \operatorname{sn} \left(\frac{K'}{\pi} \left[\log \left(\sqrt{\frac{z^2}{k^2} - 1} + \frac{z}{k} \right) + \log \frac{k}{\sqrt{b^2 + k^2} + b} \right] \middle| m^2 \right) \\ &\stackrel{\text{eq. (4.38)}}{=} \operatorname{sn} \left(\frac{K'}{\pi} \cosh^{-1} \frac{z}{k} + K \middle| m^2 \right) = \operatorname{cd} \left(\frac{K'}{\pi} \cosh^{-1} \frac{z}{k} \middle| m^2 \right). \end{aligned} \quad (4.40)$$

The last conversion can be found in [98] p.22; (eq.(122.23)). This form is very useful in numerical calculations done for example in Mathematica, because the functions are advantageously implemented.

The Limiting Cases

If we consider the two interesting limiting cases: $k \rightarrow \infty$ (strip) or $k \rightarrow 0$ (circle) we see from eq. (4.38) that in the first case $\frac{K}{K'}$ is 0, which implies $m = 0$ and in the latter case $\frac{K'}{K}$ is 0, which implies $m = 1$. Equation (4.37) becomes in the second case: $\lim_{k \rightarrow \infty} \log \mathcal{J}(z) = -(iz+b)$ and in the second $\lim_{k \rightarrow \infty} \log \mathcal{J}(z) = \log(-i\frac{w}{b})$. Since $\operatorname{sn}(z|1) = \tanh(z)$ ([98] p.21, eq.(122.09)) (then also $K' = \frac{\pi}{2}$), we see that:

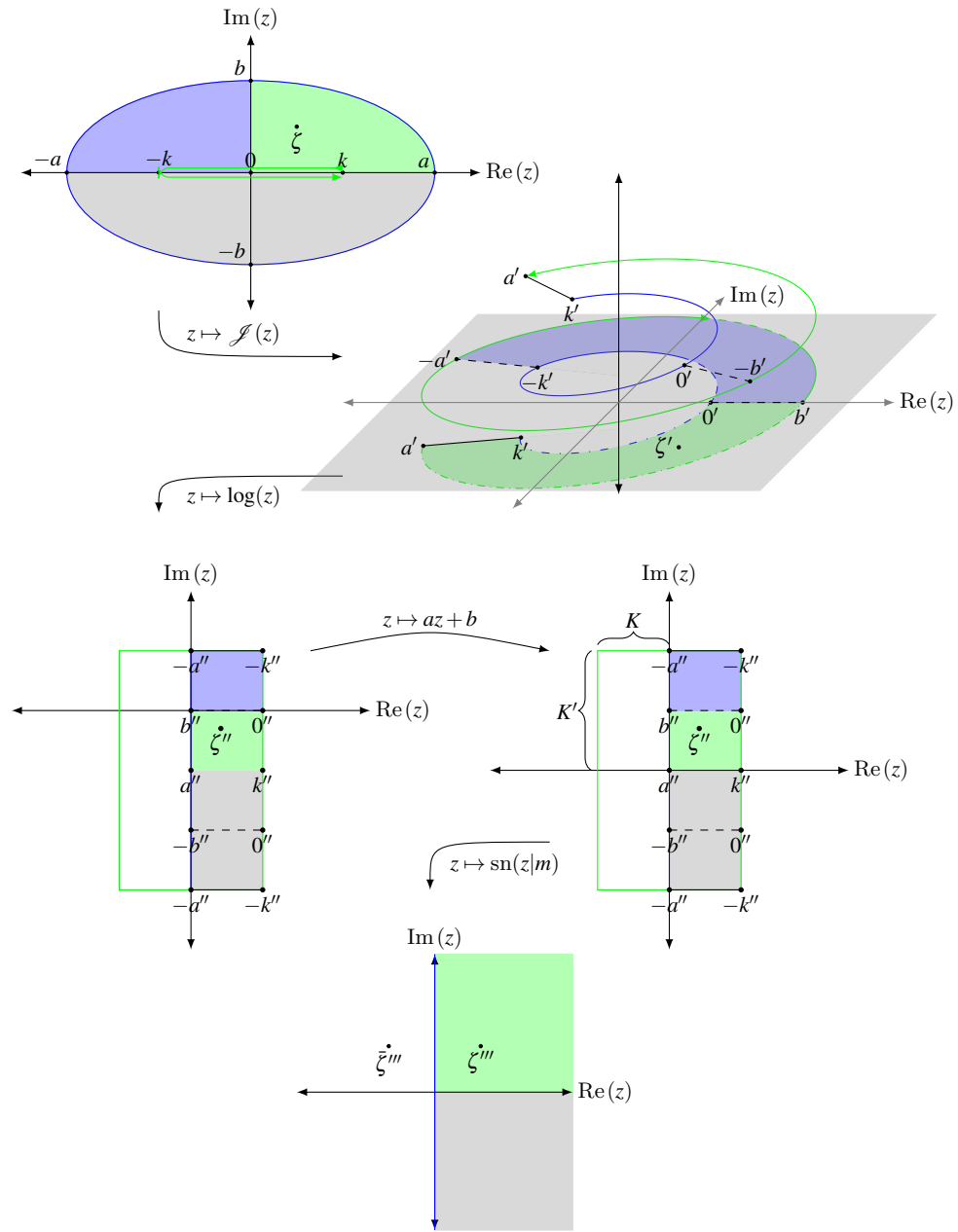


Fig. 4.4: A new mapping of the ellipse onto the negative half-plane.

$$\lim_{k \rightarrow 0} \operatorname{sn}(z, m) = \tanh \left(\frac{1}{2} \log \left(\frac{z}{b} \right) \right) = -\frac{1 + \frac{iz}{b}}{1 - \frac{iz}{b}}. \quad (4.41)$$

We identify the Cayley transform section 4.1.1 with the argument iz/b and rotated by $\pi/2$ implying the mapping of the circle with radius b onto the imaginary axis. The points $\pm k \rightarrow 0$, hence $\pm k'' \rightarrow \infty$ (they merge at infinity, horizontal semi strip). $\pm k$ (and $\pm a$) $\rightarrow \pm\infty$ implies $\pm k''$ (and $\pm a''$) $\rightarrow \pm i\infty$. The resulting domain is an infinite strip parallel to the imaginary axis namely: $\{z \mid -b \leq \operatorname{Re}(z) \leq 0\}$. The resulting transformation is the mapping of the strip onto the negative half-plane: $\lim_{m \rightarrow 1} \operatorname{sn}(K'z|m) = \sin(\pi/2z)$ (section 4.2.2).

Symmetrizing

As for the rectangle, we want to symmetrize the result. Applying the Cayley transform and eq. (4.2) yields:

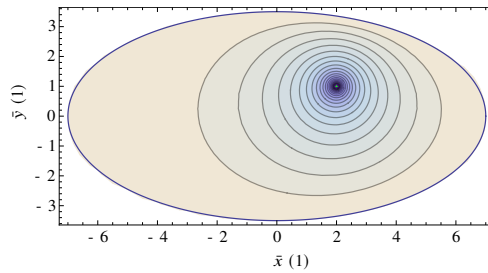
$$F(z) = \frac{2}{\sqrt{m} \operatorname{cd} \left(\frac{\cosh^{-1}(\frac{z}{k}) K(1-m^2)}{\pi} |m^2\right) + 1} - 1. \quad (4.42)$$

The function obeys: $F(0) = 0$ and $F^*(z) = F(z^*)$. The Green function using eq. (4.42) is pictured in fig. 4.5 in the case of a strongly displaced beam for the dimensions of the PS vacuum chamber ($a = 7$ cm, $b = 3.5$ cm). To generate the graphics the representation of the Green function given in appendix C.1.3 was used. For completeness we provide the mapping of an ellipse onto $\mathbb{D}$ as given by [41], (eq. 52):

$$F(z) = \sqrt{m} \operatorname{sn} \left(\frac{2K}{\pi} \sin^{-1} \left(\frac{z}{k} \right) |m^2 \right), \quad e^{-\frac{\pi K'}{K}} = \left(\frac{a-b}{a+b} \right)^2, \quad (4.43)$$

where the foci of the ellipse are located at $\pm\sqrt{m}$.

Fig. 4.5 The Green function of the Proton-Synchrotron vacuum chamber.



4.3.2 Novel n -Poles

4.3.2.1 The Quadrupole

The (constant) imaginary part of a complex number ($z = x + iy$) with an integer exponent can be interpreted as the pole face of a n -pole structure. E.g. if we take the imaginary part of $z \mapsto z^2 := u + iv$ it is clear that $\text{Im } z^2 = v = 2xy$. These are hyperbola, forming an approximation for a quadrupole. The quadrupole is mapped onto a strip $\{z \mid |\text{Im } z| \leq v\}$, whereas if we introduce a branch cut along the negative x -axis, one has to fix the Riemann sheet, where the point is located. We take the signum function of the real part to decide which Riemann sheet is used. The strip again is mapped via $z \mapsto \tanh \frac{\pi z}{4v}$ onto the circle. Taking the square root now unwinds the two sheets. Finally, we obtain:

$$F(z) = \sqrt{\tanh \frac{\pi z^2}{4v} \text{sgnRe } z}. \quad (4.44)$$

F has the properties: $F(0) = 0$ (and is there not conformal since the origin is a critical point of the mapping), $F^*(z^*) = F(z)$ and it is not defined along the imaginary axis. This is no restriction, since the values can be obtained arbitrary closely to the axis. Some special points are: $\lim_{r \rightarrow \infty} F(r) = 1$, $\lim_{r \rightarrow \infty} F(re^{i\pi}) = -1$, $\lim_{r \rightarrow \infty} F(re^{i\frac{\pi}{2}}) = i$ and $\lim_{r \rightarrow \infty} F(re^{i\frac{3\pi}{2}}) = -i$.

4.3.2.2 The n -Pole

The integer n of the power of a complex number (z^n) as already mentioned, maps a n -pole structure onto a strip, which again is mapped onto the circle and finally the n -th root has to be taken. Of course the argument is winded n times and as a consequence the n -th root reflects this symmetry. The complex plane is divided into n sectors each with an angle $2\pi/n$, each mapped onto the whole plane. Defining a function, which remembers the value of the original sector of z , we get:

$$F(z) = \sqrt[n]{\tanh \frac{\pi z^n}{4d}} e^{\left(\sum_{j=0}^{n-1} \frac{2i\pi \theta\left(\arg(w) - \frac{\pi(2(j+1)-1)}{n}\right)}{n} \right)}. \quad (4.45)$$

θ is the *Heaviside theta function* (the definition can be found in appendix C.1.6). Green functions of the first and second kind are plotted in fig. 4.7 up to the forth order (octupolar structure).

Remarks on the n -Pole

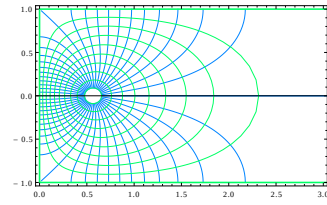
The last term can be realized e.g. with a *which* statement in *Mathematica*, telling to rotate back to the original sector. Of course on the boundaries of the sheets one has to carefully investigate the transition to the next Riemann sheet at the argument $\frac{2\pi k}{n}$ $k \in \{1, 2, \dots, n-1\}$. At the origin and for radial symmetric distributions (or distributions respecting the symmetry) the last term of eq. (4.45) can be set to 1.

4.4 Summary

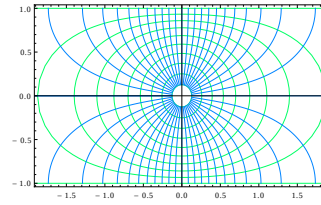
In this chapter, symmetric conformal closed-form transformations, as drawn in fig. 4.6, were derived by using the *Schwarz-Christoffel-transformation* (section 4.1). The Schwarz-Christoffel-transformation transforms arbitrary polygons onto simple domains as $\mathbb{D}$. During the application, one has to solve the parameter problem section 4.1.3. Up to four vertices, the solution can be expressed in terms of closed-forms (n denotes the number of vertices):

- $n = 1$: linear functions,
- $n = 2$: trigonometric functions,
- $n = 3$: hypergeometric functions,
- $n = 4$: elliptic functions,
- $n > 4$: numerical solutions involving the *parameter problem* and the *numerical inversion* of a complex function.

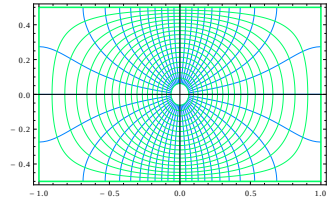
In section 4.3 the composed mappings, expressed through closed-forms, of the ellipse (novel formulation) eq. (4.42) and the n -pole family eq. (4.45) were presented. New closed-form mappings for the *combined-function magnets* of the PS are developed in chapter 6.



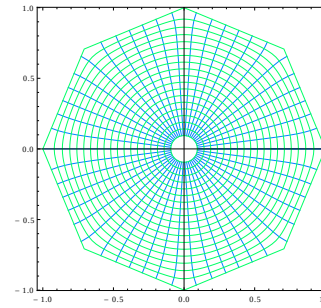
(a) The half-strip.



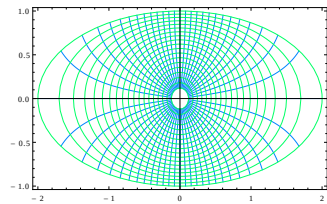
(b) The strip.



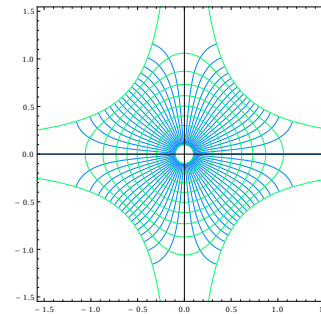
(c) The rectangle.



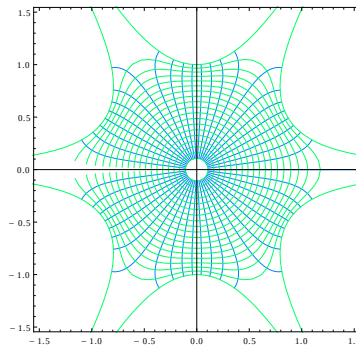
(d) The octagon.



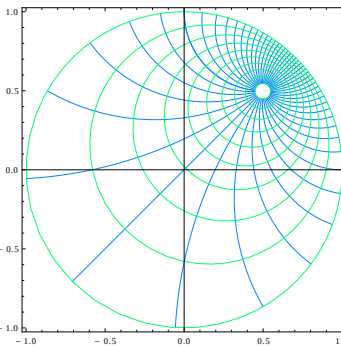
(e) The ellipse.



(f) The quadrupole.



(g) The sextupole.



(h) The unit circle.

Fig. 4.6: Different symmetric closed-form mappings onto $\mathbb{D}$. The blue lines are mapped onto rays with constant argument and the green lines are mapped onto concentric centred circles.

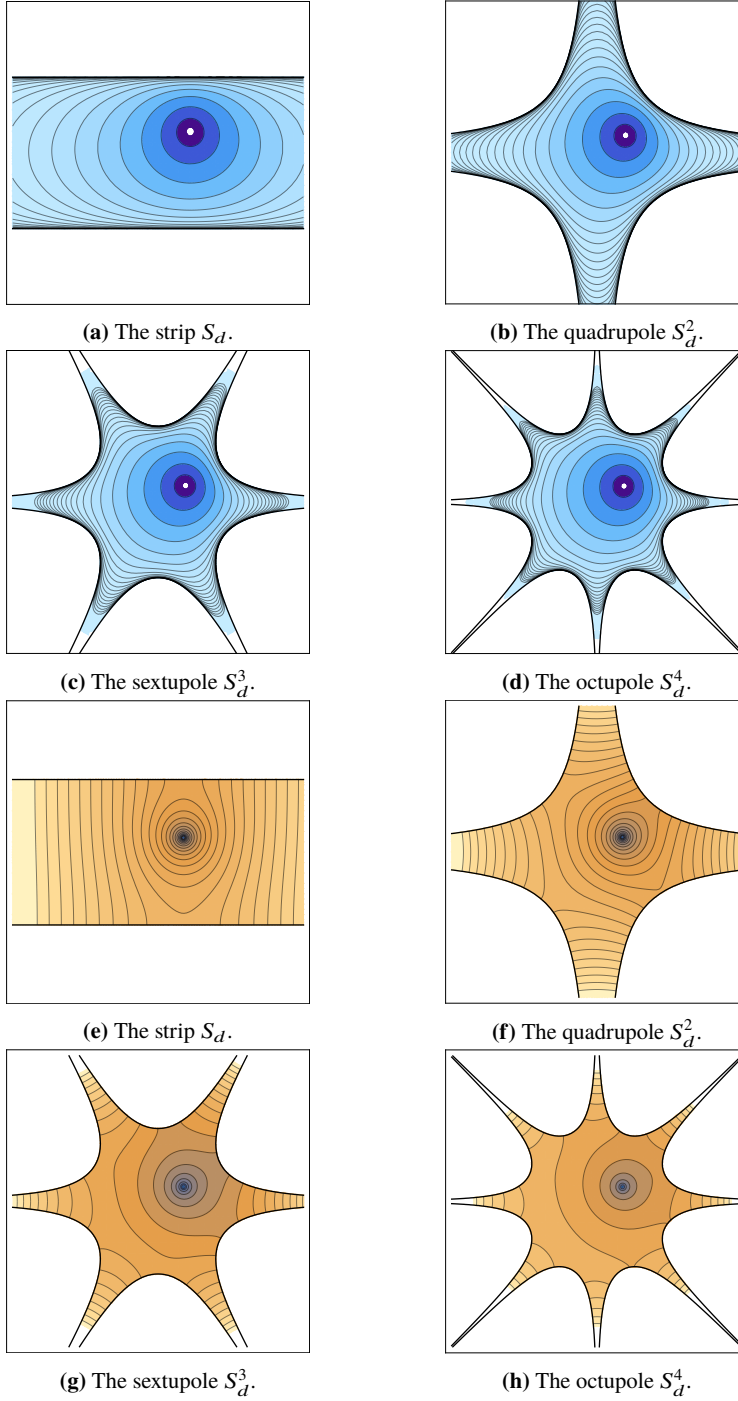


Fig. 4.7: Novel solutions of the Green function of the first and second kind of the n -pole. (Type 1 in blue, type 2 in brown).

Part III

Applications

In this part we apply the ideas to the real world.

Chapter 5

Image Operators

We think in generalities, but we live in detail.

— Alfred North Whitehead

Abstract In this chapter, the formalism of the *image tensors* is introduced. We start from the equation of motion and linearise the acting forces. It leads to a coupled system of differential equations. Decoupling yields the equation of the harmonic oscillator. The effect of the force in the decoupled system is a shift in the normal mode system, which is expressed through tensors.

Two types and three regimes are commonly distinguished: the whole beam and the single-particle effect in three frequency ranges: static, low and non-penetrating. The formalism is based on the work [2]. After formulating the image tensors, the approach is extended to formulate these as *operators of the underlying conformal mappings - the image operators*. The general indirect field is expressed through the *complex Lorentz-force*, an operator of the Green functions of the first and second kind. Formulas to estimate the tune-shift due to indirect field effects for models of different complexity are given, which are utilized in later applications.

For several common cross-sections, the off-axis normal mode coefficients are provided for the first time and are listed to serve as a reference. The extension to *arbitrary cross-sections* is addressed in the applications (chapters 6 and 7).

The units used are cgs throughout the entire chapter. Nevertheless, the final image operators are unit-less.

5.1 Classic Image Tensors

5.1.1 The Equations of Motion

To emphasize the physical meaning of the following quantities, we start from Newton's equations and later introduce a new implementation based on mapping methods. The general formulation of the electromagnetic force on a *test particle* in the transverse plane can be written as:

$$\ddot{\vec{r}} = \vec{F}. \quad (5.1)$$

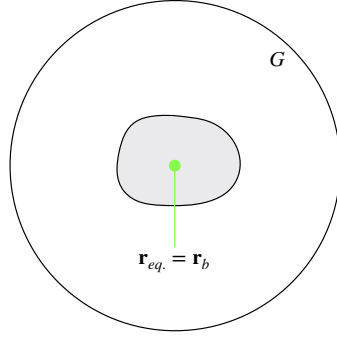


Fig. 5.1a Cross-section of the pipe for a beam in a centred configuration.

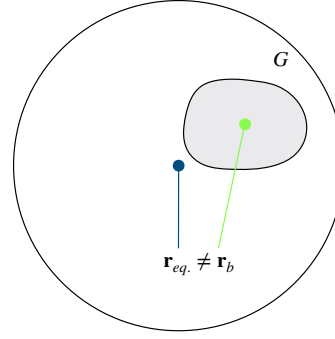


Fig. 5.1b Cross-section of the pipe for a beam in an off-centred configuration.

A test particle is driven by the following forces ([2] eq.(1)):

$$\vec{F} = \vec{f}_{d.i.} + \vec{f}_{ext.} + \vec{f}_{i.i.} \quad (5.2)$$

It is separated into three contributions: $\vec{f}_{d.i.}$ (direct interaction) describes the electromagnetic-interaction with other particles of the beam, it is also called direct space charge. The second contribution $\vec{f}_{ext.}$ stems from external fields like high-frequency (HF) fields or magnets, to accelerate and constrain the particles onto the design orbit. The third term $\vec{f}_{i.i.}$ (indirect interaction) is caused by the surrounding elements, as the vacuum chamber, which interact with the electromagnetic field of the beam particles. For simplicity, it is assumed that the field is generated by a source with the strength of the total charge located at the centre of the charge ($\vec{r}_b$). The size of the beam is generally small compared to the size of the interacting sur-

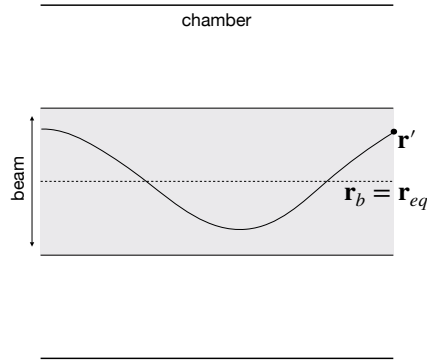


Fig. 5.2a The incoherent trajectory of test particle is located at $\vec{r}'$.

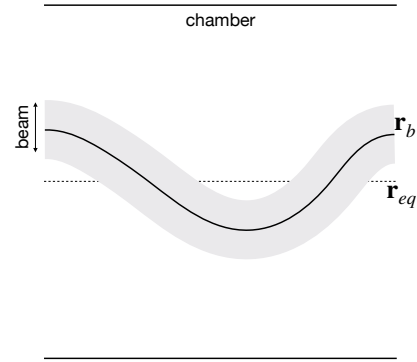


Fig. 5.2b The coherent movement of the whole beam at an off-centred position.

rounding elements. We linearise the contributions around the equilibrium position, which is characterized by $\vec{F} = 0$. Of course, this corresponds with $\vec{f}_{d.i.}$ and can be developed at the centre of charge $\vec{r}_b$:

$$\vec{f}_{d.i.}(\vec{r}, \vec{r}_b) = \underbrace{\vec{f}_{d.i.}|_{\vec{r}_b}}_{=0} + (\vec{r} - \vec{r}_b) \cdot \nabla_r \vec{f}_{d.i.}|_{\vec{r}_b} + O(2). \quad (5.3)$$

The design orbit is defined at $\vec{r}_{eq.}$ as:

$$\vec{f}_{ext.}|_{\vec{r}_{eq.}} + \vec{f}_{i.i.}|_{\vec{r}_b=\vec{r}_{eq.}} = 0. \quad (5.4)$$

This is depicted in fig. 5.1a. Developing the other contributions:

$$\vec{f}_{ext.}(\vec{r}, \vec{r}_{eq.}) = \vec{f}_{ext.}(\vec{r}_{eq.}) + (\vec{r} - \vec{r}_{eq.}) \cdot \nabla_r \vec{f}_{ext.}|_{\vec{r}_{eq.}} + O(2). \quad (5.5)$$

$$\begin{aligned} \vec{f}_{i.i.}(\vec{r}, \vec{r}_b, \vec{r}_{eq.}) &= \vec{f}_{i.i.}|_{\vec{r}_b=\vec{r}_{eq.}} + (\vec{r} - \vec{r}_{eq.}) \cdot \nabla_r \vec{f}_{i.i.}|_{\vec{r}_b=\vec{r}_{eq.}} \\ &\quad + (\vec{r}_b - \vec{r}_{eq.}) \cdot \nabla_{r_b} \vec{f}_{i.i.}|_{\vec{r}_b=\vec{r}_{eq.}} + O(2). \end{aligned} \quad (5.6)$$

For simplicity we assume $\vec{f}_{ext.}$ to be a (scaling) function of the form $f_{ext.}(\vec{r} - \vec{r}_{eq.})$, $f_{ext.} \in \mathbb{R}$. Two cases are usually distinguished (a rigorous classification can be found in, e.g. in [111]):

- $\vec{r}_b = \vec{r}_{eq.} \wedge \vec{r} \neq \vec{r}_{eq.}$:
The centre of charge of the beam is on the design orbit, whereas the test particle is displaced $\rightarrow$ *incoherent setting* (single displacement):

$$\vec{f} = (\vec{r} - \vec{r}_{eq.}) \cdot \nabla_r \left(\left(\vec{f}_{d.i.} + \vec{f}_{i.i.} \right) + f_{ext.} \right) |_{\vec{r}_b=\vec{r}_{eq.}} \quad (5.7)$$

This situation is shown in figs. 5.1a and 5.2a for the cross-section of the beam-pipe and from a direction orthogonal to the direction of the pipe, respectively. The beam is indicated by the grey shaded area and the boundary is denoted by G or chamber. The equilibrium position $\vec{r}_{eq.}$ is the position, where the indirect space charge forces (produced by a beam with the centre of charge at $\vec{r}_b$) cancel the external forces. The test particle is located at $\vec{r}'$ and its motion is classified as incoherent motion (the trajectory is indicated by the black curve).

- $\vec{r} = \vec{r}_b \neq \vec{r}_{eq.}$:
The centre of charge of the beam is displaced, whereas the test particle is at the centre of charge. $\rightarrow$ *coherent setting* (whole beam displacement):

$$\vec{f} = (\vec{r}_b - \vec{r}_{eq.}) \cdot \left((\nabla_r + \nabla_{r_b}) \vec{f}_{i.i.} + f_{ext.} \right) |_{\vec{r}=\vec{r}_b} \quad (5.8)$$

This situation is shown in figs. 5.1b and 5.2b, again for the cross-section of the beam-pipe and from a direction orthogonal to the direction of the pipe,

respectively. The meaning of the symbols is the same as in figs. 5.1a and 5.2a. Here, $\vec{r}_b \neq \vec{r}_{eq.}$, and the beam is displaced as a whole. Figure 5.2b indicates the movement of the beam along the pipe.

5.1.2 The Form Tensors

For didactic reasons, the formalism is presented close to the historical form [1] with the addition of the fact, that the indirect fields provoke also coupling, which is broadly neglected in the literature. A general approach is discussed later. The effect is usually added up over the whole ring, and it is justified to begin from the unperturbed single-particle dynamics. Starting from the smooth approximation ([77], p.171), the indirect forces can be derived from a potential and hence, the linearisation of it lead to a tensor (e.g. [52]). The linearised equation of motion in the smooth approximation has the form:

$$\left(\frac{d^2}{d\tau^2} + \Omega_c^2 \nu_0^2 \bar{\bar{U}} \right) \cdot \vec{r} = 0, \quad (5.9)$$

Where $\tau = s/c$ is the orbital coordinate, Ω_c is the (angular) circulation frequency, ν_0 is the unperturbed tune, defined by:

$$\nu_0^2 = \frac{1}{\Omega_c^2} \frac{1}{m\gamma_0} f_{ext.}, \quad (5.10)$$

with γ_0 and m the Lorentz factor and the mass, respectively. The matrix $\bar{\bar{U}}$ is defined as:

$$\bar{\bar{U}} = \bar{\bar{1}} - \frac{1}{\nu_0^2 \Omega_c^2 m \gamma_0} \begin{pmatrix} \delta_x u_x & \delta_y u_x \\ \delta_x u_y & \delta_y u_y \end{pmatrix}. \quad (5.11)$$

In the following the $\vec{e}_{x,y}$ are the canonical vectors along the coordinate axis.

- Incoherent case: The coefficients are defined as:

$$\delta_{x,y} = \partial_{x,y} \big|_{\vec{r}=\vec{r}_b=\vec{r}_{eq.}} \quad u_{x,y} = q^{-1}(\vec{f}_{d.i.} + \vec{f}_{i.i.}) \cdot \vec{e}_{x,y}, \quad (5.12)$$

and $\vec{r} := (\vec{r} - \vec{r}_{eq.})$.

- Coherent case: The coefficients are defined as:

$$\delta_{x,y} = (\partial_{x_b,y_b} + \partial_{x,y}) \big|_{\vec{r}=\vec{r}_b=\vec{r}_{eq.}} \quad u_{x,y} = q^{-1}(\vec{f}_{i.i.}) \cdot \vec{e}_{x,y}, \quad (5.13)$$

and $\vec{r} := (\vec{r}_b - \vec{r}_{eq.})$.

In terms of tune-shifts:

$$\bar{\bar{U}} = \bar{\bar{I}} + 2\nu_0^{-2} \Delta \bar{\bar{v}} \quad \Delta \bar{\bar{v}}_{ij} := - \left(\frac{NRr_0}{\pi\nu_0\beta_0^2\gamma_0 L^2} \right) \epsilon_{ij}. \quad (5.14)$$

The scalar coefficient depends only on the machine. N being the total number of particles in the beam, R the ring radius, r_0 the classical particle radius, β_0 the particle velocity, L the typical transverse dimension of the chamber, while the Laslett coefficients:

$$\epsilon_{ij} = \frac{L^2}{4\lambda} \quad (5.15)$$

depend only on the transverse pipe geometry, $\lambda = Nq/2\pi R$ being a linear charge density. The Laslett coefficients form a 2nd-rank tensor. By a transformation to the normal mode system, the tensors are diagonal and the characteristic frequencies - the betatron frequencies $\Omega_{1,2}$ - can be written in terms of the tune-shift as:

$$\Omega_{1,2} = \Omega_c(\nu_0 + \Delta\nu_{1,2}) \approx \Omega_c\nu_0 \left(1 + \frac{\Delta\nu_{1,2}}{\nu_0} \right). \quad (5.16)$$

where the $\Delta\nu_{1,2}$ can be related to normal mode Laslett coefficients,

$$\Delta\nu_{1,2} = \left(\frac{NRr_0}{\pi\nu_0\beta_0^2\gamma_0 L^2} \right) \epsilon_{1,2}, \quad (5.17)$$

where:

$$\epsilon_{1,2} = \frac{L^2}{4\lambda} \left[\frac{\delta_x u_x + \delta_y u_y}{2} \pm \sqrt{\left(\frac{\delta_x u_x - \delta_y u_y}{2} \right)^2 + \frac{\delta_x u_y \delta_y u_x}{2}} \right] \quad (5.18)$$

Even if the diagonal terms in $\bar{\bar{U}}$ are zero at some $\vec{r}_{eq.}$, the normal mode tune-shift and Laslett coefficients can nonetheless be locally different from zero, provided the off-diagonal terms do not vanish there.

5.1.3 Different Regimes

Depending on the oscillations of the centre of charge of the beam, three regimes have to be taken into account, if the electric boundary is not perfectly conducting and of finite thickness. Zotter gives a criterion for the frequency for the magnetic field [8]: if the skin depth δ fulfils $\delta < \sqrt{hd}$ then the field is non penetrating. Here h is the characteristic size of the chamber (e.g. the half-height) and d is its thickness. While the *skin depth* is determined as: $\delta = \frac{2\rho_{\text{wall}}}{\omega\mu_{\text{wall}}}$. Here the resistivity of the chamber is ρ_{wall} , its permeability is μ_{wall} and $\omega = (n - Q)\Omega$ (n is the closest integer to the tune Q). If the frequency is higher the magnetic field does not penetrate the electric

boundary and the field lines will be parallel to the boundary. It can be expressed in terms of the electric field, since the boundary conditions for the magnetic potential are of the same form as for the electric potential.

Now, we give the *Lorentz force* of a particle with velocity β_0 as an *operator of the Green functions*. The chamber cross-section is denoted by G , and the magnet cross-section is denoted by M .

$$F = 2\lambda\bar{\partial}(-g_G(z, z_0) + \beta_0^2 n_M(z, z_0)). \quad (5.19)$$

In this form the problem is reduced to the conformal mapping of the domain G and M . If the mappings are given as closed-forms, the force can be expressed through closed-forms.

Given this, we separate the contributions into static (=) and dynamic ($\approx$) terms:

$$g(z) = g_=(z, z_{\text{eq.}}) + g_{\approx}(z, z_b), \quad (5.20) \quad n(z) = n_=(z, z_{\text{eq.}}) + n_{\approx}(z, z_b). \quad (5.21)$$

Incoherent Regime

The centre is fixed, hence:

$$F^i := 2\lambda\bar{\partial}(-g_=(z, z_{\text{eq.}}) + \beta_0^2 n_=(z, z_{\text{eq.}})). \quad (5.22)$$

Coherent Regime: penetrating

The beam reveals collective oscillations around z_{eq} and contains static and dynamic parts:

$$F^{c.p.} := 2\lambda\bar{\partial}(-g(z, z_b) + \beta_0^2 n(z, z_b)). \quad (5.23)$$

Coherent Regime: non-penetrating

Since the magnetic potential can be obtained from the oscillating electric one:

$$n = q\beta_0 g_{\approx} = \beta_0(g_{\approx}(z, z_b) - g_=(z, z_{\text{eq.}})), \quad (5.24)$$

it follows:

$$F^{n.p.} := 2\lambda\bar{\partial}(-\beta_0^2 n_=(z, z_{\text{eq.}}) + \beta_0^2 g_=(z, z_{\text{eq.}}) + (1 - \beta_0^2)g(z, z_b)). \quad (5.25)$$

5.1.3.1 Bunching

So far only the setting for the coasting beam (a beam equally distributed over the full length of the accelerator) was discussed. If bunching of the beam has to be included a so-called bunching factor as introduced by Laslett ([1], [112]). λ_{max} and $\bar{\lambda}$ denotes the maximum line charge density and the average line charge density along the whole ring, respectively. The *bunching factor* B is of the form $B = \frac{\lambda_{max}}{\bar{\lambda}}$. The idea of the bunching factor is to estimate the worst-case scenario for the bunch centre. Here it has again to be distinguished whether the magnetic field penetrates the chamber or not. On top of the previous considerations, the bunch frequency plays now a role. Consequently, we take the frequency of the transverse motion as $n - \Omega$ into account. n is the closed integer to Ω , and the frequency of the bunch is $h\Omega$ and has an amplitude of $(B^{-1} - 1)$.

5.2 The Image Operators

The complex Lorentz force (eq. (3.9)) expressed through the Green functions, which depends only on the conformal mappings, is used to give the form tensor as operators of these mappings. The expressions are newly derived in this thesis and therewith presented in detail. The formalism is kept to cover the class of generic simply-connected domains in the electric and star-like regulated domains in the magnetic case. One can obtain the solution for bounded domains using the integral equation eq. (3.57), which is, due to the limited scope, not explicitly provided here. The in-depended variables are z_b and z_b^* . The complex potential and its derivatives can be written as follows:

$$\Xi := \bar{\partial}(-g(z, z_b) + \beta_0^2 n(z, z_b)) = \Xi^o(z, z_b) + \Xi^e(z, z_b^*) \quad (5.26)$$

$$=: \sum_{k=\{1,2\}} \Xi^{k,o}(z, z_b) + \Xi^{k,e}(z, z_b^*). \quad (5.27)$$

The index k denotes either the the electric, $k = 1$, or the magnetic part, $k = 2$, of Ξ , whereas the $\Xi^{k,o}(z, z_b)$ is the odd part of the mirror function and $\Xi^{k,e}(z, z_b)$ is the even part. The distinction has two reasons: first the algebra is a lot easier and second the odd part has a removable (hebbare) singularity (e.g. [113]), which has to be noticed.

For the electric part the even and the odd terms have different signs and for the magnetic part the signs are the same. We write the dependence of the conformal mapping explicitly:

$$\Xi_F^{k,o}(z, z_b) = \beta_0^{2(k-1)} \frac{1}{N} \sum_j^N \left(\frac{(z - z_b)F_j'(z) + F_j(z_b) - F_j(z)}{(z_b - z)(F_j(z_b) - F_j(z))} + \delta_{2,j} \frac{2F_j'(z)}{F_j(z) - 1} \right), \quad (5.28)$$

$$\Xi_F^{k,e}(z, z_b^*) = (-1)^k \beta_0^{2(k-1)} \frac{1}{N} \sum_j^N \frac{\tilde{F}_j(z_b^*) F_j'(z)}{F_j(z) \tilde{F}_j(z_b^*) - 1}. \quad (5.29)$$

The conjugate of the function can be expressed as a new function of the conjugate of the argument: $\tilde{F}(z^*) := F(z)^*$. In the electric case $N = 1$ in the magnetic case N depends on the degeneracy at infinity of the mapping function, which can be written as F_k mapping the desired direction to infinity as discussed in section 3.6.2. These terms reveal the form at the point $z = z_b$:

$$\begin{aligned} \hat{\Xi}_F^{k,o}(z_b) := \lim_{z \rightarrow z_b} \bar{\partial}_z \Xi_F^k(z, z_b) &= \frac{\beta_0^{2(k-1)}}{N} \sum_j^N \left[\frac{4F_j^{(3)}(z_b) F_j'(z_b) - 3F_j''(z_b)^2}{12F_j'(z_b)^2} \right. \\ &\quad \left. \delta_{2,k} \frac{2 \left(F_j'(z_0)^2 - (F_j(z_0) - 1) F_j''(z_0) \right)}{(F_j(z_0) - 1)^2} \right], \quad (5.30) \end{aligned}$$

$$\begin{aligned} \hat{\Xi}_F^{k,e}(z_b) &:= \lim_{z \rightarrow z_b} \bar{\partial}_z \Xi_F^k(z, z_b) \\ &= \frac{(-1)^k \beta_0^{2(k-1)}}{N} \sum_j^N \frac{\tilde{F}_j(z_b^*) (\tilde{F}_j(z_b^*)^2 - 1) F_j''(z_b) - \tilde{F}_j(z_b^*) F_j'(z_b)^2}{(\tilde{F}_j(z_b^*)^2 - 1)^2}, \quad (5.31) \end{aligned}$$

$$\hat{\Xi}_F^{k,o}(z_b) := \lim_{z \rightarrow z_b} \bar{\partial}_{z_b} d \Xi_F^{k,o}(z, z_b) = \frac{\beta_0^{2(k-1)}}{N} \sum_j^N \frac{2F_j^{(3)}(z_b) F_j'(z_b) - 3F_j''(z_b)^2}{12F_j'(z_b)^2}, \quad (5.32)$$

$$\hat{\Xi}_F^{k,e}(z_b) := \lim_{z \rightarrow z_b} \partial_{z_b} d \Xi_F^{k,e}(z, z_b^*) = -\frac{(-1)^k \beta_0^{2(k-1)}}{N} \sum_j^N \frac{F_j'(z_b) F_j'(z_b^*)}{(F_j(z_b) F_j(z_b^*) - 1)^2}. \quad (5.33)$$

Assuming that F_1 maps the electro-static interacting domain onto $\mathbb{D}$ and F_2 maps the magneto-static interacting domain onto $\mathbb{D}$, we can write:

$$\hat{\Xi}_{F_1, F_2}(z_b) := \sum_{k \in \{1, 2\}} \left(\hat{\Xi}_F^{k,o}(z_b) + \hat{\Xi}_F^{k,e}(z_b) \right) \quad \hat{\Xi}_{F_1, F_2}^j(z_b) := \sum_{k \in \{1, 2\}} \hat{\Xi}_F^{k,j}(z_b). \quad (5.34)$$

Using these expressions (L denotes a typical scaling length like the radius of a circular chamber), we can write eq. (5.18) as:

$$\epsilon_{1,2}^i = \pm \frac{1}{2} \left| \hat{\Xi}_{F_1, F_2}(z_b) \right|, \quad (5.35)$$

$$\xi_{1,2}^{c.p.} = \frac{1}{2} \left[-\operatorname{Re} \dot{\Xi}_{F_1, F_2}^e(z_b) \pm \sqrt{\left| \dot{\Xi}_{F_1, F_2}^e(z_b) + \dot{\Xi}_{F_1, F_2}^o(z_b) \right|^2 - \operatorname{Im}^2 \dot{\Xi}_{F_1, F_2}^e(z_b)} \right], \quad (5.36)$$

$$\xi_{1,2}^{c.n.} = \frac{1}{2} \left[-\bar{\beta}^2 \operatorname{Re} \dot{\Xi}_{F_1, 0}^e(z_b) \pm \sqrt{\left| \dot{\Xi}_{F_1, F_2}^e(z_b) + \dot{\Xi}_{F_1, 0}^o(z_b) \right|^2 - \bar{\beta}^2 \operatorname{Im}^2 \dot{\Xi}_{F_1, 0}^e(z_b)} \right]. \quad (5.37)$$

with $\bar{\beta}_0^2 = 1 - \beta_0^2$. $\epsilon_{1,2}^i$ are the incoherent coefficients and $\xi_{1,2}^{c.p.}$ and $\xi_{1,2}^{c.n.}$ are the penetrating and non penetrating coherent coefficients, respectively. Of course all values can be obtained for arbitrary source points as well. As previously pointed out, taking the values at z_b is based on the fact that the cross-section of the beam is very small compared to the beam pipe dimensions. The image fields of the beam are approximated as originating from a point source, the centre of charge of the beam. The error by doing so is very small for standard applications, since the beam size is generally small compared to the geometry of the surrounding elements. The full coefficients will be given and can be calculated straightforward.

5.2.1 Integrating the Indirect Space Charge Effects in more Complex Models

The integration of the indirect field effects to arbitrary order would require advanced methods of normal form theory [71] or could be treated via tracking methods (as discussed in section 6.4.4). Due to the accessible nature of our solutions - as complex Lorentz force - the implementation is simple.

In our applications, the linear approximation without coupling effects is sufficient to explain the observed physical effects. For completeness, we refer to the method of Edward-Teng [114] if using the mapping approach to include coupling effects. The full perturbative approach, dealing with this subject, can be found in [76, 115]. These methods are not discussed further due to the limited scope of this thesis. Nevertheless, we provide the full normal mode off-axis image coefficients.

By a projection of these coefficients onto the symmetry axis, the identification with the classical Laslett coefficients becomes: $\epsilon_{1,2} \Leftrightarrow \epsilon^{h,v}$ and $\xi_{1,2} \Leftrightarrow \xi^{h,v}$, which is applied in the following formulas. Tedious but straight forward computations yield for the total tune-shift:

The r_0 is the classical proton-radius $r_0 = 1.5347 \times 10^{-18}$, $\gamma_0 = (1 - \beta_0^2)^{-1/2}$, L the length of the accelerator and B the bunching factor as defined in section 5.1.3.1:

$$\Delta Q_{x/y}^{inc.} = -\frac{1}{\gamma_0} \frac{r_0}{\pi} \frac{1}{L} \left[\int_0^L ds \beta_{x/y}(s) \right]$$

$$\left\{ \varepsilon_1^{h/v}(s, x, y) + \varepsilon_2^{h/v}(s, x, y) + \varepsilon_1^{h/v}(s, x, y) \left(\frac{\beta_0^{-2} - 1}{B} \right) \right\} \quad (5.38)$$

$$\Delta Q_{x/y}^{coh.} = -\frac{1}{\gamma_0} \frac{r_0}{\pi} \frac{1}{L} \left[\int_0^L ds \beta_{x/y}(s) \right. \\ \left. \left\{ \xi_1^{h/v}(s, x, y) + \xi_2^{h/v}(s, x, y) + \xi_1^{h/v}(s, x, y) \left(\frac{\beta_0^{-2} - 1}{B} \right) \right\} \right] \quad (5.39)$$

In the next section we use three different approximations of the full equations using (eq. (5.38)) and (eq. (5.39)), is ascending complexity:

1. The *smooth approximation*: averaging the β -function over the whole ring with centred form factors ($x=y=0$):

$$\beta_{x/y}(s) \Rightarrow \bar{\beta}_{x/y}. \quad (5.40)$$

2. The *centred approximation*: take form factors on-axis centred ($x=y=0$), but take the beta function $\beta_{x/y}$ at s (either measured or calculated):

$$\varepsilon(x, y) \Rightarrow \varepsilon, \quad (5.41)$$

$$\xi(x, y) \Rightarrow \xi. \quad (5.42)$$

3. The *closed orbit (c.o.)*: on top of the centred approximation the form factors are calculated at the position of the closed orbit ($x_{(c.o.)}(s), y_{(c.o.)}(s)$).

The model is also refined by increasing the resolution of the longitudinal integration.

5.3 Form Factors for Particular Cases

In this section, we present the normal mode coefficients off-axis of several geometries in terms of the image operators, which then define the image coefficients for the different regimes as discussed in section 5.1.3. These solution are novel and have been derived in the context of this thesis. Some normal mode coefficients are calculated for special cases: the regular polygon $n = 8$ and n -pole, $n = 4$. In both cases there is no restriction for arbitrary n .

In the rectangular case an error was found in the literature and was corrected. All other existing results in the literature (largely only on-axis) show agreement with our calculations, which is an excellent indication for correctness of the method and the calculations. By employing the Schwarz-Christoffel-transformation, arbitrary

cross-section geometries can be approximated and as an example in chapter 7, the polygonal approximation of the LHC beam screen is discussed. In this case, bound for the error made by the approximation can be given (see section 3.4). In chapter 6, the polygonal approximation of magnetic elements is discussed.

In addition, former expressions of the literature are given to have a possibility of direct comparison. All calculations were restricted to the coordinate axes.

The comparison to the literature

The incoherent form factors where introduced by [9] and are given by:

$$\varepsilon_1^h = -\varepsilon_1^v = h^2 \frac{\partial^2 \phi}{\partial x^2} \Big|_{x=x_0} \Rightarrow 4\text{Re } \bar{\partial}_z^2 \phi \Big|_{z=x_0} \quad (5.43)$$

$$\varepsilon_2^h = -\varepsilon_2^v = g^2 \frac{\partial^2 A}{\partial x^2} \Big|_{x=x_0} \Rightarrow 4\text{Im } \bar{\partial}_z^2 A \Big|_{z=x_0}. \quad (5.44)$$

h and g denote a specific distance to the electric and magnetic boundary, respectively. ϕ is the electric image potential and A is the magnetic image potential⁹. Of course these factors are the real part of the first Laurent-coefficients of the corresponding complex Lorentz force. Along the x-axis the coherent form factor can be calculated via:

$$\xi_1^h = h^2 \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x_0} \right) \frac{\partial \phi}{\partial x} \Big|_{x=x_0}, \quad \xi_1^v = h^2 \left(\frac{\partial}{\partial y} + \frac{\partial}{\partial y_0} \right) \frac{\partial \phi}{\partial y} \Big|_{x=x_0}, \quad (5.45)$$

$$\xi_2^h = g^2 \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x_0} \right) \frac{\partial A}{\partial x} \Big|_{x=x_0}, \quad \xi_2^v = g^2 \left(\frac{\partial}{\partial y} + \frac{\partial}{\partial y_0} \right) \frac{\partial A}{\partial y} \Big|_{x=x_0}, \quad (5.46)$$

$$\Rightarrow \quad \Rightarrow \quad 4\text{Re } (\bar{\partial}_z + \bar{\partial}_{z_0} + \partial_{z_0}) \bar{\partial}_z f(z, z_0, z_0^*), \quad 4\text{Im } (\bar{\partial}_z + \bar{\partial}_{z_0} - \partial_{z_0}) \bar{\partial}_z f(z, z_0, z_0^*). \quad (5.47)$$

x_0 and y_0 denote the source point coordinates. Since the Lorentz force is expressed through to conformal invariant Green functions, only the mapping of a specific shape has to be known to calculate these quantities, which was treated in detail in the previous section. In the following, the normal mode image tensors are plotted for arbitrary beam positions for almost all discussed cross-sections. In all figures, on-axis solutions are drawn as solid purple and blue lines and the boundary is represented as red shape.

⁹ ϕ and A are the Green functions of the first and second kind, respectively in real notation scaled to physical units as discussed in sections 3.3.1 and 3.3.2.

5.3.1 The Rectangular Cross-Section

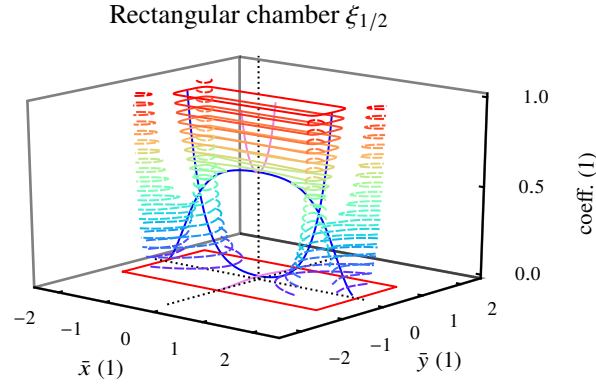


Fig. 5.3a: The coherent electric image tensor of the rectangular chamber.

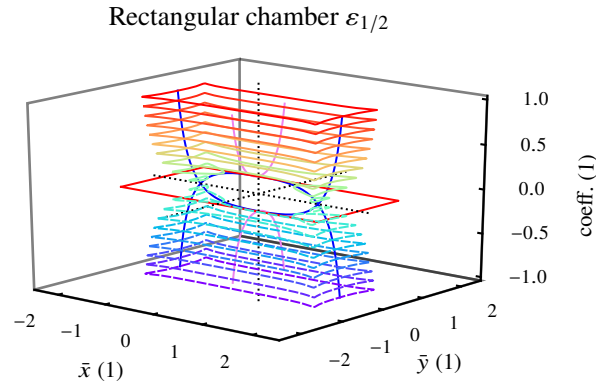


Fig. 5.3b: The incoherent electric image tensor of the rectangular vacuum chamber.

Since for the rectangular cross-section there was a disagreement for the on-axis coefficients with the literature [6], [7] and [51], the rectangular case is explained in detail, whereas the off-axis solution is new. The rectangular cross-section of the vacuum chamber has the dimension half-width h and half-height w . The parameter m is a function of h and w and is implicitly formulated in eq. (4.17). A visualization of this dependency as a function of the ratio of the two side-lengths can be found in fig. C.1. The mapping function was given in eq. (4.21). Inserting into eq. (5.34)

yields:

$$\hat{\Xi}_{F1,0}(z_b) = \frac{K^2}{48b^2} \left(m^2 \text{cn}_{z_b}^2 \left(\frac{12 \text{dn}_{z_b}^2 \text{sn}_{z_b}^2}{(m^2 \text{snc} - 1)^2} - \frac{3m^2 \text{sn}_{z_b}^2}{\text{dn}_{z_b}^2} + \frac{12}{m^2 \text{snc} - 1} + 8 \right) - \frac{3 \text{dn}_{z_b}^2 \text{sn}_{z_b}^2}{\text{cn}_{z_b}^2} + \left(\frac{12}{m^2 \text{snc} - 1} + 8 \right) \text{dn}_{z_b}^2 + 10m^2 \text{sn}_{z_b}^2 \right). \quad (5.48)$$

$$\hat{\Xi}_{F1,0}^e(z_b) = -\frac{K^2 m \text{cn}_{z_b}^2 \text{dn}_{z_b}^2}{4b^2 (\text{snc} - 1)^2}, \quad (5.49)$$

$$\hat{\Xi}_{F1,0}^o(z_b) = \frac{1}{48b^2} K^2 \left[m^2 \text{cn}_{z_b}^2 \left(-\frac{3m^2 \text{sn}_{z_b}^2}{\text{dn}_{z_b}^2} - 2 \right) - \frac{3 \text{dn}_{z_b}^2 \text{sn}_{z_b}^2}{\text{cn}_{z_b}^2} - 2 \text{dn}_{z_b}^2 + 2m^2 \text{sn}_{z_b}^2 \right]. \quad (5.50)$$

cn_{z_b} , dn_{z_b} and sn_{z_b} denote the Jacobi elliptic functions with the argument: $\frac{(z_b + ib)K'}{2b}$. snc denotes the product: $m \text{sn}_{z_b} \text{sn}_{z_b^*}$. The definition of the elliptic integrals and Jacobi elliptic functions is provided in appendices C.1.1 and C.1.2. These results were compared to the literature. The coefficients agree along the coordinate axes with the results of [7] p.10, eq.(6.1-6.2). For [7] p.10, eq.(6.3), the results are different. Recalculating the results of [7] revealed an analytical disagreement, which was corrected. The corrected coefficient matches numerically the directly calculated values using eq. (5.37). Since these two in-depended methods are in agreement, this is regarded to be sufficient to assume that the new coefficient is correct. We found simpler expressions for the on-axis coefficients, starting from the real electric potential ϕ of the rectangular shape is (also defined in appendix C.1.3):

$$\phi = 2 \log \left| \frac{\text{sn}(\frac{K}{2}(z + ih|m^2) - \text{sn}(\frac{K}{2}(z_0 + ih|m^2))}{\text{sn}(\frac{K}{2}(z + ih|m^2) - \text{sn}(\frac{K}{2}(z_0^* + ih|m^2))} \right|. \quad (5.51)$$

Where $m'^2 := 1 - m^2$ is used. The rectangle is centred and w is parallel to the x-axis and sn is the Jacobi elliptic function as defined in appendix C.1.2. $K := K(m)$ and $K' := K(1 - m^2)$ (see appendices C.1.1 and C.1.2). After some tedious algebra the result obtained at the beam location has the form:

$$\varepsilon_1^h = -\varepsilon_1^v = \frac{1}{24} K^2 \left[3m'^2 \left(\frac{1}{\zeta} - \zeta \right) - 2(m^2 + 1) \right] \quad (5.52)$$

$$\xi_1^h = \frac{K^2 m'^2}{4} \left(\frac{1}{\zeta} - \zeta \right) \quad (5.53)$$

$$\xi_1^v = \frac{K^2 m'^2}{4} \zeta \quad (5.54)$$

$\zeta := \frac{dn-m}{dn+m}$. dn is the Jacobi elliptic function $dn(K'(1-g)|m'^2)$. g is the distance from the centre along the x-axis. All results can be shown to be identical with the literature, although eq. (5.53) differs ([7] p.10, e.q(6.3)). At the centre of the chamber $\zeta = \frac{1-m}{1+m}$ ($dn(0|m) = 1$) the coefficients are:

$$\varepsilon_1^h = -\frac{1}{12}K^2(m^2 - 6m + 1), \quad (5.55)$$

$$\xi_1^h = K^2m, \quad (5.56)$$

$$\xi_1^v = \frac{1}{4}K^2(m-1)^2. \quad (5.57)$$

in agreement with Ng ([7]). If the limit $w \rightarrow \infty$ is taken, we obtain the strip. It follows $m \rightarrow 0$, $K \rightarrow \frac{\pi}{2}$ and $\zeta \rightarrow 1$, which gives

$$\varepsilon_1^h = -\frac{\pi^2}{48}, \quad (5.58)$$

$$\xi_1^h = 0, \quad (5.59)$$

$$\xi_1^v = \frac{\pi^2}{16}. \quad (5.60)$$

These results agree with Zotter ([9]) and Ng ([7]). The coefficients are plotted in figs. 5.3a and 5.3b. The red shape indicates the boundary (the vacuum chamber). The blue and line, coloured in a pale shade of pink, indicated the on-axis coefficients.

5.3.2 The Parallel Plates

The plates have a distance $2d$. $z \mapsto \tanh \frac{\pi z}{4d}$ maps a strip with half-width d onto the circle. Employing eq. (5.34) yields:

$$\hat{\Xi}_{1,2}(z_b) = \beta_0^{1\mp 1} \frac{\pi^2 \left(1 \mp 3 \sec^2 \left(\frac{\pi \text{Im}(z_b)}{2d} \right) \right)}{48d^2}, \quad (5.61)$$

$$\hat{\Xi}_{1,2}^e(z_b) = \mp \beta_0^{1\mp 1} \frac{\pi^2}{8d^2 \left(\cos \left(\frac{\pi \text{Im}(z_b)}{d} \right) + 1 \right)}, \quad (5.62)$$

$$\hat{\Xi}_{1,2}^o(z_b) = -\beta_0^{1\mp 1} \frac{\pi^2}{48d^2}. \quad (5.63)$$

The first sign corresponds to the electric the second to the magnetic solution. And hence:

$$\epsilon_1^i = \pm \pi^2 \frac{2 - 6 \sec^2 \left(\frac{\pi \text{Im}(z)}{2d} \right)}{192d^2}, \quad \epsilon_2^i = -\beta_0^2 \frac{\pi^2 \left(3 \sec^2 \left(\frac{\pi \text{Im}(z)}{2d} \right) + 1 \right)}{96d^2} \quad (5.64)$$

$$\epsilon_1^{c.m.} = \frac{\pi^2}{8d^2 \left(\cos \left(\frac{\pi \text{Im}(z)}{d} \right) + 1 \right)}, \quad \epsilon_2^{c.m.} = \beta_0^2 \frac{\pi^2 \sec^2 \left(\frac{\pi \text{Im}(z)}{2d} \right)}{16d^2} \quad (5.65)$$

$$\epsilon_{1,2}^{c.p.} = 0. \quad (5.66)$$

At the centred beam location we obtain:

$$\varepsilon_2^h = -\frac{\pi^2}{24}, \quad (5.67)$$

$$\xi_2^h = 0, \quad (5.68)$$

$$\xi_2^v = \frac{\pi^2}{16}. \quad (5.69)$$

These results are identical to Zotter ([9]). These coefficients have also been calculated for $z \neq z_0$. All results agree with the literature and are listed in the following. We start with the electric potential:

$$\phi = 2 \log \frac{e^{\frac{\pi z_0^*}{2h}} + e^{\frac{\pi z}{2h}}}{e^{\frac{\pi z}{2h}} - e^{\frac{\pi z_0}{2h}}}. \quad (5.70)$$

The coefficients are:

$$\varepsilon_1^h = -\frac{1}{32} \left[\pi^2 \left(\text{csch}^2 \frac{\pi}{4h} (x - x_0) + \text{sech}^2 \frac{\pi}{4h} (x - x_0) \right) - \frac{16h^2}{(x - x_0)^2} \right], \quad (5.71)$$

$$\xi_1^h = 0, \quad (5.72)$$

$$\xi_1^v = \frac{1}{16} \pi^2 \text{sech}^2 \frac{\pi}{4h} (x - x_0). \quad (5.73)$$

These results were shown to be identical to the results of Zotter ([9]). At the beam location we obtain the values given in eqs. (5.58) to (5.60). The magnetic potential of a strip with half-width g is (same reference system as for the electric strip):

$$A = 2 \log \frac{\cosh \frac{\pi}{4g} (z - z_0^*) \sinh \frac{\pi}{4g} (z - z_0)}{z - z_0}. \quad (5.74)$$

The magnetic form factors are:

$$\varepsilon_2^h = -\frac{g^2}{2(x_0 - x)^2} - \frac{\pi^2}{4(1 - \cosh \frac{\pi}{g} (x_0 - x))}, \quad (5.75)$$

$$\xi_2^h = 0, \quad (5.76)$$

$$\xi_2^v = \frac{1}{16} \pi^2 \text{sech}^2 \frac{\pi}{4g} (x - x_0). \quad (5.77)$$

These results were shown to be identical to Zotter ([9]).

5.3.3 The Cut Parallel Plates - on-axis

As an approximation for a dipole magnet the half-strip of half width g centred and parallel to the x -axis can be used

$$A = 2 \log \frac{\left(\sinh \frac{\pi z}{2g} - \sinh \frac{\pi z_0}{2g} \right) \left(\sinh \frac{\pi z_0^*}{2g} + \sinh \frac{\pi z}{2g} \right)}{z - z_0}. \quad (5.78)$$

The coefficients are:

$$\varepsilon_2^h = \frac{\pi^2}{8} \left(\frac{4g^2}{\pi^2(x - x_0)^2} - \operatorname{csch}^2 \frac{\pi}{2g}(x - x_0) - \operatorname{csch}^2 \frac{\pi}{2g}(x + x_0) \right), \quad (5.79)$$

$$\xi_2^h = \frac{\pi^2}{4} \operatorname{csch}^2 \frac{\pi}{2g}(x + x_0), \quad (5.80)$$

$$\xi_2^v = \frac{\pi^2}{4} \frac{\left(\cosh \frac{\pi}{2g}x \cosh \frac{\pi}{2g}x_0 + 1 \right)}{\cosh \left(\frac{\pi}{2g}x + \cosh \frac{\pi}{2g}x_0 \right)^2}. \quad (5.81)$$

At the beam location we obtain:

$$\varepsilon_2^h = \frac{\pi^2}{24} \left(3 \operatorname{csch}^2 \frac{\pi x_0}{g} - 1 \right), \quad (5.82)$$

$$\xi_2^h = \frac{\pi^2}{4} \operatorname{csch}^2 \frac{\pi x_0}{g}, \quad (5.83)$$

$$\xi_2^v = \frac{\pi^2}{16} \left(\operatorname{sech}^2 \frac{\pi x_0}{2g} + 1 \right). \quad (5.84)$$

5.3.4 The n -Pole Structure - on-axis

As an approximation for the n -pole characterized by the distance of the pole shoes of g at $x = 1$ with the beam centred at the origin ($z_0 = 0$). The mapping is provided in eq. (4.45) and the potential for this symmetric setting can be found in section 3.6.2.3. In the general case only ξ_2 for the centred beam can be given:

$$\xi_2^h = \frac{4g^2 + \pi x^n \left(g(n-1) \sinh \frac{\pi x^n}{g} - \pi n x^n \right) \operatorname{csch}^2 \frac{\pi x^n}{2g}}{8x^2}. \quad (5.85)$$

ξ_2^v vanishes at the centre (=beam location).

5.3.5 The Combined-Function Magnet - on-axis

Due to symmetry reasons the on-axes potential of the combined-function magnet can be written as:

$$A = 2 \log \frac{\sinh \frac{\pi}{2g} (x^2 - x_0^2)}{2} (x - x_0). \quad (5.86)$$

g is the distance to the magnetic pole-shoes at x_0 and x_0 denotes the shift from the centre of symmetry of the magnet. Its corresponding coefficients are:

$$\varepsilon_2^h = \frac{\pi}{2} \left(\frac{g^2}{\pi^2} (x - x_0) - \frac{g \sinh \frac{\pi}{g} (x_0^2 - x^2) + 2\pi x^2}{\cosh \frac{\pi}{g} (x^2 - x_0^2) - 1} \right), \quad (5.87)$$

$$\xi_2^h = \frac{\pi}{4} \operatorname{csch}^2 \frac{\pi}{2g} (x_0^2 - x^2) \left(g \sinh \frac{\pi}{g} (x_0^2 - x^2) + 2\pi x (x - x_0) \right), \quad (5.88)$$

$$\xi_2^v = -\frac{\pi}{4} \operatorname{csch}^2 \frac{\pi}{2g} (x_0^2 - x^2) \left(g \sinh \frac{\pi}{g} (x_0^2 - x^2) + 2\pi x \left(x - x_0 \cosh \frac{\pi}{2g} (x_0^2 - x^2) \right) \right). \quad (5.89)$$

At the beam location we obtain:

$$\varepsilon_2^h = \frac{\pi^2 x_0^2}{6} - \frac{g^2}{8x_0^2}, \quad (5.90)$$

$$\xi_2^h = \frac{g^2}{4x_0^2}, \quad (5.91)$$

$$\xi_2^v = \frac{\pi^2 x_0^2}{4} - \frac{g^2}{4x_0^2}. \quad (5.92)$$

5.3.6 The Octagonal Chamber (HL-LHC Beam Screen)

Of course it is possible to calculate arbitrary regular polygons with the transformation eq. (4.26) from a circle to a regular polygon with radius r . We restrict ourself on the special case of the octagon, since parts of the beam screen of the HL-LHC have this form, which is treated in section 7.5. F^{-1} is the inverse of eq. (4.26) and $r = 1$. The analytical results are:

$$\hat{\varepsilon}_{F1,0}(z_b) = -\frac{\Gamma\left(\frac{3}{8}\right)^2 \Gamma\left(\frac{9}{8}\right)^2}{3\sqrt{2}\pi \left(1 - F^{(-1)}(z_b)^8\right)^{3/2} \left(F^{(-1)}(z_b) F^{(-1)}((z_b)^*) - 1\right)^2} \left[-22F^{(-1)}(z_b)^7 F^{(-1)}((z_b)^*) + \left(14F^{(-1)}(z_b)^8 - 3\right) F^{(-1)}((z_b)^*)^2 \right]$$

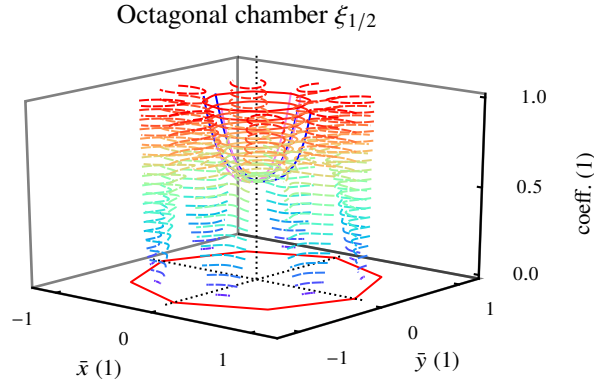


Fig. 5.4a: The coherent electric image tensor of the octagonal vacuum chamber (explanation in section 5.3.1).

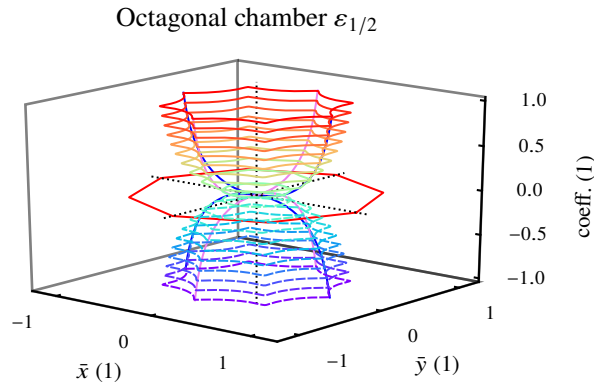


Fig. 5.4b: The incoherent electric image tensor of the octagonal vacuum chamber (explanation in section 5.3.1).

$$+ \left(14 - 3F^{(-1)}(z_b)^8 \right) F^{(-1)}(z_b)^6 \Big]. \quad (5.93)$$

$$\hat{\Xi}_{F1,0}^e(z_b) = \frac{\Gamma\left(\frac{3}{8}\right)^2 \Gamma\left(\frac{9}{8}\right)^2 \sqrt[4]{1 - F^{(-1)}(z_b)^8} \sqrt[4]{1 - F^{(-1)}((z_b)^*)^8}}{\sqrt{2}\pi (F^{(-1)}(z_b) F^{(-1)}((z_b)^*) - 1)^2}, \quad (5.94)$$

$$\hat{\Xi}_{F_o,0}^o(z_b) = \frac{7\Gamma\left(\frac{3}{8}\right)^2 \Gamma\left(\frac{9}{8}\right)^2 F^{(-1)}(z_b)^6}{3\pi\sqrt{2-2F^{(-1)}(z_b)^8} (F^{(-1)}(z_b)^8 - 1)}. \quad (5.95)$$

The coefficients are plotted in figs. 5.4a and 5.4b.

5.3.7 The Quadrupole - Hyperbolic Pole Shoes

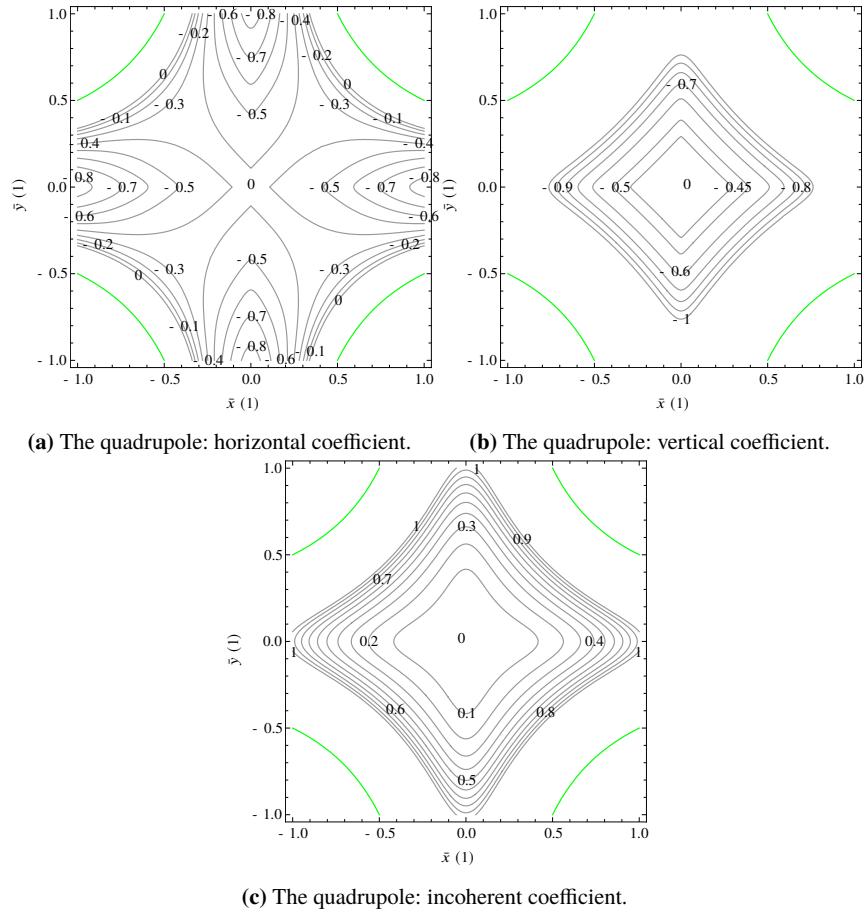


Fig. 5.5: The image coefficients of the quadrupolar structure in terms of closed-forms. The green curves indicate the boundary of the shape.

The results for the quadrupole are very tedious and are given for the sake of completeness. The corresponding parameters and transformations are given in section 4.3.2.1.

$$\begin{aligned}
\hat{\Xi}_{0,F_2}(z_b) = & \frac{1}{3072d^2z_b^2(\sqrt{\coth(\frac{\pi z_b^2}{4d})} - \sqrt{\tanh(\frac{\pi z_b^{*2}}{4d})})^3} \\
& \left[16(\pi^2 z_b^4 - 3d^2) \coth^{\frac{7}{2}}(\frac{\pi z_b^2}{4d}) + 48(3d^2 - \pi^2 z_b^4) \sqrt{\tanh(\frac{\pi z_b^{*2}}{4d})} \coth^3(\frac{\pi z_b^2}{4d}) \right. \\
& + 240(\pi^2 z_b^4 - 3d^2) \tanh(\frac{\pi z_b^{*2}}{4d}) \coth^{\frac{5}{2}}(\frac{\pi z_b^2}{4d}) \\
& + 80(3d^2 - \pi^2 z_b^4) \tanh^{\frac{3}{2}}(\frac{\pi z_b^{*2}}{4d}) \coth^2(\frac{\pi z_b^2}{4d}) \\
& + 16(10\pi^2 z_b^4 - 30d^2 + \text{csch}^2(\frac{\pi z_b^2}{4d})(7\pi^2 z_b^4 + 6d\pi \tanh(\frac{\pi z_b^{*2}}{4d})z_b^2 - 3d^2)) \coth^{\frac{3}{2}}(\frac{\pi z_b^2}{4d}) \\
& - 48\sqrt{\tanh(\frac{\pi z_b^{*2}}{4d})}(10\pi^2 z_b^4 - 30d^2 + \text{csch}^2(\frac{\pi z_b^2}{4d})(3\pi^2 z_b^4 + 2d\pi \tanh(\frac{\pi z_b^{*2}}{4d})z_b^2 - 3d^2)) \\
& \quad * \coth(\frac{\pi z_b^2}{4d}) - \text{csch}^3(\frac{\pi z_b^2}{4d}) \text{sech}^2(\frac{\pi z_b^2}{4d}) \text{sech}(\frac{\pi z_b^{*2}}{4d}) \\
& \quad * \left[60\pi^2 \cosh(\frac{\pi(3z_b^2 + z_b^{*2})}{4d})z_b^4 - 19\pi^2 \cosh(\frac{\pi(5z_b^2 + z_b^{*2})}{4d})z_b^4 \right. \\
& \quad - 36d\pi \sinh(\frac{\pi(z_b^2 - z_b^{*2})}{4d})z_b^2 - 84d\pi \sinh(\frac{\pi(z_b^{*2} - 3z_b^2)}{4d})z_b^2 \\
& \quad + 132d\pi \sinh(\frac{\pi(z_b^2 + z_b^{*2})}{4d})z_b^2 + 12d\pi \sinh(\frac{\pi(3z_b^2 + z_b^{*2})}{4d})z_b^2 + 2(19\pi^2 z_b^4 + 69d^2) \cosh(\frac{\pi(z_b^2 - z_b^{*2})}{4d}) \\
& \quad + 14(\pi^2 z_b^4 - 3d^2) \cosh(\frac{\pi(z_b^{*2} - 5z_b^2)}{4d}) + 3(45d^2 - 43\pi^2 z_b^4) \cosh(\frac{\pi(z_b^{*2} - 3z_b^2)}{4d}) + 20(\pi^2 z_b^4 - 12d^2) \cosh(\frac{\pi(z_b^2 + z_b^{*2})}{4d}) \\
& \quad \left. - 144d^2 \cosh(\frac{\pi(3z_b^2 + z_b^{*2})}{4d}) + 57d^2 \cosh(\frac{\pi(5z_b^2 + z_b^{*2})}{4d}) \right] \sqrt{\coth(\frac{\pi z_b^2}{4d})} \\
& + \text{csch}^3(\frac{\pi z_b^2}{4d}) \text{sech}^2(\frac{\pi z_b^2}{4d}) \text{sech}(\frac{\pi z_b^{*2}}{4d}) \\
& \left[32\pi^2 \cosh(\frac{\pi(3z_b^2 + z_b^{*2})}{4d})z_b^4 - 13\pi^2 \cosh(\frac{\pi(5z_b^2 + z_b^{*2})}{4d})z_b^4 - 36d\pi \sinh(\frac{\pi(z_b^2 - z_b^{*2})}{4d})z_b^2 - 84d\pi \sinh(\frac{\pi(z_b^{*2} - 3z_b^2)}{4d})z_b^2 \right. \\
& + 132d\pi \sinh(\frac{\pi(z_b^2 + z_b^{*2})}{4d})z_b^2 + 12d\pi \sinh(\frac{\pi(3z_b^2 + z_b^{*2})}{4d})z_b^2 + 18(\pi^2 z_b^4 - 5d^2) \cosh(\frac{\pi(z_b^2 - z_b^{*2})}{4d}) \\
& + (6d^2 - 2\pi^2 z_b^4) \cosh(\frac{\pi(z_b^{*2} - 5z_b^2)}{4d}) + (33d^2 - 95\pi^2 z_b^4) \cosh(\frac{\pi(z_b^{*2} - 3z_b^2)}{4d}) + 12(\pi^2 z_b^4 - 18d^2) \cosh(\frac{\pi(z_b^2 + z_b^{*2})}{4d}) \\
& \quad \left. - 60d^2 \cosh(\frac{\pi(3z_b^2 + z_b^{*2})}{4d}) + 39d^2 \cosh(\frac{\pi(5z_b^2 + z_b^{*2})}{4d}) \right] \sqrt{\tanh(\frac{\pi z_b^{*2}}{4d})} \Big]. \quad (5.96)
\end{aligned}$$

$$\hat{\Xi}_{0,F_2}^e(z_b) = \frac{3\pi^2 z_0^4 \text{csch}^2 \frac{\pi z_0^2}{2d} - 4(3d^2 + \pi^2 z_0^4)}{48d^2 z_0^2} \quad (5.97)$$

$$\hat{\Xi}_{0,F_2}^o(z_b) = - \frac{\pi^2 z_b z_b^* \text{csch}^2 \frac{\pi z_b^2}{4d} \text{sech}^2 \frac{\pi z_b^{*2}}{4d}}{16d^2 \sqrt{\coth \frac{\pi z_b^2}{4d}} \sqrt{\tanh \frac{\pi z_b^{*2}}{4d}} \left(\sqrt{\coth \frac{\pi z_b^2}{4d}} - \sqrt{\tanh \frac{\pi z_b^{*2}}{4d}} \right)^2} \quad (5.98)$$

The corresponding coefficients are plotted in fig. 5.5.

5.3.8 The Circular Chamber

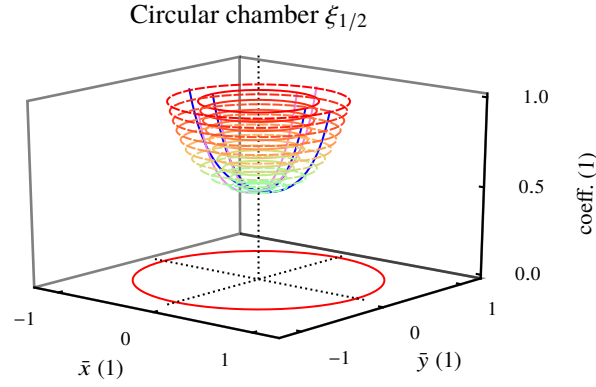


Fig. 5.6a: The coherent electric image tensor of the the circular chamber (explanation in section 5.3.1).

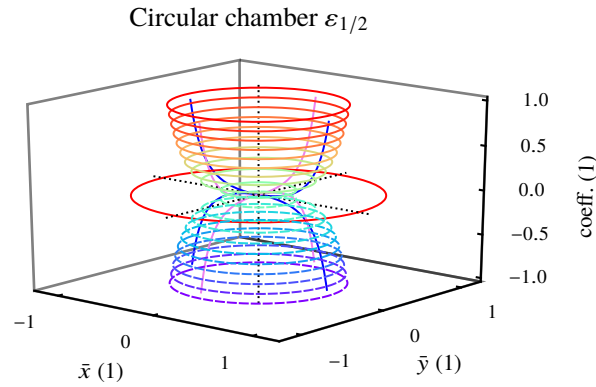


Fig. 5.6b: The incoherent electric image tensor of the the circular chamber (explanation in section 5.3.1).

A circle with radius h and centred has the electric image potential (the source singularity is subtracted):

$$\phi = 2 \log(z z_0^* - h^2). \quad (5.99)$$

A circular hole drilled into a ferromagnetic material with the relative permeability μ_r and a radius g and centred has the magnetic image potential (the source singularity is subtracted, see e.g. [22]):

$$A = \frac{2\lambda\beta_0}{\mu} \log(zz_0^* - g^2), \quad (5.100)$$

where $\mu := \frac{\mu_r+1}{\mu_r-1}$ and in the limit of a perfect perfect permeability of the ferromagnetic material $\mu \rightarrow 1$. From this we get the coefficients out of the electrical coefficients by setting $g = h$. The coefficients are:

$$\varepsilon_1^h = \mu\varepsilon_2^h = -\frac{h^2x_0^2}{2(h^2 - xx_0)^2}, \quad (5.101)$$

$$\xi_1^h = \mu\xi_2^h = \frac{h^2(h^2 + x_0^2)}{2(h^2 - xx_0)^2}, \quad (5.102)$$

$$\xi_1^v = \mu\xi_2^v = \frac{h^2(h^2 - x_0^2)}{2(h^2 - xx_0)^2}. \quad (5.103)$$

These electrical results are in agreement with Zotter ([9]). At the beam location we obtain:

$$\varepsilon_1^h = \mu\varepsilon_2^h = -\frac{h^2x_0^2}{2(h^2 - x_0^2)^2}, \quad (5.104)$$

$$\xi_1^h = \mu\xi_2^h = \frac{h^2(h^2 + x_0^2)}{2(h^2 - x_0^2)^2}, \quad (5.105)$$

$$\xi_1^v = \mu\xi_2^v = \frac{h^2}{2(h^2 - x_0^2)}. \quad (5.106)$$

The coefficients are plotted in figs. 5.6a and 5.6b.

5.3.9 The Elliptic Chamber

The electric potential of the centred elliptic chamber with the half-height h (parallel to the y-axes) and half height of w is provided in eq. (C.6). The mapping F is given in eq. (4.42). We use the abbreviations: $\text{cdC} := \text{cd}_{z_b^*} + \text{cd}_{z_b}$. And cd_{z_b} , nd_{z_b} and sd_{z_b} are the Jacobi elliptic functions with the argument $F(z_b)$. m obeys the implicit equation $\frac{\sin^{-1} \frac{h}{k}}{\pi} = \frac{K}{K'}$ (depicted in fig. C.1 for the ratio of width and height), $m'^2 = 1 - m^2$, $k^2 := w^2 - h^2$, and $K' := K(m')$, where K is the complete elliptic integral of the first kind. The definition of the elliptic integrals and Jacobi elliptic functions is provided in appendices C.1.1 and C.1.2.

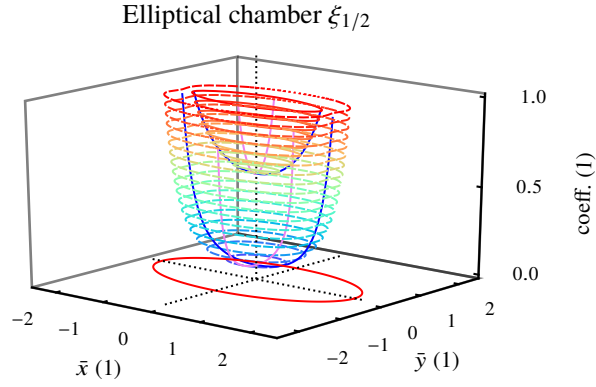


Fig. 5.7a: The coherent electric image tensor of the elliptical CERN PS vacuum chamber (explanation in section 5.3.1).

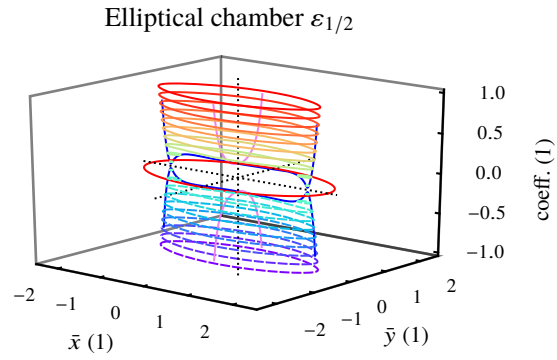


Fig. 5.7b: The incoherent electric image tensor of the elliptical CERN PS vacuum chamber (explanation in section 5.3.1).

$$\begin{aligned} \hat{\Xi}_{F1,0}(z_b) = & \\ \frac{1}{12} \left[- \frac{nd_{z_b}^2 F'(z_b)^2 \left(3cdC^2 cd_{z_b}^2 - 4cdCm'^2 sd_{z_b}^2 \left(cd_{z_b}^* - 2cd_{z_b} \right) - 12m'^4 sd_{z_b}^4 \right)}{cdC^2 sd_{z_b}^2} \right. & \\ + \frac{cd_{z_b}^* \left(2m^2 F'(z_b)^4 \left(5cd_{z_b}^2 + 2m'^2 sd_{z_b}^2 \right) - 3F''(z_b)^2 + 4F^{(3)}(z_b) F'(z_b) \right)}{cdCF'(z_b)^2} & \left. \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{6\text{nd}_{z_b} F''(z_b) \left(\text{cdCcd}_{z_b} - 2m'^2 \text{sd}_{z_b}^2 \right)}{\text{cdCsd}_{z_b}} \\
& + \frac{\text{cd}_{z_b} \left(2m^2 F'(z_b)^4 \left(5\text{cd}_{z_b}^2 - 4m'^2 \text{sd}_{z_b}^2 \right) - 3F''(z_b)^2 + 4F^{(3)}(z_b) F'(z_b) \right)}{\text{cdCF}'(z_b)^2} \\
& + \frac{6m^2 \text{cd}_{z_b} \text{sd}_{z_b} F''(z_b)}{\text{nd}_{z_b}} - \frac{3m^4 \text{cd}_{z_b}^2 \text{sd}_{z_b}^2 F'(z_b)^2}{\text{nd}_{z_b}^2} \Big]. \quad (5.107)
\end{aligned}$$

$$\hat{\Xi}_{F_1,0}^e(z_b) = - \frac{m'^4 \text{nd}_{z_b} \text{sd}_{z_b} \text{nd}_{z_b}^* \text{sd}_{z_b}^* F'(z_b) F'(z_b^*)}{\text{cdC}^2} \quad (5.108)$$

$$\begin{aligned}
\hat{\Xi}_{F_1,0}^o(z_b) = 12F'(z_b)^2 & \left[\frac{\text{cd}_{z_b}^2 F'(z_b)^4 \left(-3m^4 \text{sd}_{z_b}^4 + 2m^2 \text{nd}_{z_b}^2 \text{sd}_{z_b}^2 - 3\text{nd}_{z_b}^4 \right)}{\text{nd}_{z_b}^2 \text{sd}_{z_b}^2} \right. \\
& \left. - 3F''(z_b)^2 + 2F'(z_b) \left(F^{(3)}(z_b) + m'^2 F'(z_b)^3 \left(m^2 \text{sd}_{z_b}^2 + \text{nd}_{z_b}^2 \right) \right) \right] \quad (5.109)
\end{aligned}$$

At the centre with centred beam we obtain:

$$\epsilon_1^i = \pm \frac{(m(m+6)+1)K'^2 - 2\pi^2}{12\pi^2 k^2}, \quad (5.110)$$

$$\epsilon_1^{c.p.} = \frac{(m+1)^2 K (1-m^2)^2 - \pi^2}{4\pi^2 k^2}, \quad (5.111)$$

$$\epsilon_1^{c.n.} = \frac{\pi^2 - 4mK (1-m^2)^2}{4\pi^2 k^2}. \quad (5.112)$$

All coefficients are in numerical agreement to [7] p.6, eq.(4.1)-eq.(4.3). The coefficients are plotted in figs. 5.7a and 5.7b.

5.4 Summary

In this chapter, we derived the *formalism of the image tensors*, as operators of the underlying *conformal mappings* in terms of the *Green function* of the first and second kind. If we know the closed-form conformal mapping for specific geometries on the unit circle, we can calculate the impact of the indirect fields for these geometries in terms of closed-forms. It also allows us to include novel semi-analytical solutions for *arbitrary geometries* for the electric (as demonstrated for the beam-screen in section 7.2.1) and the magnetic case (as demonstrated for the PS magnets in section 6.2). The *complete fields* are formulated through the *complex Lorentz force* eq. (3.9).

Different regimes of the interaction phenomena were discussed. Formulas to estimate the tune shift for models of varying complexity, as used later in the applications, were provided (eqs. (5.38) to (5.39)).

Explicit normal form closed-form coefficients were calculated, including several *new chamber/magnet geometries for arbitrarily displaced beams*, as the *rectangle*, the *ellipse*, the *half-strip*, the *n-pole magnets*, explicitly for the *quadrupole* and the *octagonal chamber*.

The results were compared to the (on-axis) results provided in the literature to avoid errors in the derivations. It allowed correcting an error of a coefficient of the rectangular shape, that was found in the literature.

Chapter 6

Applications to the CERN Proton Synchrotron (PS)

The true sign of intelligence is not knowledge but imagination.
— Albert Einstein

6.1 The Machine

The CERN Proton Synchrotron (PS) is a key component in CERN's accelerator complex, where it usually accelerates either protons delivered by the Proton Synchrotron Booster or heavy ions from the Low Energy Ion Ring (LEIR). In the course of its history, it has juggled many different kinds of particles, feeding them directly to experiments or more powerful accelerators. With a circumference of 628 meters ($2\pi * 100$ m), the PS has 277 conventional (room-temperature) electromagnets, including 100 *combined-function* magnets to bend and focus the beams around the ring. The accelerator operates at up to 26 GeV/c. In addition to protons, it has accelerated helium, oxygen, lead, xenon and sulphur nuclei, electrons, positrons and antiprotons [116].

6.2 The PS Magnets

The PS main *magnetic unit* (MU) is a normal conducting, *combined-function magnet* used to bend and focus the particle beam. It is composed of two half-units: a focusing (F) and defocusing (D) half which are rigidly joined together and introduce an alternating-gradient focusing. Each half-unit consists of five laminated, *C-shaped* iron blocks of either "closed" (fig. 6.1a) or "open" (fig. 6.1b) hyperbolic pole profiles arranged in such a manner that a magnet has an arc shape with a bending radius $r_0 = 70,07$ m and the overall length of 4,260 m along the orbit. In the PS ring, there are 100 magnets of four types. These types differ from each other by the placement of the iron back-leg with respect to the beam orbit. Additionally, the order of the focusing and the defocusing half-unit changes. The four different configurations have the naming R, S, T and U as defined in table 6.1 and shown schematically in fig. 6.2. The outside units (upper drawings) have the main coil on the inside of the ring, while for the inside units the opposite is the case. Two MUs are separated by a straight

section, so the pattern is ‘FOFDOD’ (‘O’ stands for a straight section). In fig. 6.3 a photograph of magnetic unit of type T can be seen.

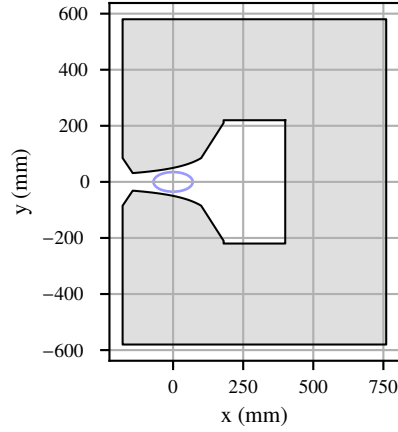


Fig. 6.1a The cross-section of a “closed” block of the PS magnetic units (grey). The position of the vacuum chamber is shown (blue).

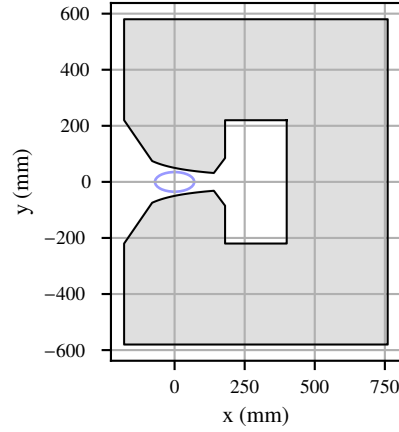


Fig. 6.1b The cross-section of an “open” block of the PS magnetic units (grey). The position of the vacuum chamber is shown (blue).

Table 6.1: The nomenclature of the magnetic units in the PS. The types are drawn schematically in fig. 6.2.

Type	Configuration	Number of installed units
R	Yoke outside, defocusing-focusing	35
S	Yoke outside, focusing-defocusing	15
T	Yoke inside, defocusing-focusing	35
U	Yoke inside, focusing-defocusing	15

6.3 The Modelling of the Elements

The PS model consists of three components, the vacuum chamber and the two magnetic half-units also-called blocks. The straight sections between the magnetic units are used for placing accelerating cavities, beam diagnostic devices, injection and extraction elements, and magnetic lenses. For simplicity within this study, these regions are treated as drift regions, where only the vacuum chamber has an impact

Fig. 6.2 The different MUs of the PS. Taken from [116].

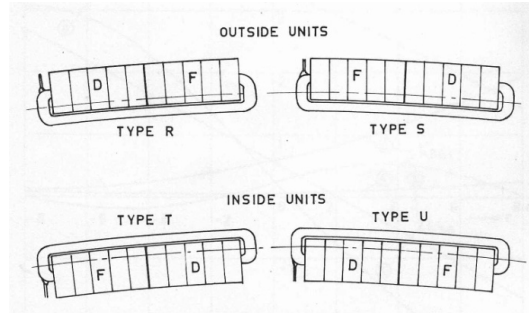


Fig. 6.3 MU of the PS of the type T (see fig. 6.2). Taken from [117].



onto the beam. This is no significant limitation since these elements have a negligible contribution compared to the vacuum chamber and the magnetic units.

6.3.1 The Vacuum Chamber

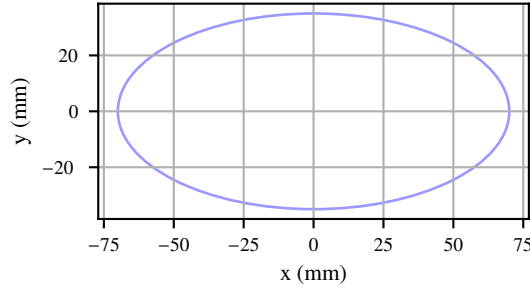
6.3.1.1 Geometric Setting

In the PS the cross-section of the vacuum chambers is varying with the elliptic geometry being the standard chamber. Consequently, the vacuum chamber is approximated as an ellipse with the dimensions: horizontal semi-axis $a = 7$ cm and vertical semi-axis $b = 3.5$ cm. An illustration of the chamber can be seen in fig. 6.4.

6.3.1.2 The Image Tensors

The electric image tensor of the geometry as shown in fig. 6.4 was derived in section 5.3. The image tensors of this shape can be expressed through closed-forms and were derived for off-axis calculations for the first time in section 5.3.9. The

Fig. 6.4 An approximation of the PS vacuum chamber geometry.



incoherent off-axis coefficients for the PS vacuum chamber are shown in fig. 5.7b, the coherent images tensors can be seen in fig. 5.7a.

6.3.2 The Model of the Combined-Function Magnets

In the context of this thesis, several models, including closed-forms of the PS combined function magnets are presented for the first time. So far, these elements have been treated as parallel plates. In the following, a polygonal model of the true shape of the PS magnet elements is formulated. It is more accurate but also more complicated to compute. It is then compared to analytic closed-form approximations of the PS magnet elements. These solutions are easy and fast to calculate, although there is a discrepancy to the true solution if the approximated shape differs too strongly from the true shape.

6.3.2.1 Polygonal Modelling

The true shape (cross-section) of the “closed” and the “open” PS magnet block is shown in grey in fig. 6.1a and fig. 6.1b, respectively. Both shapes are modelled as polygons in order to establish a model of mathematically limited complexity. The Schwarz-Christoffel-transformation mapping was employed, and the corresponding parameter problem was solved as described in section 4.1. The hyperbolic pole-profiles are approximated using nine vertices as shown in fig. 6.6 for the “closed” and in fig. 6.5 for the “open” block. In both cases, a total number of $n = 30$ vertices has been used to obtain sufficient accuracy. As discussed in sections 3.6.3 and 4.1.4, the Neumann function of such a shape is a conformal invariant (lemma 3.4). The exterior of the polygon is mapped onto the interior of $\mathbb{D}$ and the sink at infinity is moved to the origin. This model serves as a reference model. It appraises the validity of the simpler models as described in the following.

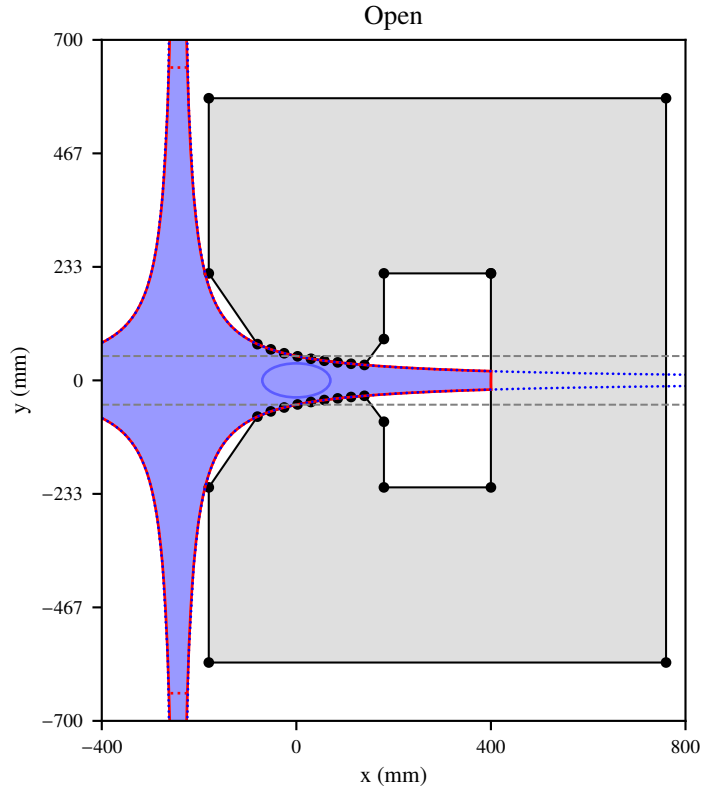


Fig. 6.5: The approximations of the PS open magnet block as described in section 6.4.1.

6.3.2.2 Closed-Form Expression for the Magnetic Blocks

A closed-form for the solution of the boundary problem is achieved by simplifying the shape. Following the ideas in section 4.3.2, the mapping of an n -polar structure is mapped onto $\mathbb{D}$ via the n^{th} -root. If we cut the tails (the parts of the shape extending to infinity) of an n -pole, a Jacobi elliptic function is involved, since the n^{th} -root maps the cut n -pole onto a rectangle (solution is provided in eq. (4.21)). The branch cut of the mapping lies in every odd tail. According to fig. 6.7, where the cut 4-pole mapping onto a circle is shown, by rotation the cut can be moved either on the x or the y -axis. The structure is opened by moving either the blue (fig. 6.7, (b)) or the dashed line (fig. 6.7, (b1)) to ∞ . As described in section 3.6, by opening a shape to ∞ , it is possible to find a closed-form for the mapping. Figure 6.7 shows that the

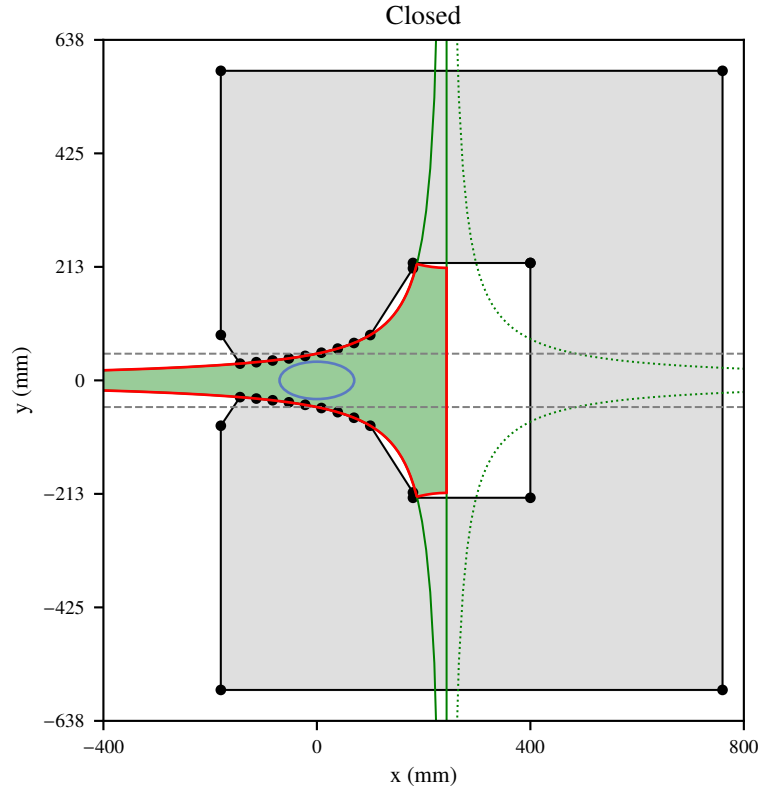


Fig. 6.6: The approximations of the PS closed magnet block as described in section 6.4.2.

cut 4-pole is mapped onto a Riemann manifold with two sheets. The sign of the real part is used to keep the information of the corresponding sheet. It allows restoring the information of the corresponding sheet onto the initial area (coloured grey and white).

The half-quadrupole can in principle be found, as shown in fig. 6.8. The difference to the full quadrupole is the mapping onto a strip and afterwards on the upper half-plane. The branch cut remains as a small stub. The final mapping, although represented as a closed-form, is involved. Instead, there is the possibility to take advantage of the symmetry. The source, shown as a red point in fig. 6.9, can be mirrored along the y -axis. It results in the full quadrupole structure with two sources - the source itself and its image - and can be treated in the same way as the quadrupole.

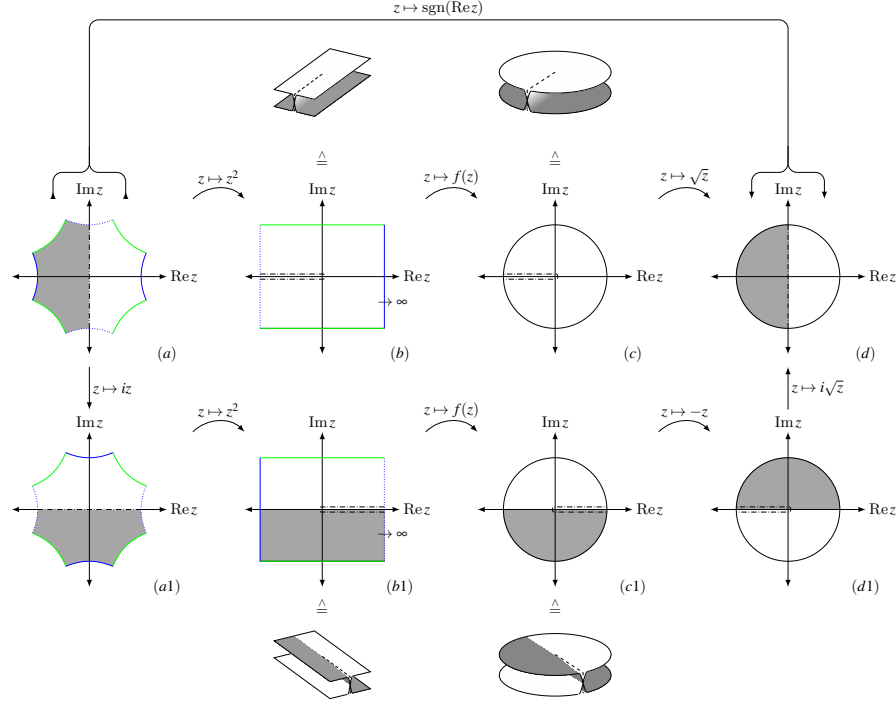


Fig. 6.7: The novel mapping of the combined-function magnet structure. f denotes an Jacobi elliptic function (see section 4.2.2).

6.4 Results

The attempt of approximating the magnets of the PS as two infinite plates fails when the form factors are calculated off-centred due to the failure in symmetry. The parallel plates are invariant against a translation in x direction. As a consequence, the potential and hence the field is invariant as well. In reality, the indirect fields do not show this behaviour. As shown in the last section, there are simple solutions for the problem, avoiding the complex calculation of the inverse mapping of the Schwarz-Christoffel-transformation. To benchmark the results, as described, the Schwarz-Christoffel-transformation for a sufficiently accurate polygonal modelling with 30 vertices is calculated and the solution of an exterior domain for the Neumann function (lemma 3.3) was used. Both approaches, as mentioned, are novel in this context, and so far only the strip was known.

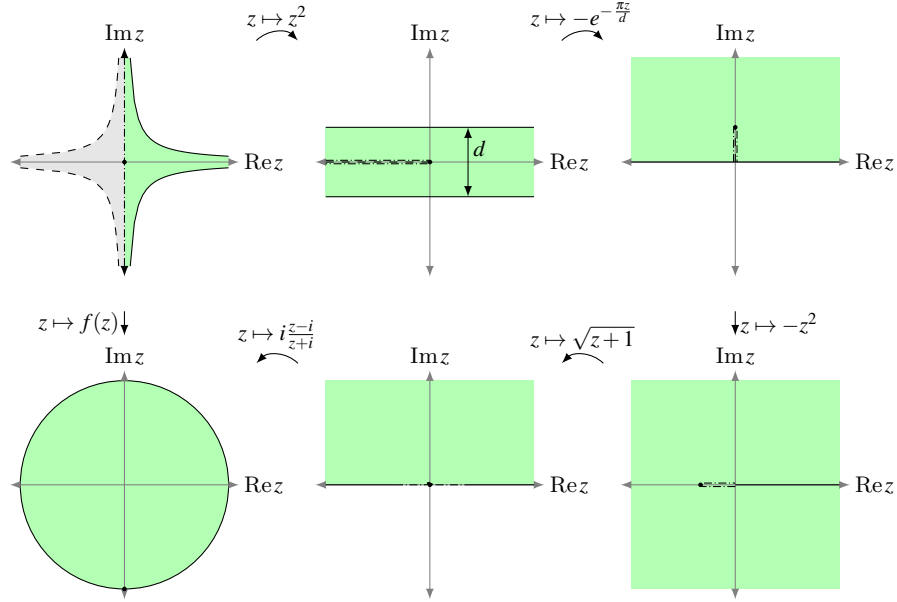


Fig. 6.8: The composition of conformal mapping of the half-quadrupole onto $\mathbb{D}$.

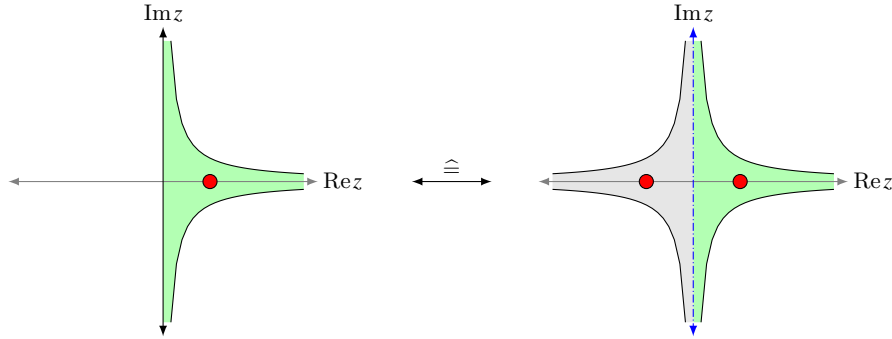


Fig. 6.9: The solution of the half-quadrupole can be constructed either by the long path sketched in fig. 6.8 or by a simple trick.

6.4.1 The Open Magnetic Block

The different models of the open magnetic block are drawn in fig. 6.5. As a first closed-form approximation, the quadrupole (FQ - full quadrupole) is developed (dotted blue line). The tails of the quadrupole overlap the true boundary. The result is a significant error in the calculated field. If all the tails are cut, labelled as FQ_{closed} (red shape - all tails are cut symmetrically at the position, where the right tail starts to overlap the polygonal model, as indicated for the upper and lower tail as dashed red

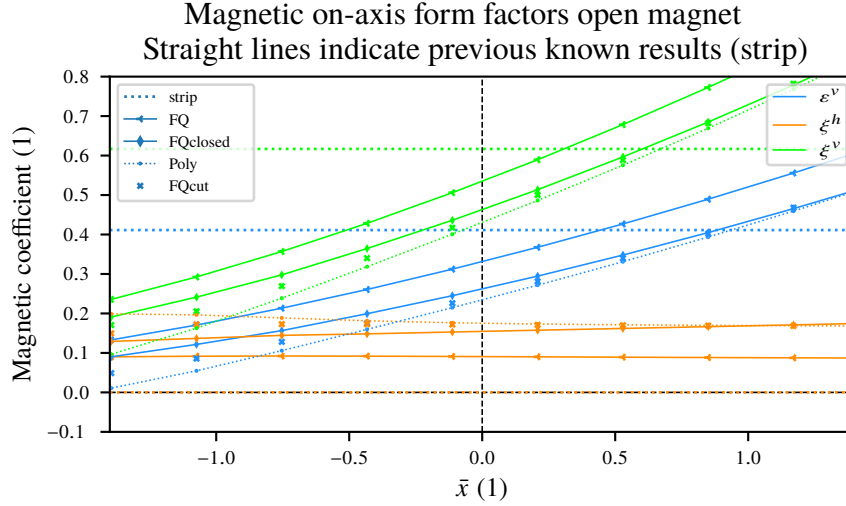


Fig. 6.10: The magnetic coefficients of the open magnetic block as defined in eqs. (5.35) to (5.37). Figure 6.5 shows the corresponding models.

line. Also the left tail is cut which is not visible in fig. 6.5.), the *novel integral-equation* eq. (3.57) has to be employed to obtain the field and the magnetic coefficients. Hence, no closed-form solution can be obtained using this model. Opening only the upper and lower tails of the shape leads (the upper and lower tails are opened to infinity, while the right and due to symmetry reasons also the left are cut) to a closed-form as proved in eq. (3.62). We term this model the *FQcut* (red shape with blue filling). By doing so, the error is further reduced w.r.t. the polygonal model (*Poly*).

The different magnetic Laslett coefficients of these models along the x -axis are presented in fig. 6.10. The parallel plate model (*strip*, indicated as parallel grey dashed lines in fig. 6.5) shows an offset at the position of the centre of the vacuum chamber. The model leads to the wrong field along the symmetry axis and hence to the wrong estimation of the caused tune-shift. The *FQ* still exhibits a strong offset, while showing the correct quantitative behaviour off-centred. By cutting all tails (*FQclosed*), the offset is further modified, while the area outside the magnet is reduced too strongly and no closed-form exists. Opening the tails outside the magnet (*FQcut*) results in an accurate model for the magnet block - the best closed-form expression found so far. At the opening of the magnet, the difference of the model to the true shape becomes obvious, and the magnetic coefficients deviate strongly from the polygonal solution and for largely off-centred beams the model has to be taken with care.

The *FQcut* closed-form expression for the Neumann function is (for an on-axis beam as used for the calculations):

$$\begin{aligned}
N_{d.i.}(z, z_0) = & \frac{1}{2\pi} \log \left[\operatorname{csch} \frac{\pi x_0}{2d} \sinh \frac{\pi(x_0 + z^2)}{2d} + 1 \right] \\
& \left[\sqrt{\frac{2}{\operatorname{csch} \frac{\pi x_0}{2d} \sinh \frac{\pi(z_0^2 + x_0)}{2d} + 1}} - 1 \sqrt{\frac{2}{\operatorname{csch} \frac{\pi x_0}{2d} \sinh \frac{\pi(x_0 + z^2)}{2d} + 1}} - 1 \right] \\
& \left[\sqrt{\frac{2}{\operatorname{csch} \frac{\pi x_0}{2d} \sinh \frac{\pi(x_0 + z^2)}{2d} + 1}} - 1 - \sqrt{\frac{2}{\operatorname{csch} \frac{\pi x_0}{2d} \sinh \frac{\pi(x_0 + z_0^2)}{2d} + 1}} \right]. \quad (6.1)
\end{aligned}$$

d is the distance of the pole-shoes at the centre of the vacuum chamber and the offset from the origin is x_0 .

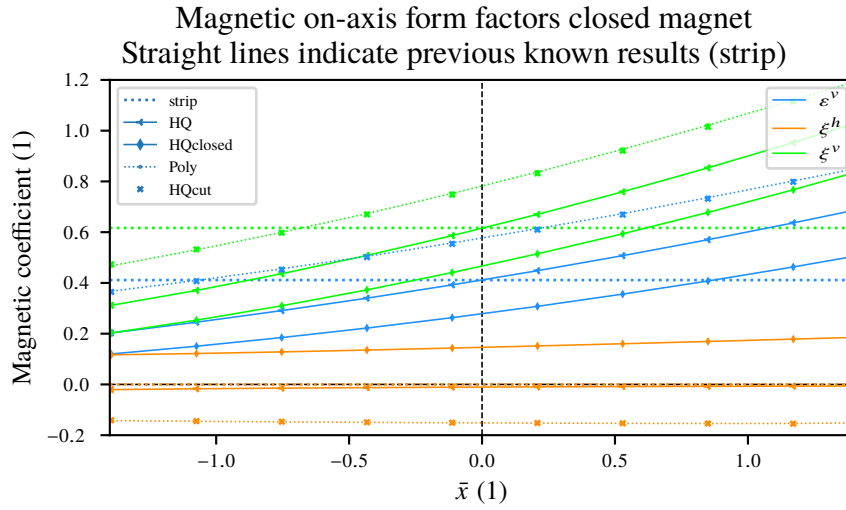


Fig. 6.11: The magnetic coefficients of the closed magnetic block as defined in eqs. (5.35) to (5.37). Figure 6.6 depicts the corresponding models.

6.4.2 The Closed Magnetic Block

The situation for the closed magnetic block is a bit more complex. Figure 6.11 shows all magnetic coefficients and the corresponding models are shown in fig. 6.6. Again, as for the open magnetic block, there is a large offset for the centred case for the simple strip model (*strip*, indicated as parallel grey dashed lines in fig. 6.6) compared to the polygonal model. The quadrupole model (green lines in fig. 6.6) would have a considerable overlap with the true shape leading to a large field error. Hence, a

different model has to be used. One way to reduce this error is to take half of the quadrupole model, denoted by *HQ* (solid left green shape in fig. 6.6). This model shows still a notable offset compared to the polygonal model (*Poly* - in grey). Cutting all tails (*HQclosed* - red shape with all tails cut. The cut of the tail extending to the negative x -axis is not visible in fig. 6.6) involving the solution of the integral-equation eq. (3.62) does not improve this significantly. Opening the half-tails (*HQcut* - red shape with green filling) decreases the deviation to the polygonal solution further. By doing so an excellent closed-form solution was found.

The final closed-form expression (*HQcut*) for the Neumann function has the form (provided here for an on-axis beam as used for the calculations):

$$N_{f.i.}(z, z_0) = \frac{1}{2\pi} \log \left| \frac{\operatorname{csch}^2 \frac{\pi x_0}{2d} \left[\cosh \frac{\pi(x_0 - z^2)}{d} - \cosh \frac{\pi(x_0 - z_0^2)}{d} \right]}{2 \left[\operatorname{csch} \frac{\pi x_0}{2d} \sinh \frac{\pi(x_0 - z_0^2)}{2d} + 1 \right]^2} \right|. \quad (6.2)$$

d is the distance of the pole-shoes at the centre of the vacuum chamber and the offset from the origin is x_0 .

6.4.3 The Multi-Turn Extraction Tune-Shift Closed-Form Model

Efforts to find a suitable replacement for the *continuous transfer* (CT) extraction mode, which has been the technique used to transfer the 14 GeV/c proton beams from the PS to the Super Proton Synchrotron (SPS) for the fixed-target physics program, converged on the proposal of a novel method of beam manipulation named Multi-Turn Extraction (MTE) [118]. This technique is based on transverse beam splitting induced by crossing a stable resonance in the horizontal phase space and solves the issue related to the unavoidable beam losses of the CT extraction. In the case of MTE, the stable fourth-order resonance is used, which generates four beamlets around a central core in phase space. In general, any resonance (stable or unstable) can be considered to design a multi-turn extraction scheme. Furthermore, MTE provides an improved betatron matching of the PS beam at injection in the SPS compared to CT. The MTE method became operational in 2015.

In the following, we calculate the tune-shift in its simple form as shifting the betatron tune. The change in tune is obtained from a model simulated using the code *MAD-X*, which stands for methodical accelerator design [119]. A model of the arrangement of the elements of the accelerator can be found in [120]. The β -functions and the transverse position of the four beamlets, with a periodicity of four turns are shown in fig. 6.12. The corresponding lattice functions were calculated using *MAD-X* and the essential extension *Polymorphic Tracking Code* (PTC - [121]), which is a more advanced tracking code.

Equations (5.38) and (5.39) show how the β -function at a given position and the corresponding form factor contribute to the tune-shift. The different models are ordered

in terms of the complexity, following the scheme as given in section 5.2.1. The simplest case is the smooth approximation, where the average β -function along the ring is taken, and the image field is taken at the centre of the cross-section of the elements (denoted by *smooth*). A more sophisticated model is to calculate the β -function at several positions along the ring, also by increasing the longitudinal resolution of the model. This model is denoted by *centred* and the coefficients are listed in table 6.2. The corresponding form factors are calculated at a centred position. These two cases are compared with the model of two parallel plates, denoted as *strip*. The electric contribution is given by plates of a distance of 3.5 cm, which is the distance to the vacuum chamber at the centre of the geometry. The magnetic contribution is the average of the two plates at a distance of 5 cm, the average distance of the magnetic elements at the centre of the geometry.

In tables 6.3 and 6.4, the values for the different tune-shifts are listed at the extraction energy of 14 GeV for the individual contributions (separated into electric and magnetic) and the full model, respectively. The term *exact* indicates that we use the closed-form expressions as discussed in sections 6.4.1 and 6.4.2 and the labels *smooth*, *centred* and *closed orbit* are defined in section 5.2.1. The term *closed orbit* indicates that the β -function is calculated as in the *centred* model, while the image coefficients are calculated at the transverse position of the closed orbit. The error made by the *strip* approximation is large, especially for the horizontal coherent tune-shift ΔQ_h^{coh} . The reason is the translation symmetry of the strip in this dimension, which is not the case in the real application. For the strip model the coefficient vanishes, while for the *smooth* model the error compared to the *centred* model is still noticeable. For the vertical coherent tune-shift ΔQ_v^{coh} the error made by this approximation is not negligible, but not as significant as for the coherent horizontal tune-shift. For the incoherent tune-shifts, the vertical values, ΔQ_v^{inc} , show a strong disagreement. The horizontal tune-shifts ΔQ_h^{inc} reveal the smallest deviation. An interesting observation is the fact that the alternating magnetic structure of the PS of "open" and "closed" blocks leads in average to the cancellation of their impact. In table 6.2 the average of such a combination is provided and is termed with the suffix *average*. Hence the error by approximating the magnets as parallel plates is relatively small in the smooth approximation.

The contribution of the indirect electric fields are spatially resolved and depicted in fig. 6.13 for different models. Notable differences can be seen for the coherent horizontal effect (lime) as the coefficients are calculated for the ellipse instead of parallel plates and primarily, due to the massive orbit changes, the *closed orbit* results (lowest plot). The same holds for the magnetic effect (fig. 6.14), while the closed-form solutions were used. Concluding, the closed-form approximations (exact modelling for the vacuum chamber and the combined-function magnets), seem to be valid and principally allowing scaling law studies. Further notable is that the effects for the magnetic and the electric indirect interaction are of the same magnitude (see table 6.3).

The results of table 6.4 show that the tune-shift is in the order of 0.1×10^{-3} , which is not critical for operation but is related to the position of the beamlet, which is decisive at the multi-turn-extraction. This is the topic of the following section.

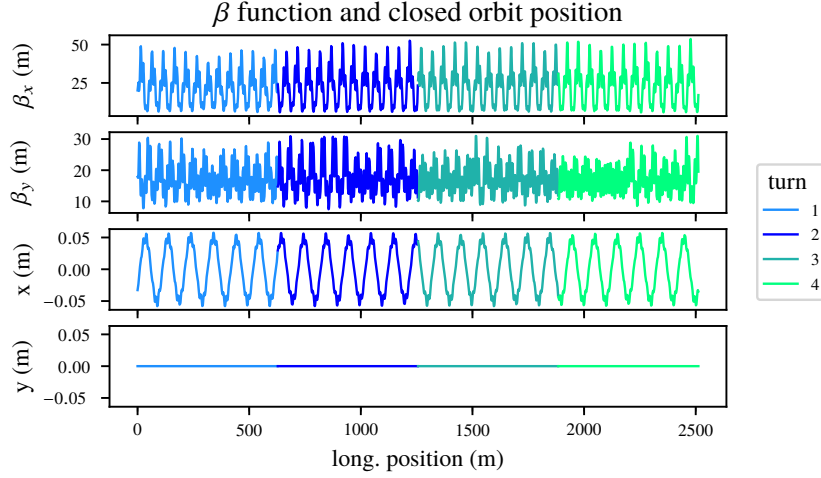


Fig. 6.12: The simulated closed orbit and β -function of the four turn orbit in the PS during the multi turn extraction in the transverse plane.

Table 6.2: The form factors for the different elements at the centred position. The average of the focusing and defocussing magnets are close to the value of the strip.

Form factor/model	ϵ_h (1)	ϵ_v (1)	ξ_h (1)	ξ_v (1)
magnetic strip	-0.411	0.411	0.000	0.617
magnetic average exact	-0.406	0.406	0.010	0.607
magnetic average polygonal	-0.403	0.403	0.012	0.602
magnetic defocussing exact	-0.245	0.245	0.172	0.446
magnetic focusing exact	-0.568	0.568	-0.151	0.768
magnetic defocussing polygonal	-0.240	0.240	0.175	0.439
magnetic focusing polygonal	-0.566	0.566	-0.151	0.765
electric exact	-0.352	0.352	0.167	1.222
electric strip	-0.206	0.206	0.000	0.617

6.4.4 The Multi-Turn Extraction Intensity-Position-Dependence Studies

In this section a *tracking* is discussed. The beam position of so-called beamlets is shifted by intensity depended effects, which is of high importance concerning the operation. In this study, we provided the indirect fields (in closed-form Green functions) instead of the form factors to track the evolution of the beam. A need for an accurate indirect field models becomes evident since these effects, as now shown, are dominating.

A measurement campaign [122] showed that there is a linear dependence of the position of the beamlets with the intensity. A numerical investigation indicates

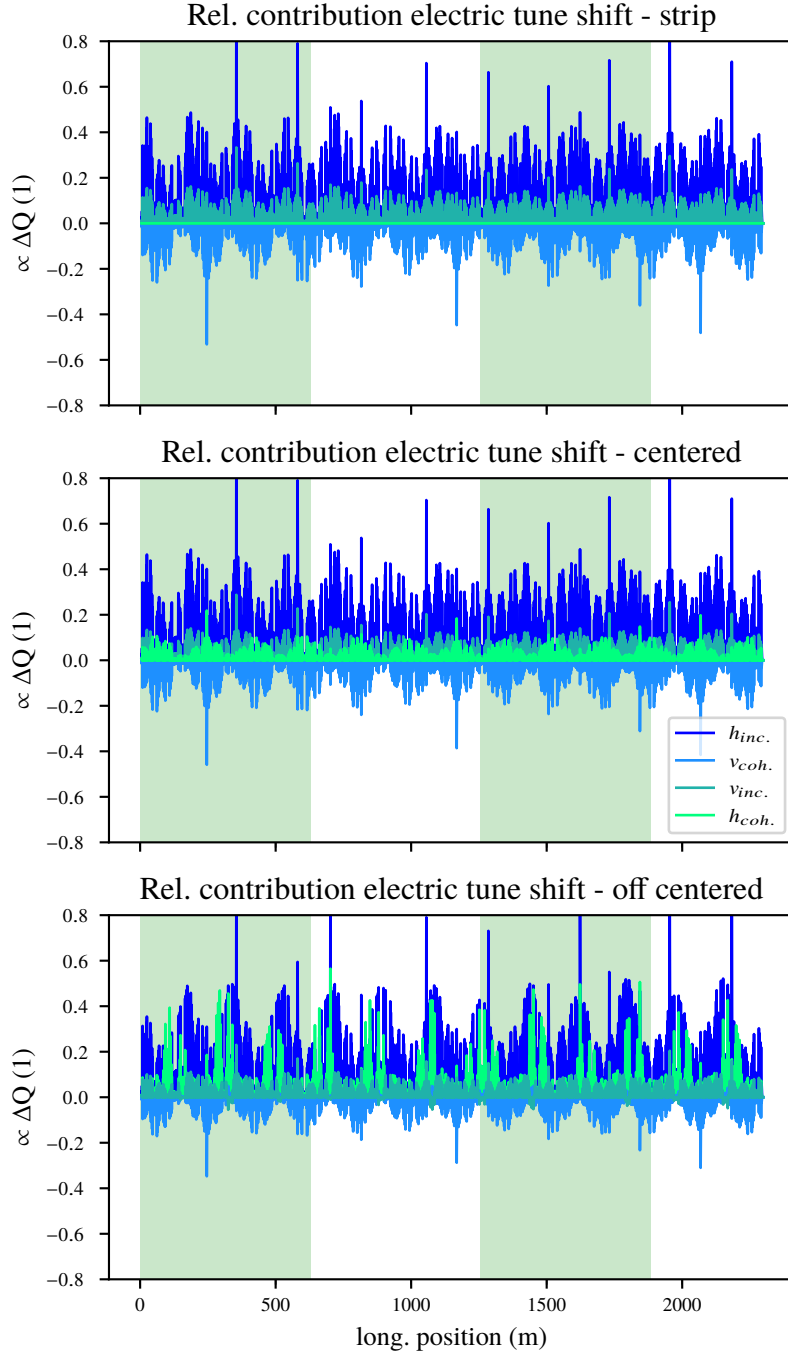


Fig. 6.13: The local contribution to the tune-shift caused by the electric boundaries of different models. The tune-shift is normalized to the maximal peak of the overall contributions.

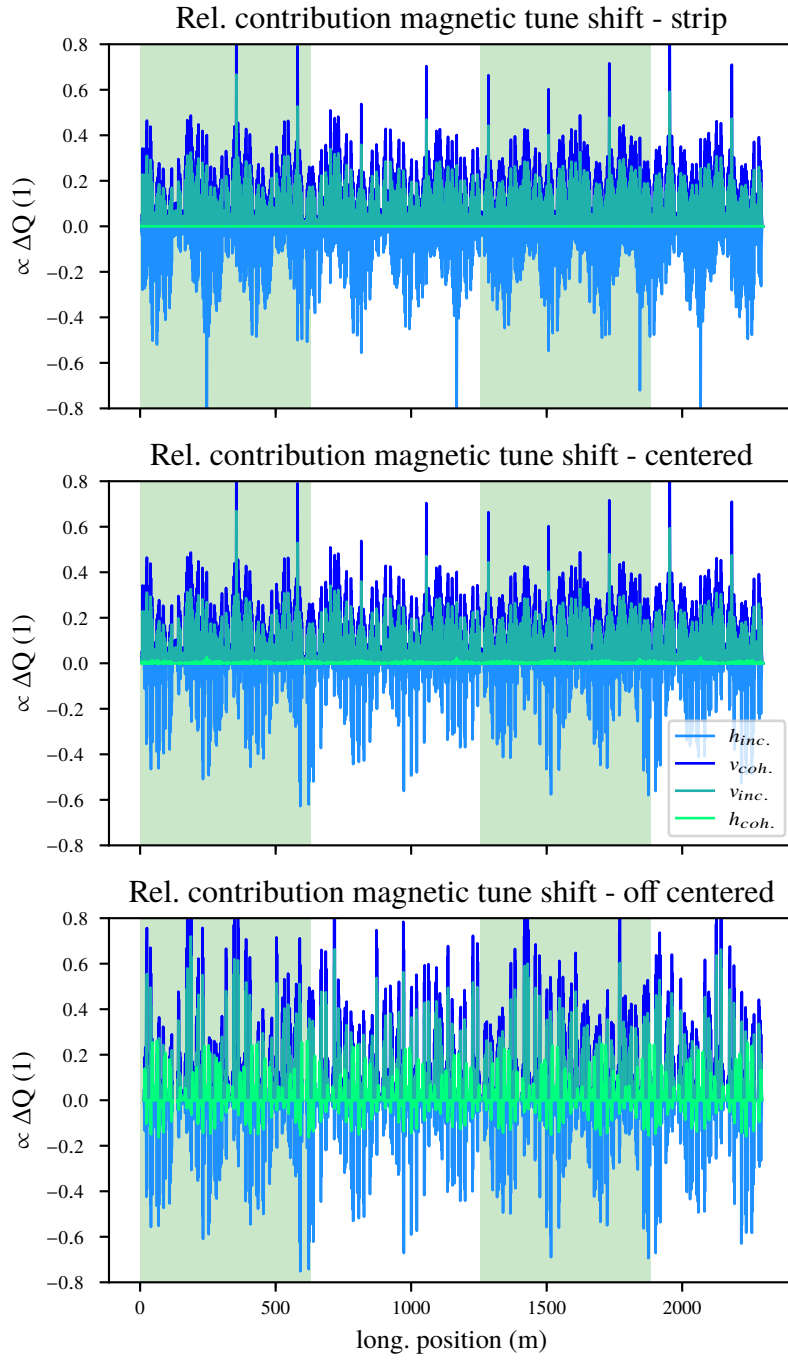


Fig. 6.14: The local contribution to the tune-shift caused by the magnetic boundaries. The tune-shift is normalized to the maximal peak of the overall contributions.

Table 6.3: The tune-shifts of the PS per proton. The different models, calculated at the energy of 14 GeV, are separated into their different contributions (electric and magnetic). The labelling is defined in section 5.2.1 and the text.

tune-shift/model	$\Delta Q_h^{inc.}$ (10^{-16})	$\Delta Q_v^{inc.}$ (10^{-16})	$\Delta Q_h^{coh.}$ (10^{-16})	$\Delta Q_v^{coh.}$ (10^{-16})
magnetic smooth exact	0.929	-0.929	-0.024	-1.090
magnetic smooth polygonal	0.921	-0.921	-0.027	-1.081
electric smooth exact	1.009	-0.793	-0.480	-2.757
magnetic strip	0.940	-0.940	-0.000	-1.108
electric strip	1.204	-0.947	-0.000	-2.840
magnetic centred exact	0.860	-0.786	-0.093	-1.147
magnetic centred polygonal	0.852	-0.781	-0.097	-1.139
electric centred exact	1.009	-0.793	-0.480	-2.757
magnetic closed orbit exact	0.880	-0.801	-0.093	-1.170
magnetic closed orbit polygonal	0.857	-0.790	-0.099	-1.154
electric closed orbit exact	0.948	-0.716	-1.548	-3.915

Table 6.4: The tune-shifts of the PS of the different approximations at the energy of 14 GeV. The centred case takes usage of the newly derived closed-form expressions. The labelling is defined in section 5.2.1 and the text.

tune-shift/model	$\Delta Q_h^{inc.}$ (10^{-16})	$\Delta Q_v^{inc.}$ (10^{-16})	$\Delta Q_h^{coh.}$ (10^{-16})	$\Delta Q_v^{coh.}$ (10^{-16})
smooth exact	1.937	-1.722	-0.503	-3.847
centred exact	1.869	-1.579	-0.573	-3.904
closed orbit exact	1.828	-1.517	-1.641	-5.085
closed orbit polygonal	1.805	-1.506	-1.647	-5.068
magnetic strip	1.949	-1.733	-0.480	-3.865
all strip	2.144	-1.887	-0.000	-3.948
For a typical number of $N=2 \times 10^{13}$ protons:				
model/shift	$\Delta Q_h^{inc.}$ (10^{-3})	$\Delta Q_v^{inc.}$ (10^{-3})	$\Delta Q_h^{coh.}$ (10^{-3})	$\Delta Q_v^{coh.}$ (10^{-3})
smooth exact	0.387	-0.344	-0.101	-0.769
centred exact	0.374	-0.316	-0.115	-0.781
closed orbit exact	0.366	-0.303	-0.328	-1.017
closed orbit polygonal	0.361	-0.301	-0.329	-1.014
magnetic strip	0.390	-0.347	-0.096	-0.773
all strip	0.429	-0.377	-0.000	-0.790

that indirect field effects cause the main component of this dependence. The corresponding calculations were carried out with the space-charge simulation code *SIMPSONS* [123] with an extension to include the closed orbit calculation for the outer islands as well as the indirect field effects. The approximation of the effect of the fields was done using the formulas provided in this work (eqs. (D.10) and (D.11)) to include the model of two infinite plates. Also, the rectangular model for the electric interaction with the boundaries was investigated and the used formula was derived

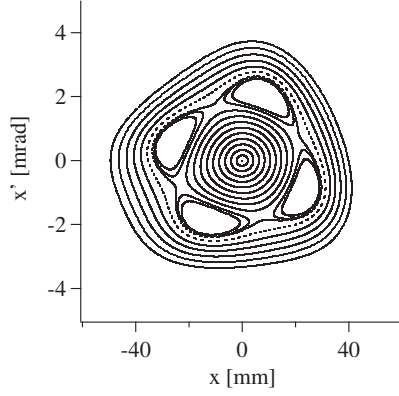


Fig. 6.15a The Phase space portrait taken from [17]. No space-charge effects are included.

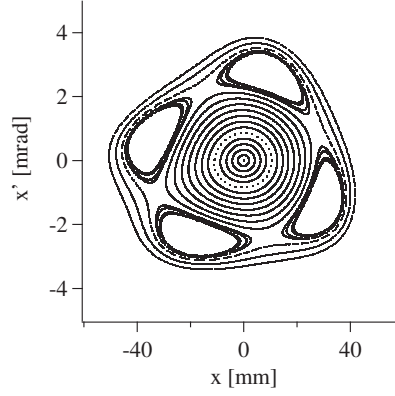


Fig. 6.15b The Phase space portrait taken from [17]. Space-charge effects are included as described in the text.

within this thesis (eq. (4.21)). These approximations were justified because the beam spends only a small fraction of its time near the vertical boundaries.

The impact of the intensity is shown in figs. 6.15a and 6.15b. The left plot shows the centre and four islands in the horizontal phase space portrait without the effect of space-charge. The right plot shows the impact of space-charge of five equally charged beamlets for a total intensity of 3.28×10^{13} p. Electric boundary conditions modelled as parallel plates are included (eq. (D.10)) and the beam momentum is 14 GeV/c - the MTE extraction energy of the PS. The position of the islands moves outwards. Higher-order magnets, namely octupoles, can in principle influence the direction, but due to the limited scope of this work, we refer for details to [17].

The configuration is calculated starting from space-charge-free orbits as initial condition and for a few turns including the fields by taking point charges located at the centre of the core and the centre of the island into account. So, oscillations are provoked around the initial orbits. Taking their averages act as new initial orbits. The procedure is repeated until a convergence criterion is met. The fields are calculated at specific locations around the ring only. In fig. 6.16 the different components of the intensity-dependent shift in position are shown. The indirect space-charge plays the major role, while the direct space-charge shows no significant impact. Interestingly, the electric and magnetic contribution of the indirect field are of similar strength as already observed in the related tune-shift study (section 6.4.3). The approximation through parallel plates breaks down, if the charge distribution centre is too far from the centre of the chamber as shown in fig. 6.17, where the beam is generated in a stable island with a large amplitude. There the impact of the model is compared to a rectangular approximation. The corresponding electric potentials are shown for a strongly off-centred source point in figs. 6.18 and 6.19 (the explicit Green function is provided in appendix C.1.3). The importance of a good model for the magnetic boundaries was also shown in [124], since only by including the magnetic interac-

Fig. 6.16 The different contributions to the position shift of the island due to the intensity. Taken from [17].

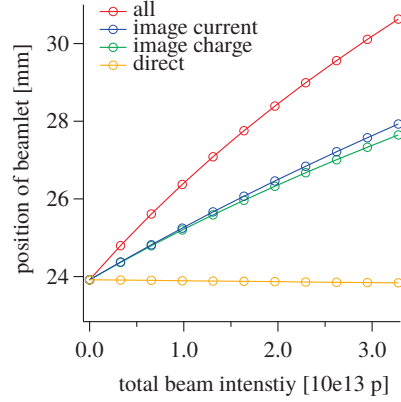
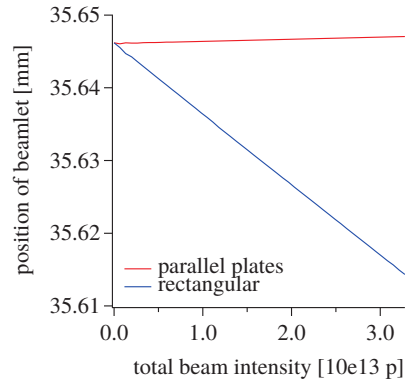


Fig. 6.17 Parallel plates and rectangular vacuum chamber models at a strongly off-centred position (eqs. (4.21) and (D.10)). Taken from [17].



tion, the simulations fit the measured bunched beamlet position.

The previous considerations attest the importance and the relevance of having simple forms of fields as in tracking algorithms. Still, the closed-forms of the combined-function magnets (eqs. (6.1) and (6.2)), as well as the elliptic boundary eq. (C.6), have to be included into this MTE model. We clearly see in the comparison of the different models figs. 6.10 and 6.11, that the field coefficients differ notably.

Fig. 6.18 The vacuum chamber approximated as two parallel plates (eq. (D.10)).

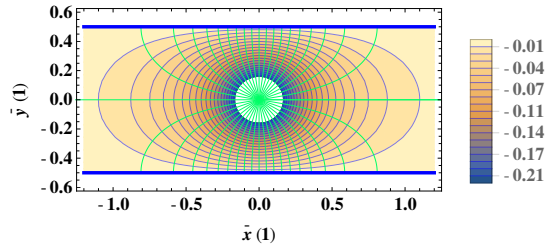
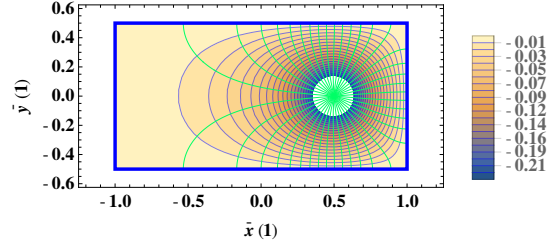


Fig. 6.19 The vacuum chamber approximated as a rectangle (eq. (4.21)) for a strongly displaced beam.



6.5 Summary

In this chapter, the results of studies concerning the CERN PS were presented. Entirely new *closed-form* expressions for several models of increasing unprecedented accuracy of the PS *combined-function* magnets were derived. The opening of a domain to infinity led to these closed-forms as discussed in section 3.5.4. It was necessary to find these solutions since only the parallel plate solution was known beforehand.

These and novel closed-form solutions (section 5.3.9) to model the off-axis elliptical vacuum chamber were used to compute, for the first time, the *intensity-dependent indirect tune-shift* during the MTE. The different approximations for the true geometries were compared. Significant differences can be observed during the MTE process, due to the largely off-centred beam. The alternating pattern of open and closed magnetic blocks tend to compensate for the magnetic effects.

On top of the closed-form solutions, a new semi-analytical approximative and more accurate solution for the PS-magnets was given. The magnets were modelled as polygons, which is novel in this context and can be used to calculate the Neumann function for arbitrary unclosed magnetic yoke structures and hence other machines as well. These precise calculations were used to justify the usage of the closed-forms since there was so far no better model available.

Finally, during the MTE, the position of the beamlets of the split beam reveals an *intensity-dependent* behaviour (measurements). A detailed study has shown that the main contribution stems from the indirect fields. It confirms the importance of accurate modelling of indirect field influence as used in tracking codes. All closed-form expressions for the fields, as used in these simulations, were derived in the context of this thesis.

Chapter 7

Applications to the CERN Large Hadron Collider (LHC)

First you guess. Don't laugh, this is the most important step. Then you compute the consequences. Compare the consequences to experience. If it disagrees with experience, the guess is wrong. In that simple statement is the key to science. It doesn't matter how beautiful your guess is or how smart you are or what your name is. If it disagrees with experience, it's wrong. That's all there is to it. "

— Richard P. Feynman

7.1 The Machine

The Large Hadron Collider (LHC) is the largest circular accelerator of the world and has a length of 27 kilometres. LHC reuses the tunnel of its predecessor LEP (Large Electron Positron collider). The shape of the LHC is not entirely circular; it consists of eight arcs, and eight straight sections termed insertions. The ring is about 100 m underground along the French-Swiss border close to Geneva and is run by CERN. The LHC is a hadron-hadron collider and consists of two counter-rotating beams crossing at four points where huge detectors track the interaction products [125] as shown in fig. 7.1 which gives a schematic overview of the accelerator. The eight octants are labelled by 1-8 and the four interaction points with their detectors, namely CMS, LHCb, ATLAS and ALICE, where Beam 1 (blue) and Beam 2 (red) collide. Two thousand eight hundred eight bunches of protons (each having a possible intensity of 1.15×10^{11} charges) are circulating in the LHC, which collide 40 million times per second. A system of the radio frequency cavities is used to accelerate the beam to the top energy, which is about 15 times higher than the injection energy of 450 GeV yielding a final value of 7 TeV and a centre of mass energy of 14 TeV. In fig. 2.1, the injector chain of the LHC is pictured. The LHC operates with protons and ions. The protons start in a linear accelerator (LINAC), the LINAC2 (which is now replaced by LINAC4), and the ions start in LINAC3. The injectors, as discussed in section 2.1.1, are required to achieve the LHC injection energy of 450 GeV. In the following we discuss the elements of the LHC, which are important for the tune-shift model, since these elements interact electromagnetically with the beam.

7.1.1 The Magnets

To force the beam on the circular orbit, 1232 15 m long superconducting dipole magnets are placed along the ring; each magnet bends the beam by an 8.33 T strong

Fig. 7.1 A schematic layout of the LHC.

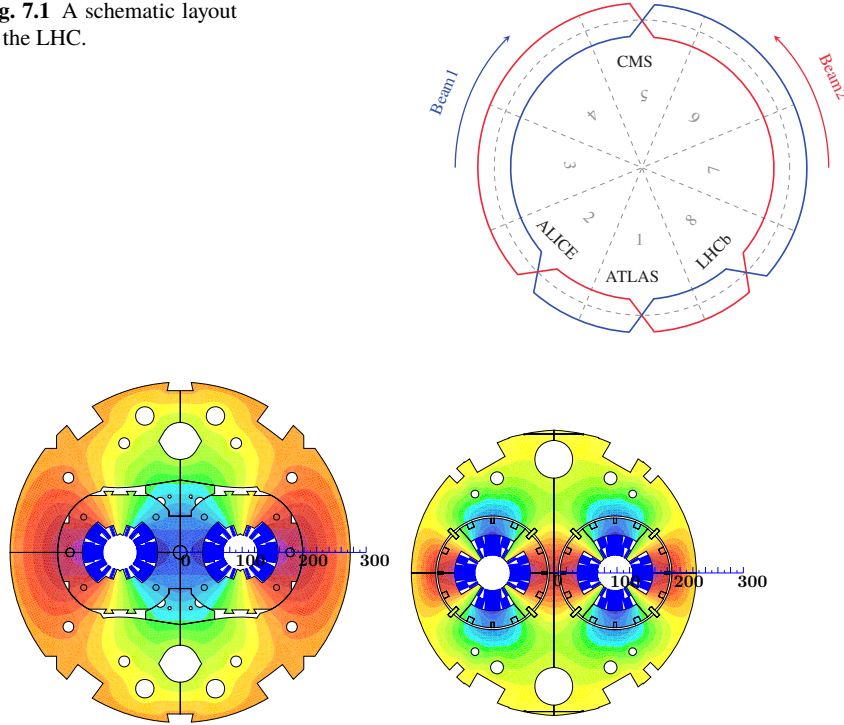


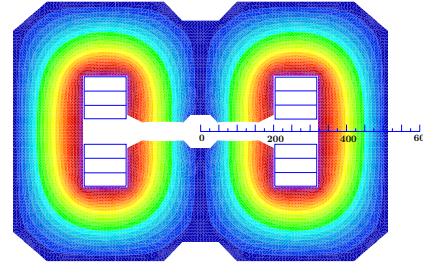
Fig. 7.2: The cross-section of the main superconducting magnets of the LHC, the dipole left (MB) and the quadrupole (MQ) right. Taken from [90].

magnet field. A special alloy of niobium and titanium is used in the magnets to generate the strong field and these elements need to be cooled down to around 1.9 K. These magnets occupy around two-thirds of the total length of the LHC. The cross-section of these magnets is pictured in fig. 7.2 (left plot). Two empty circles on the horizontal line display the beam pipe which is surrounded by a stainless steel collar and an iron yoke. The super-conducting coils are drawn in blue and the coloured domain is the iron yoke. Colours indicate the magnetic potential generated in these elements.

To keep the beam focused 392 quadrupole magnets are installed along the LHC (fig. 7.2, right plot). There are also several higher-order magnets to correct the higher-order effects, which in total sums up to 36 different types of superconducting magnets.

Normal conducting magnets are present as well. In total 15 different types of normal conducting magnets were used. An example of a dipole magnet, the *separation dipole MBW* type, as used in the tune-shift model, is shown in fig. 7.3. A detailed treatment of the magnetic infrastructure can be found in [90].

Fig. 7.3 The cross-section of a conventional warm magnet of the LHC, the separation dipole MBW. Taken from [90].



7.1.2 The LHC Beam-screen

The purpose of the beam-screen is mainly to protect the elements from damage due to the synchrotron radiation generated by the particle beam. A cross-section of an LHC-dipole is shown in fig. 7.4, providing an overview of all involved elements. The LHC beam-screen has a cross-section, which is called *rectellipse* which is a combination of an ellipse and a rectangle. It consists of circles cut by central placed rectangles (fig. 7.5). The size and the orientation (a 90-degree rotation) of the beam-screen varies along the ring. In fig. 7.5, the largest and smallest height to width ratio are depicted. They are presented, since their image field effect differs the most within the rectelliptical shapes. The beam-screen with the dimensions shown in blue covers almost the full ring, dominating with more than 94% of the total length.

7.2 The Modelling of the Elements

To establish a model of limited mathematical complexity, domains of complicated arrangements of elements as in the interaction regions are not explicitly taken into account. The interaction with these elements is negligible compared to the elements close to the beam.

The main contribution of the indirect field effects, as shown in this chapter, is caused by the electromagnetic interaction of the beam with the beam-screen since it is the closest element to the beam. As illustrated in fig. 7.7, its cross-section is approximated in our model as *in-* and *out-scribed* polygons. An increasing number of vertices is used, which in the limit approaches the correct solution. As one of the key results of this work (section 3.4) it was proved that the correct Green function of an arbitrary simply-connected domain is bounded by the solution of in- and out-scribed domains, here polygons. Although there are alternative numerical methods to estimate the impact of the beam-screen (e.g. [5]), here we want to demonstrate the advantage of having an error estimate as introduced in section 3.4 and a sufficiently accurate closed-form solution for this problem is presented.

Besides the beam-screen, as mentioned, the LHC consists of magnetic elements. The main dipole and quadrupole magnets are superconducting magnets, which are en-

closed by an iron yoke. In a simplified approximation the yoke is described as a block of iron with a drilled hole of a diameter of 80 mm, and the solution of the circular problem eq. (3.55) was used. This type of magnets gives the main contribution of the total length of the magnetic components. The second-largest part stems from the non-super-conduction magnets, the so-called warm magnets. Because of their minor contribution, they are approximated as parallel plates (strips), although arbitrary geometries can be calculated via polygonal approximations as demonstrated for the combined-function magnets section 6.2, where no closed-forms exist any-more.

In the context of this thesis, in total, around 300 different shapes were used in the model to calculate the indirect space-charge tune-shift with sufficient and unprecedented accuracy. The elements leading to an electric boundary problem included circles, ellipses, and rectangles calculated as closed-forms and the rectelliptical parts, which were approximated with the help of polygons. The formulas and graphical representations of the closed-form image tensors were provided in section 5.3.

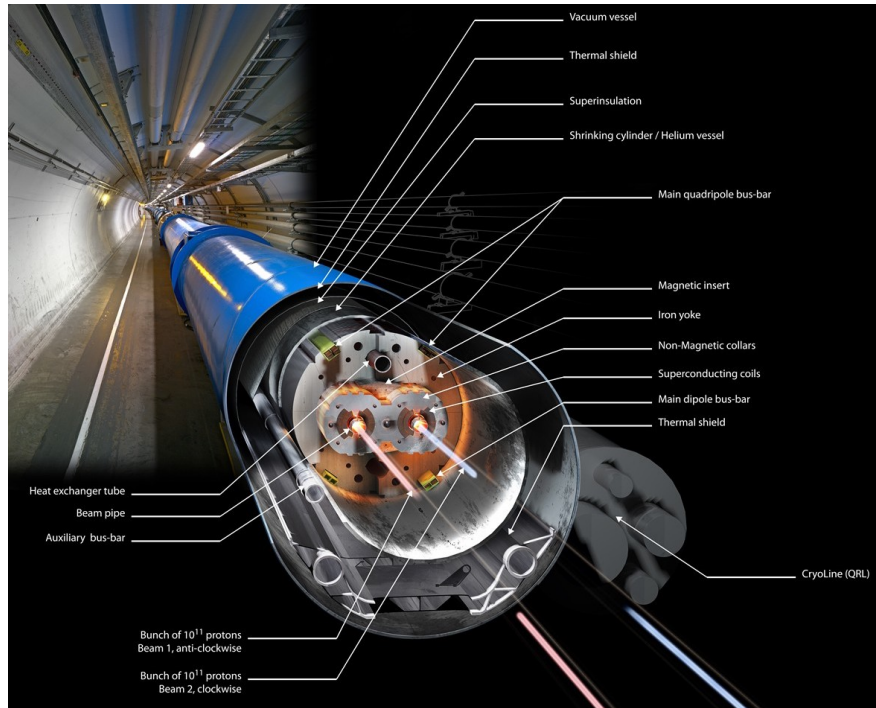


Fig. 7.4: The structure of an LHC dipole. Taken from [126].

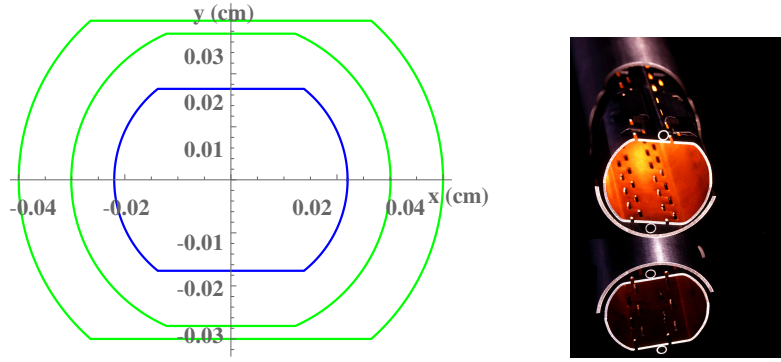


Fig. 7.5: The cross-section of the LHC beam-screen. The left graphic shows the biggest and smallest height to width ratio of the different beam-screen geometries. The right picture shows a photograph of the beam-screen taken from [127].

7.2.1 The LHC Beam-screen as a Polygon

As described above, the geometry of the LHC beam-screen is modelled as a polygon. Instead of using a polygonal shape, a circular arc shape could be utilised. A study, carried out in the scope of this work, showed that this formulation leads - by employing the Schwarzian [30, 41, 128] - to Heun's equation [129] in terms of hypergeometric functions. An exact algorithmic approach was used [130] to find the involved series coefficients. These coefficients depend in our case recursively on two start values. Nevertheless, another general approach was finally chosen since these techniques are long-winded and restricted to specific shapes - the *circular arcs*. By using the Schwarz-Christoffel-transformation, we can approximate arbitrary cross-sections as polygons to arbitrary precision (aside of possible numerical difficulties).

7.2.2 Theoretical Convergence Studies

The polygonal approximation of a circle is shown in fig. 7.6. A Green function, expressed in terms of hypergeometric functions, was formulated (using eq. (4.31)). A rate of convergence $p = 2$ was found using analytic techniques (expressed through hypergeometric series). This convergence behaviour can be explained by the fact that the chosen method increases the number of polygonal points at the n^{th} step by a factor four. Colours indicate the approach to the true shape from bright to dark. As proven in section 4.2.3, the conformal mapping of the regular polygon converges to the mapping of circle if the number of sides goes to infinity.

In fig. 7.7, the situation is presented for the rectellipse. In this graphics the constructed in/out-scribed polygonal approximations are shown. In the n^{th} approximation step the out-scribed polygons have $4n + 2$ vertices, while the in-scribed polygons have $4n$

vertices. Again, we observe a rate of convergence of $p = 2$ in numerical studies. In such studies, the true convergence behaviour can only be estimated, without knowing the true value [131, 132]. However, we know the error bound for any number of vertices by comparing the in and out-scribed polygonal solution, permitting us to end at sufficient accuracy without the need for additional convergence studies.

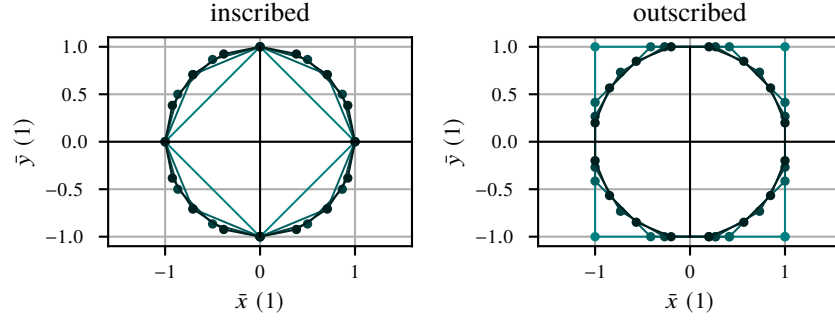


Fig. 7.6: The circle approximated as in/out-scribed regular polygons.

7.3 Measurements

Measurement data of the *Base Band Tune* (BBQ) measurement system and beam position monitors collected in 2012 during the LHC runs allowed to study the impact of the intensity on the transverse tunes [43]. The injection oscillations were monitored with beam position monitors during the injection phase. Then the data was carefully analysed and benchmarked among the different devices to extract the tune-shift due to the intensity with reasonable accuracy. These results build the basis of the validation of our theoretical model.

Since 2016, the correction of this effect is implemented into a feed-forward system of the LHC (and the SPS), using an empirically determined value for the tune-shift correction at injection. The operationally measured values are in excellent agreement with our theoretical predictions [133]. Except for the fact that the authors claim a value of 0.25 for the Laslett coefficient of the incoherent magnetic effect (by using parallel plates of width = 2.5 cm), while, as deduced here, the magnetic effect is almost zero and the electric equivalent caused by the beam-screen is calculated as ≈ 0.27 , which explains the measured value of 0.25.

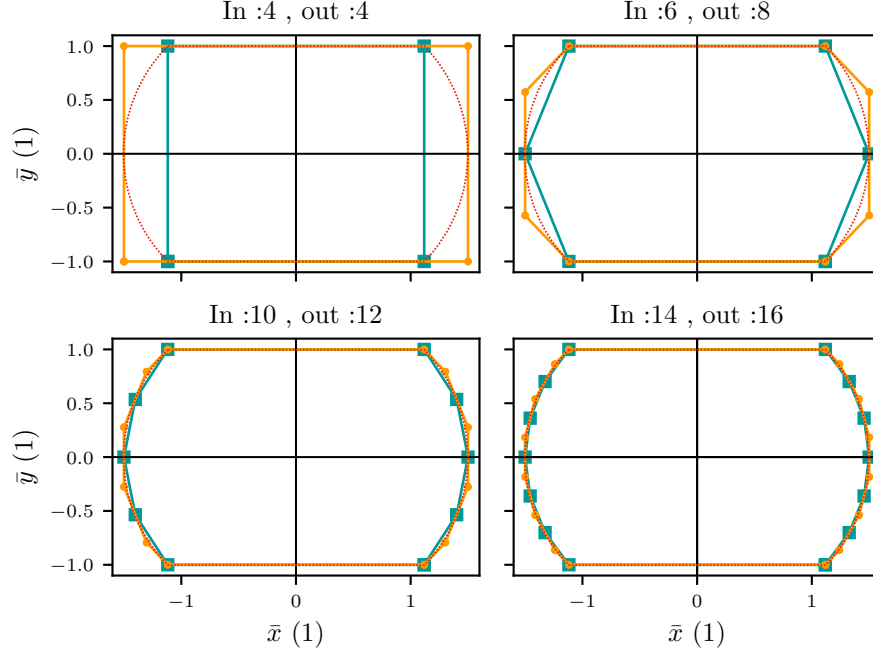


Fig. 7.7: The LHC beam-screen modelled as in/out-scribed polygons with increasing accuracy.

7.4 Results

As explained in section 5.2, different models of increasing complexity were used to provide prediction of the intensity-dependent indirect tune-shift. Lastly, these models were benchmarked with the measurements. As in the case of the PS, the codes MAD-X [119] and PTC [121] were employed to calculate the orbit and the β -function. In section 5.2.1 the different calculations to estimate the indirect tune-shifts are listed. The simplest estimate was obtained by the so-called *smooth approximation*. In this case the β -function is averaged along the whole ring, which is labelled *smooth* in table 7.1. The form factors are calculated at the centre of the geometry of the corresponding elements along the ring. The next improvement of the model is to take the calculated β -functions at the longitudinal position along the ring, while calculating the form factors still at the geometrical centre of the elements, which is named *centred* in table 7.1. Additionally, the β -function was measured along the ring to gain more accuracy. The label for this case is *meas. β* . As an extra step, the longitudinal resolution along the ring was increased, which can be found in table 7.1 (distinction of *low res.* and *high res.*). Finally, the closed orbit was measured, and the form factors were calculated at the transverse position of the beam at the corresponding element and longitudinal position, which is indicated by the label *meas. c.o.*.

All models of the beam-screen, except the parallel plate approximation, predict values within the error-band of the measurements.

The main contribution of the tune-shift is originated in the interaction of the beam with the LHC beam-screen. The circular iron yokes of the superconducting magnets were calculated to contribute about four to five orders of magnitude lower than the beam-screen. The effect of the warm magnet elements is about three orders lower than the beam-screen and hence justifying their modelling as parallel plates. Due to their insignificant impact, magnets are not further discussed. Moreover, the superconducting elements do not interact significantly with the beam.

In table 7.1 the tune-shifts per proton are summarized for the different models. Here the beam-screen is taken as an out-scribed polygon with 132 vertices. One sees that the *smooth approximation* already yields a reliable indication of the overall tune-shift. If the resolution of the model is increased, the tune-shift slightly decreases. Taking the measured β into account does not change the values sharply, a good indication that the model is quite accurate. Including the closed orbit also does not change the values considerably, which can be explained by the fact that the beam is only weakly off-centred.

The tune-shift is increasing during the filling process from the SPS due to the increasing circulating intensity. At the end of the filling process the tune-shift calculates to around 0.009 (with 2808 bunches each 1.15×10^{11} protons). This clearly demonstrates the necessity for a correction of this effect.

In fig. 7.8 the measurement of the horizontal and vertical incoherent tune-shift of the two counter-rotating beams is compared to different models of the beam-screen. The true shape of the beam-screen is approached using in-, and out-scribed polygonal models with an increasing number of vertices starting from the parallel plates ($n=2$). The exact solution is between the in and out-scribed polygonal solution as proven. It is indicated for the incoherent horizontal tune-shift of beam 1 by the orange line (out-scribed) and dark green line (in-scribed). We see that the rectangular shape provides an excellent model to approximate the solution as a closed-form. Calculations have been performed to a maximum number $n = 130$ for the inscribed and $n = 132$ for the out-scribed polygon, since the error is negligible compared to the error of the measurement even earlier. The round markers indicate closed-form solutions. The solid lines show the measured values inclusive error bounds (dotted lines). In other studies only the strip (parallel plates [10]) was used.

The full off-axis image tensors of the beam-screen (of an inscribed polygonal approximation with $n = 132$ vertices) are plotted in figs. 7.9a and 7.9b for the coherent and incoherent case, respectively.

7.4.1 A Closed-Form Model: The Rectellipse as a Rectangle

Since the polygonal solution and the measurement agree excellently with the rectangular approximation, (see fig. 7.8), it is adequate to approximate the rectellipse as a rectangle, which can be expressed through closed-forms.

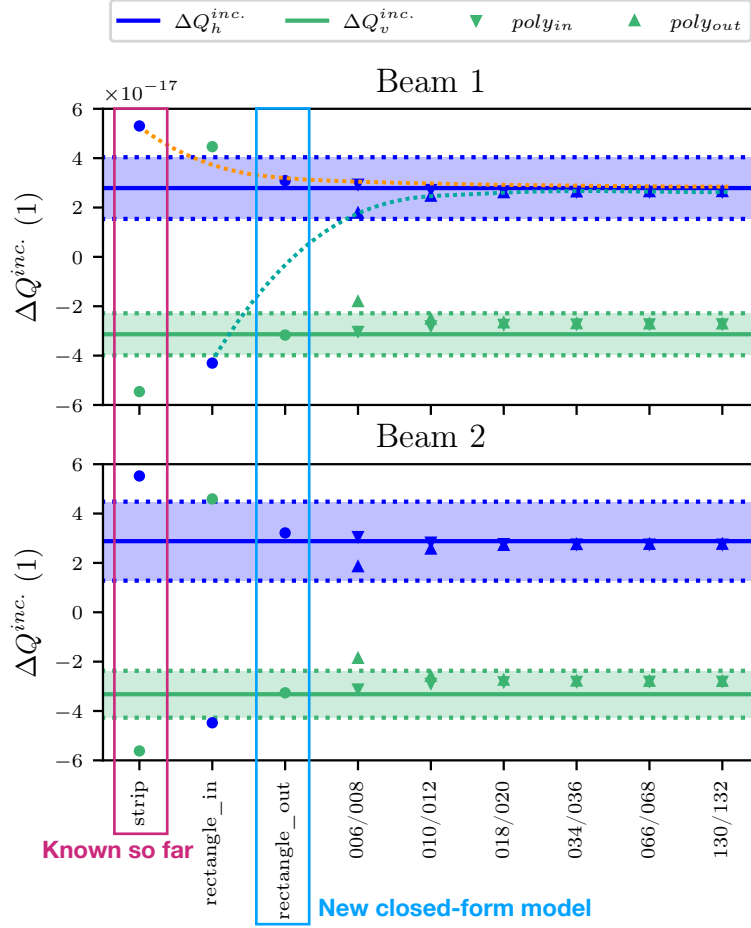


Fig. 7.8: The indirect tune-shift per proton $\Delta Q^{inc.}$ of the LHC calculated with different models (the numbers on the x -axis indicate the number of vertices to approximate the beam-screen - see fig. 7.7) compared to the measured values (solid lines including error bounds). The dotted green and orange lines indicate the convergence behaviour according to lemma 3.1.

A comparison of the electric potentials for a centred beam is depicted in figs. 7.10 and 7.11 for the rectangle and the rectellipse, respectively. In the upper plot of fig. 7.12 the difference of the Green functions of a centred beam and in the lower plot, the difference between the two incoherent coefficients is pictured. Because the potential difference of the two shapes reveals only a very small variation around the centre of the geometry, the incoherent coefficients do not differ considerably there. So, the approximation of the rectellipse as an out-scribed rectangle is justified for

beams near the centre of geometry as it is the case during the operation in the LHC. The closed-form image tensors of the rectangle are plotted in figs. 5.3a and 5.3b.

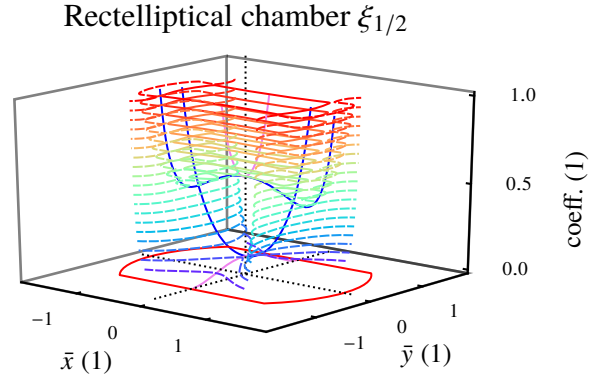


Fig. 7.9a: The coherent electric image tensor of the rectelliptical CERN LHC beam-screen (explanation in section 5.3.1).

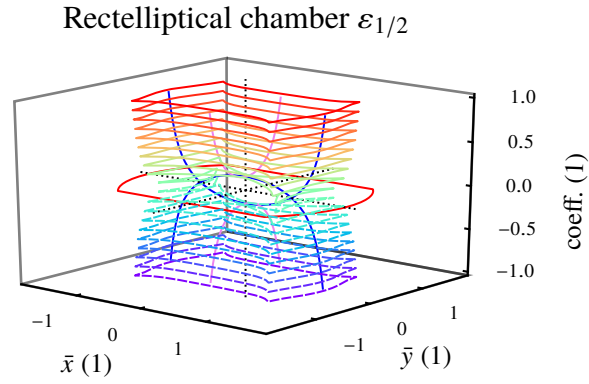


Fig. 7.9b: The incoherent electric image tensor of the rectelliptical CERN LHC beam-screen (explanation in section 5.3.1).

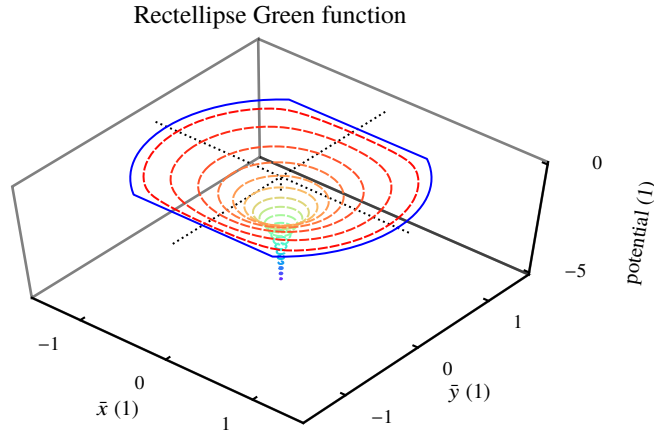


Fig. 7.10: The Green function for a centred source of the rectelliptical shape.

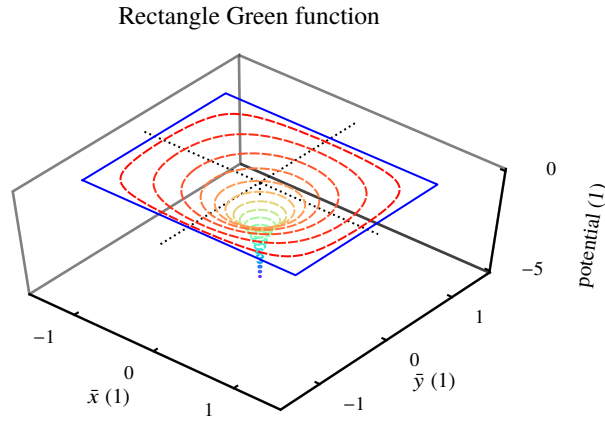
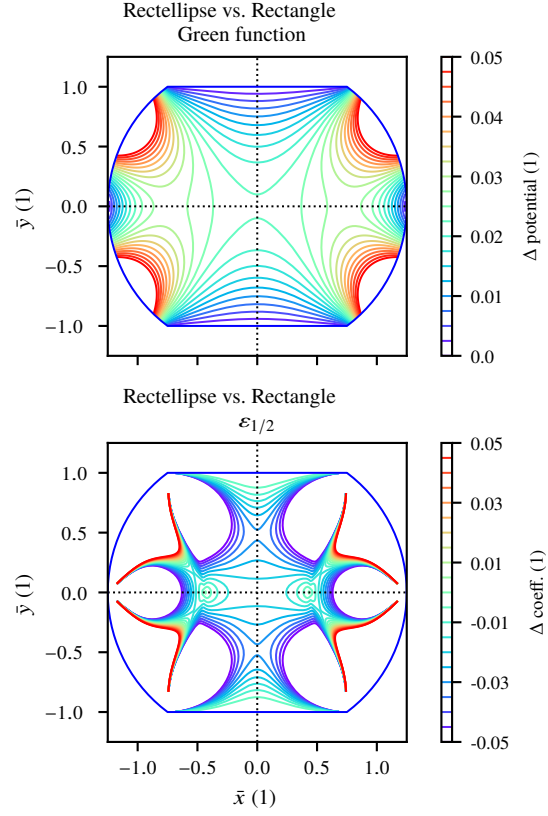


Fig. 7.11: The Green function for a centred source of the rectangular shape.

7.5 Future Application: CERN (H)igh-(L)uminosity - LHC

As explained in section 2.1.1, the upgrade of the LHC to the HL-LHC has the final goal to increase the rate of collisions by a factor of five. In order to achieve such a high performance, several challenging upgrades have to be installed, which will take place in the following decade. The model of HL-LHC changes only slightly with respect to the LHC model, as used in this thesis to calculate the indirect space-charge tune-shift. A new octagonal beam-screen will be installed in the aperture of

Fig. 7.12 A comparison of the rectellipse and the outscribed rectangle. In the upper plot, the difference between the potentials of a centred beam is shown. The lower plot shows the deviation of the two incoherent image coefficients.



superconducting magnets nearby the ATLAS and CMS experiments at the location

Table 7.1: The tune-shifts of the CERN LHC - different studies.

Per proton at injection energy:				
beam tune-shift	beam 1 $\Delta Q_h^{inc.} (1)$	$\Delta Q_v^{inc.} (1)$	beam 2 $\Delta Q_h^{inc.} (1)$	$\Delta Q_v^{inc.} (1)$
experimental	$2.79(\pm 1.25)e-17$	$-3.16(\pm 0.85)e-17$	$2.9(\pm 1.6)e-17$	$-3.32(\pm 0.95)e-17$
low res.				
smooth	$2.650e-17$	$-2.737e-17$	$2.614e-17$	$-2.796e-17$
centred	$2.428e-17$	$-2.404e-17$	$2.427e-17$	$-2.405e-17$
meas. β	$2.498e-17$	$-2.620e-17$	$2.525e-17$	$-2.598e-17$
high res.				
smooth	$2.635e-17$	$-2.716e-17$	$2.592e-17$	$-2.782e-17$
centred	$2.386e-17$	$-2.363e-17$	$2.385e-17$	$-2.366e-17$
meas β	$2.379e-17$	$-2.371e-17$	$2.399e-17$	$-2.379e-17$
meas. c.o.	$2.387e-17$	$-2.385e-17$	$2.384e-17$	$-2.396e-17$

of the mini- β quadrupole magnets. Instead of the rectellipse, the shape will be octagonal, as illustrated in fig. 7.13. The $Q3$ (left) and the $Q1$ (right) shapes are shown. The Green function of this geometry can be solved exactly as given in section 5.3.6 and the off-axis form factors are visualized in figs. 5.4a and 5.4b.

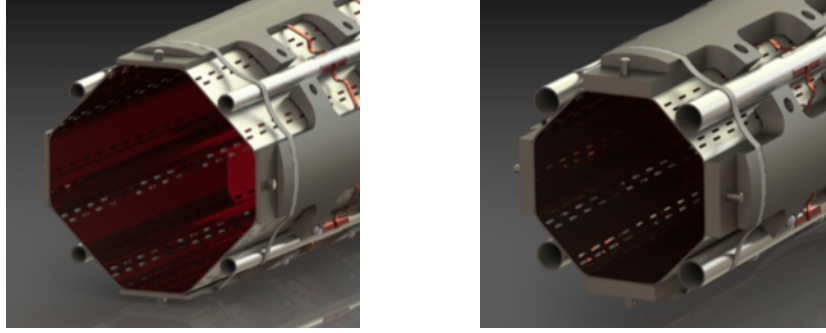


Fig. 7.13: The new HL-LHC beam-screen. Graphics taken from [134].

7.5.1 Results

The results of the calculations of the tune-shift estimations per proton are summarized in table 7.2. Four different models were compared: the *smooth* and the *centred* model with two different longitudinal resolutions (denotes as *low res.* and *high res.*). The *smooth* and the *centred* calculations were performed as described in section 5.2.1. A small decrease of the tune-shift can be seen when the resolution of the model is increased. It can be noticed that for the HL-LHC model, the *smooth approximation* differs more significantly from the *centred* model than in the LHC model.

For the HL-program the intensity of the beam in the LHC will increase by a factor two. The approximation through the rectangular shape still is valid since only small parts of the beam pipe will be replaced, but the intensity-dependent indirect tune-shift increases by about a factor two as a result of the higher intensity. At the end of a standard LHC filling procedure, the maximal tune-shift would be in the order of 0.035, assuming an increase of a factor two of the number of protons with respect to a standard filling according to the design values given in section 7.4.

Table 7.2: The tune-shifts of the CERN HL-LHC of the different approximations.

Per proton at injection energy:				
beam	beam 1		beam 2	
tune-shift	$\Delta Q_h^{inc.}(1)$	$\Delta Q_v^{inc.}(1)$	$\Delta Q_h^{inc.}(1)$	$\Delta Q_v^{inc.}(1)$
low res.				
smooth	2.724e-17	-2.786e-17	2.705e-17	-2.828e-17
centred	2.450e-17	-2.336e-17	2.447e-17	-2.343e-17
high res.				
smooth	2.711e-17	-2.765e-17	2.684e-17	-2.813e-17
centred	2.419e-17	-2.306e-17	2.416e-17	-2.312e-17

7.6 Summary

In this chapter, the intensity-dependent indirect tune-shift of the LHC at injection energy was calculated with unprecedented accuracy. This effect has to be taken into account during the filling of the machine during operation to keep the beam stable. The computations agree remarkably well with measurements, increasing the accuracy of previous methods by more than a factor of two. It was shown that the main contribution of the effect is originated in the electric interaction with the beam-screen, and other interactions are insignificant (all form factors were calculated using closed-forms).

A novel semi-analytical method to calculate the approximative electric field caused by the off-centred beam with the beam-screen was developed. This method provides an error bound of the approximation as proven in this thesis in section 3.4. Additionally, it was shown that the problem could be formulated through a closed-form expression, namely by approximating the beam-screen via a rectangular shape. Different models were compared based on these methods, exploiting different levels of complexity. The longitudinal resolution of the model was increased, the β -function was measured, and the image effects were calculated at the position of the measured closed orbit.

The conclusion is that the *smooth approximation* using the closed-form serves the purpose of estimating the intensity-dependent tune-shift adequately.

As a future application, the intensity-dependent tune-shift for the HL-LHC was calculated. The estimation of this effect is of particular interest for the project, since it will be approximately a factor two higher than for LHC. Modifications of the beam-screen to an octagonal shape, which was solved exactly, show a minor impact on the final result. Consequently, the closed-form approach developed for the LHC, by using a rectangular approximation of the beam-screen, can be applied to the HL-LHC.

Part IV

Summary

Chapter 8

Summary and Outlook

The main goal of this work was to provide estimates for the influence of *indirect space-charge driven* (ISCD) *effects*, which are crucial for stable performance of accelerators as, e.g. the LHC, in *simple closed-form expressions*. In contrast to numerical methods, which were considered in chapters 2 and 6, involving expensive calculations, simpler forms can reveal some insights into the behaviour of the system on an analytical level and describe phenomena observed in the PS, the LHC and possibly other accelerators.

Therefore a theoretical framework to formulate a new *Lorentz force operator* of the fundamental solutions, namely the Green functions, for simply-connected domains of the perfect 2D electro and magnetostatic boundary value problem, based on conformal mappings, was established on the *Riemann-sphere* (an overview of the mathematical structure is explicated in fig. 3.2). It allowed for gathering insights which revealed new simple mathematical expressions.

One of the main results was the derivation of *closed-forms* for *unbounded star-like domains* eq. (3.62) for the magnetostatic problem. It was used to estimate the influence of the magneto-static interaction of the beam with the *combined-function magnets* of the PS and were compared to a - in this context also novel - highly accurate polygonal approximation to validate the simpler approximations. Different ISCD *tune-shift* models for the largely *off-centred beamlets*, as generated during the *Multi-Turn Extraction* (MTE) were studied, resulting in an accurate closed-form description. The effect of the electro and magnetostatic interaction were about the same strength. The beamlets reveal also an *intensity-dependent positional behaviour* in measurements. This observation could be explained by ISCD *effects* in simulations based on *closed-forms* expressions as provided in this work and shows the importance of such studies.

The *electrostatic case*, which leads to the *Dirichlet problem*, was solved using the *Green function* of the first kind. Because of its conformal invariance, the solution was shown to be much simpler than in the magnetostatic case. Here, one major result was the proof of an approximative solution of the indirect fields based on *Schwarz-Christoffel transformation* for *arbitrary simply-connected domains*, including a *novel error bound estimation* (section 3.4). Having an error bound has the advantage if

compared to alternatively existing methods, that convergence studies are not needed since the error to the true solution is known (which is usually not the case). Besides, the approximative method is stopped if the accuracy is sufficient.

This technique was used to calculate the ISCD *tune-shift* of the LHC, which has to be corrected during the accelerator operation, showing unprecedented *agreement with the measurement* and justifying the approximation of the *LHC beam-screen* with a *closed-form*, the rectangular shape in terms of elliptic functions. This approach improves previous attempts by a factor of two and describes the phenomena for the first time theoretically. Furthermore, it was shown that the principal component of the ISCD tune-shift is the interaction with the beam-screen (>99%) and that the impact of the magnetic interaction could be neglected. In this manner, several different models were compared, including *measurements* of the β -functions and the *closed orbits* without demonstrating significant changes, showing evidence that the used model is sufficiently accurate. Since 2016, the measured tune-shift was corrected at injection via a feed-forward system based on empirical data. Using the estimates derived in this thesis, this could be achieved in the future in a much more elegant and faster way.

As another case study, the HL-LHC was treated. In the HL-LHC era, the *intensity* will *increase by a factor of two*, and in turn, also the indirect tune-shift grows by a factor of two. Although parts of the standard rect-elliptical shape are foreseen to be replaced by an octagonal cross-section, for which an exact solution was provided, the influence is mainly governed by the remaining rect-elliptical part. As in the LHC, the beam-screen can be approximated by the *closed-form* of the rectangle.

Along the way of developing the theory, several *new representations of the Green function* were found as a novel *integral representation* of the Neumann function for regular simply-connected domains (section 3.5.2). The convergence problems arising using the method of images were discussed in a dedicated chapter (appendix D.1), where a new convergence proof can be found.

Finally, closed-form solutions based on the newly introduced *off-axis image operators for several new cross-sections* were presented, which can be utilised in other models and accelerators. For existing solutions, a detailed comparison with the literature was performed to sort out discrepancies and to provide a reliable reference. Algorithmic routines to calculate the image operators for arbitrary cross-sections - if possible expressed through closed forms - were developed to avoid the tedious and error-prone calculations by hand and to make the process more transparent.

In future studies, the following topics could be addressed:

- A detailed study of the intensity-dependent beamlet position, including the novel closed-form expressions of the combined-function magnets.
- Including coupling effects in the LHC and the HL-LHC, especially since the intensity, and consequently, the ISCD effects increase in future applications.
- The calculation of smooth boundaries and their boundary behaviour compared to the polygonal approximations.
- The implementation of arbitrary charge distributions and higher-order terms of the indirect field in tracking codes based on the developed Green functions as provided in this thesis.

Appendices

Appendix A

The Complex Version of the Electro and Magnetostatic Equations

We take a closer look onto the physics behind eq. (3.2). We assume $M \subseteq \bar{\mathbb{C}}$ is a 1-dimensional connected complex manifold with ∂M as a Jordan contour (section 3.2.1). F be a complex valued function of a complex argument. The theorem of Green-Riemann is written in its complex version:

$$\int_{\partial M} F dz = \int_M \bar{\partial} F d\bar{z} \wedge dz. \quad (\text{A.1})$$

We show now:

$$\text{Re} \int_{\partial M} F dz \rightarrow \text{circulation of } \bar{F}, \quad (\text{A.2})$$

$$\text{Im} \int_{\partial M} F dz \rightarrow \text{flow of } \bar{F}. \quad (\text{A.3})$$

Proof. Here a more technical notation is used to highlight the familiar form of the equations as found in the technical literature. Writing the right side in the classical form, where $F = u + iv$, u and v are real valued functions of a complex argument, one obtains:

$$\int_{\partial M} \bar{F} dz = \int_{\partial M} (u - iv)(dx + idy) \quad (\text{A.4})$$

$$= \int_{\partial M} \left[(u dx + v dy) + i(-v dx + u dy) \right] \quad (\text{A.5})$$

$$= \int_{\Gamma} ds \left[\underbrace{(u, v) \cdot \vec{s}_{\Gamma}}_{\text{circulation}} + i \underbrace{(u, v) \cdot \vec{n}_{\Gamma}}_{\text{flow}} \right]. \quad (\text{A.6})$$

Γ denotes the parametrized boundary ∂M and $\vec{s}_{\Gamma}$ and $\vec{n}_{\Gamma}$ denote the mathematical positively oriented tangential and outwards pointing normal vector of Γ . If we now take a look on the right side of eq. (3.2), we find (using the curl and divergence operator defined in the standard way, e.g. [30], p.40):

$$\int_M \bar{\partial} \bar{F} \, d\bar{z} \wedge dz = \frac{1}{2} \int_M (\partial_x + i\partial_y)(u - iv) \, d\bar{z} \wedge dz \quad (\text{A.7})$$

$$= i \int_M [\partial_x u + \partial_y v + i(-\partial_x v + \partial_y u)] \, dx dy \quad (\text{A.8})$$

$$= \int_M [(\partial_x v - \partial_y u) + i(\partial_x u + \partial_y v)] \, dx dy \quad (\text{A.9})$$

$$= \int_M [\text{curl}(u, v) \cdot \vec{n} + i(\text{div}(u, v))] \, dx dy. \quad (\text{A.10})$$

□

Appendix B

Dirac Sequences as Used for the Proof in theorem 3.11

Definition B.1. A Dirac sequence or approximative identity is a sequence of functions $\{\Phi_r\}_{0 < r < 1}$ satisfying the following three conditions (see e.g. [18], p. 88):

- $\int_0^1 \Phi_t(\theta) d\theta = 1$ for all t
- $\sup_t \int_0^1 |\Phi_t(\theta)| d\theta < \infty$
- for all $\delta > 0$ one has $\int_{|x| > \delta} |\Phi_t(\theta)| d\theta \rightarrow 0$ as $t \rightarrow 1$.

As shown in fig. 3.18 the mapping F_r^{-1} maps the coloured circular domains onto same coloured domains, which grow with increasing r . The points $\{e^{i\varphi_k} | k \in \{1, 2, 3\}\}$ are mapped in the limit $r \rightarrow 1$ onto the north-pole on $\tilde{\mathbb{C}}$. The boundary of the domain enclosed by $F_r e^{i\varphi^{-1}}$ is growing to ∞ around the points in the limit $\lim_{r \rightarrow 1} F_r^{-1}(r e^{i\varphi_k})$. We divide the boundary into three sub domains along the green lines. The three in this way generated boundaries extend from the points p_1 to p_2 from p_2 to p_3 and from p_3 to p_1 , whereas the points z_1, z_2 and z_3 are located in between these points, respectively. Lets denote these boundaries as $\Gamma_{1 \rightarrow 2}, \Gamma_{2 \rightarrow 3}$ and $\Gamma_{3 \rightarrow 1}$. The points are chosen such that the length of the paths γ_i is equal.

If we now regard an function of the form:

$$K_r(\tau) = \frac{|\partial f_r^{-1}(\tau)|}{|\partial M_r|} \quad (\text{B.1})$$

Now we can split the numerator into equal parts:

$$K_r(\tau) = \frac{|\partial f_r^{-1}(\tau)|}{\sum_k^N \int_{\Gamma_k \rightarrow \Gamma_{\text{mod}(k+1, N)}} |\tau| |f_r^{-1}(\tau)|} = \frac{|\partial f_r^{-1}(\tau)|}{N \underbrace{\int_{\Gamma_1 \rightarrow \Gamma_2} |\tau| |f_r^{-1}(\tau)|}_{:= L_r}} \quad (\text{B.2})$$

Now splitting the nominator:

$$\kappa_r^k(\tau) := |\partial f_r^{-1}| \Theta(\arg(\tau) - \varphi_k) \Theta(\varphi_{\text{mod}(k+1, N)} - \arg(\tau)) \quad (\text{B.3})$$

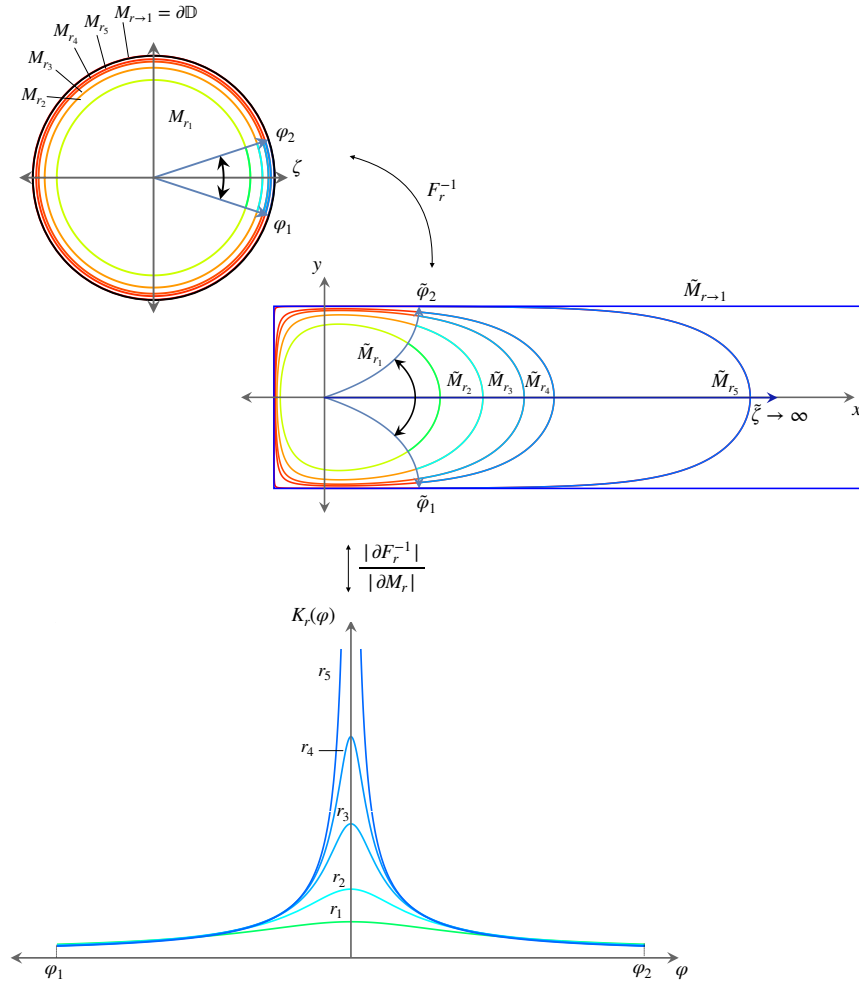


Fig. B.1: An unbounded star-like domain M with one opening to the north-pole.

hence $|\partial F_r^{-1}(\tau)| = \sum_k^N \kappa_r^k(\tau)$ and:

$$K_r(\tau) = \frac{1}{N} \sum_k^N \frac{\kappa_r^k(\tau)}{L_r} \quad (\text{B.4})$$

Each term $\frac{\kappa_r^k(\tau)}{L_r}$ full fills all three conditions of definition B.1:

- $\int \frac{\kappa_r^k(\tau)}{L_r} |\tau|$

Consequently we can write:

$$\lim_{r \rightarrow 1} K_r(\tau) = \sum_k^N \delta(\tau - e^{i\varphi_k}) \quad (\text{B.5})$$

Another example is shown in fig. B.1. There the opening to the north-pole is reached following the ray with $\varphi = 0$. The domains with increasing r mapped by F_r^{-1} onto the strip-like shapes, are coloured from yellow to red outside of the arc $\Gamma_{\varphi_1 \rightarrow \varphi_2}$ and green to blue on the inside. If $r \rightarrow 1$ the shape is the so-called half-strip. The length of the mapped arcs (by F_r^{-1}) from φ_1 to φ_2 is drawn in the lowest plot. There the form of the Dirac sequence depended on r is clearly visible and in the limit $r \rightarrow 1$ the result is the well-known Dirac delta distribution. The mapping was explicitly calculated in section 4.2.1.

Appendix C

Special Functions Used in the Text

A listing of functions used in the text.

C.1 Used Elliptic Integrals and Functions

C.1.1 Elliptic Integrals

To avoid complications, we have to mention, that the elliptic integrals and functions are used with the convention where the second argument is the parameter not the modulus. The elliptic integral of the first kind $F(\phi|m)$ is defined as:

$$F(\phi|m) := \int_0^\phi \frac{d\theta}{\sqrt{1-m\sin^2\theta}} = \int_0^{\sin\phi} \frac{dt}{\sqrt{(1-t^2)(1-mt^2)}}, \quad -\frac{\pi}{2} < \phi < \frac{\pi}{2}. \quad (\text{C.1})$$

The complete elliptic integral of the first kind $K(m) := F(\frac{\pi}{2}|m)$ and $K'(m) := K(m')$, with $m'^2 = 1 - m^2$. The elliptic integral of the second kind $E(\phi|m)$ is given by:

$$E(\phi|m) = \int_0^\phi \sqrt{1-m(\theta\sin^2)} d\theta = \int_0^{\sin\phi} \frac{\sqrt{1-mt^2}}{\sqrt{1-t^2}} dt \quad -\frac{\pi}{2} < \phi < \frac{\pi}{2}. \quad (\text{C.2})$$

The complete elliptic integral of the second kind $E(m) := E(\frac{\pi}{2}|m)$.

C.1.2 Jacobi Elliptic Functions

The amplitude for Jacobi elliptic functions $\text{am}(u|m)$ is the inverse of the elliptic integral of the first kind. If $u = F(\phi|m)$, then $\phi = \text{am}(u|m)$. The Jacobi elliptic functions $\text{sn}(u|m)$ and $\text{cn}(u|m)$ are given respectively by $\text{sn}(u|m) = \sin(\phi)$ and $\text{cn}(u|m) = \cos(\phi)$, where $\phi = \text{am}(u|m)$.

In addition, other used functions are defined as, , where $\phi = \text{am}(u|m)$:

- $\text{dn}(u|m) = \sqrt{1 - m \sin^2(\phi)}$,
- $\text{cd}(u|m) := \frac{\cos(\phi)}{\sqrt{1 - m \sin^2(\phi)}}$,
- $\text{nd}(u|m) := \frac{1}{\sqrt{1 - m \sin^2(\phi)}}$,
- $\text{sd}(u|m) := \frac{\sin(\phi)}{\sqrt{1 - m \sin^2(\phi)}}$.

C.1.3 The Parameter of the Green Function of the Rectangle and the Ellipse

Using eqs. (3.45), (4.17) and (4.21) one gets:

$$g = \frac{1}{2\pi} \log \left| \frac{\text{sn}(\frac{K}{2}(z + i\frac{b}{2})|m^2) - \text{sn}(\frac{K}{2}(z_0 + i\frac{b}{2})|m^2)}{\text{sn}(\frac{K}{2}(z + i\frac{b}{2})|m^2) - \text{sn}(\frac{K}{2}(z_0^* + i\frac{b}{2})|m^2)} \right| \quad (\text{C.3})$$

$$\frac{K'(m^2)}{K(m^2)} = \frac{2b}{a} \quad (\text{C.4})$$

for the Green function of the rectangle with the width and height a and b respectively. Using eqs. (3.45), (4.38) and (4.42) one gets:

$$g = \frac{1}{2\pi} \log \left| \frac{\text{cd}(\frac{K'}{\pi} \cosh^{-1} \frac{z}{k}|m^2) - \text{cd}(\frac{K'}{\pi} \cosh^{-1} \frac{z_0}{k}|m^2)}{\text{cd}(\frac{K'}{\pi} \cosh^{-1} \frac{z}{k}|m^2) + \text{cd}(\frac{K'}{\pi} \cosh^{-1} \frac{z_0^*}{k}|m^2)} \right| \quad (\text{C.5})$$

$$\frac{K(m^2)}{K'(m^2)} = \frac{|\log(\frac{k}{b + \sqrt{b^2 + k^2}})|}{\pi} = \frac{|\sinh^{-1} \frac{b}{k}|}{\pi} \quad (\text{C.6})$$

for the Green function of the ellipse with the width and height a and b respectively.

$$k := \sqrt{a^2 - b^2}.$$

In fig. C.1 the parameter is plotted for the ellipse (blue) and the rectangle (green). The ellipse starts from a circle and the rectangle from a square and both end up in a strip of width 1. The parameter of the rectangle decreases much slower than the parameter of the ellipse, meaning that the after a ratio of around 1.5 for the ellipse the change of the field is not as strong as in the case of an rectangle. In both cases the eqs. (4.17) and (4.38) have to be solved numerically.

C.1.4 Q-Pochhammer Symbol

The Q-Pochhammer product is defined as:

$$(a; q_\infty) := \prod_{k=0}^{\infty} (1 - aq^k), \quad (\text{C.7})$$

$$(a; q_n) := \frac{(a; q)_\infty}{(aq^n; q)_\infty}. \quad (\text{C.8})$$

C.1.5 Hypergeometric Function and Gamma Function

The Gamma function is given by:

$$\Gamma(s) := \int_0^{\infty} t^{s-1} e^{-t} dt, \quad \text{Re}(s) > 0. \quad (\text{C.9})$$

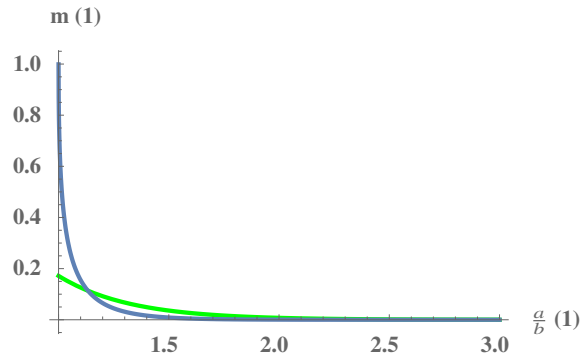
Using the rising factorial Pochhammer $(a)_n := a(a+1) \dots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$, the regular Hypergeometric function is defined as:

$${}_2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad a, b, c \in \mathbb{C}. \quad (\text{C.10})$$

C.1.6 Heaviside Function

$$H(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases} \quad (\text{C.11})$$

Fig. C.1 The parameter m as a function of the width a of height b ratio for the ellipse (blue) and the rectangle (green).



See [135] p.16, eq.(1.8).

Appendix D

On Issues of the Method of Images

D.1 The Convergence of the Image Method Applied to the Problem of Two Infinite Plates

D.1.1 The Problem of One Infinite Plate

Here we discuss the *method of images*, which is a common approach to solve magneto- and electrostatic boundary problems as presented in standard textbooks on this topic. Although it is a very intuitive method, it can only be applied in highly symmetrical configurations as, e.g. one or two infinite plates, a rectangle a circle or a wedge.

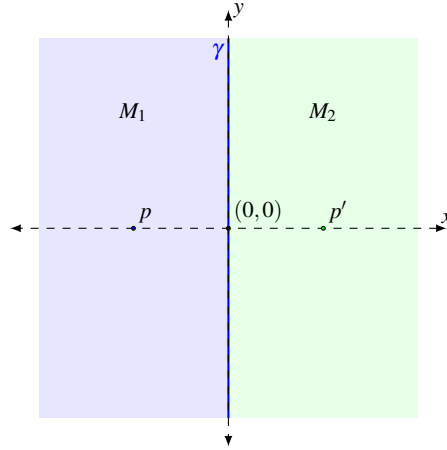
The method also shows pitfalls concerning the convergence of the obtained infinite series, which will be solved in this study using physical arguments.

As shown, the method gets rapidly fairly complicated, but it is still successfully used in mathematical physics to obtain new series representations of special functions (see e.g. [24]). In this spirit, we derive a new representation of a product representation of elliptic functions appendix D.1.3.

For the moment we step back from the assumption of perfect boundaries and assume materials with finite dielectric or magnetic properties, which means for the material that $\mu < \infty$ and $\varepsilon < \infty$ (as defined in sections 3.3.1 and 3.3.2).

The key idea is that the field caused by the fundamental singularity, as demonstrated in the last chapter, has to full fill boundary conditions at a boundary γ of a material M_1 , having the material property either ε_1 or μ_1 in the electric or magnetic case, respectively. Along a straight wall γ (the y -axis), there will be a change to a material M_2 with a ε_2 or μ_2 in the electric or magnetic case, respectively. The generating source at $p = (-\Delta, 0)$ (the wire carrying a current is located at a distance of Δ from the origin on the x -axis) will be located on the x -axis. The situation is illustrated in fig. D.1. Where p be now reflected at γ on the x -axis to $p' = (\Delta, 0)$. After short algebra (see e.g. [74], pp.111-113 for the electric case), one can determine the strength of the image p' subject to eqs. (3.25a) and (3.25b) in the magneto static case and eqs. (3.33a) and (3.33b) in the electrostatic case:

Fig. D.1 An infinite plane wall γ separating two regions M_1 (blue) and M_2 (green) with different material properties.



$$\text{Electrostatic image strength: } q = 1 \rightarrow q' = -\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + \varepsilon_1} := -e^\alpha, \quad (\text{D.1})$$

$$\text{Magnetostatic image strength: } q = 1 \rightarrow q' = \frac{\mu_2 - \mu_1}{\mu_2 + \mu_1} := e^\alpha. \quad (\text{D.2})$$

We introduced a damping factor $\alpha < 0$ with the value: $\log \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + \varepsilon_1}$ or $\log \frac{\mu_2 - \mu_1}{\mu_2 + \mu_1}$ in either the electrical or the magnetic case.

The validity of the solution can be easily verified by inserting this into the continuity conditions eqs. (3.25a) and (3.25b) and eqs. (3.33a) and (3.33b) for the magnetic and electric case, respectively. Hence the fields are generated by the source q and the mirror image of q' with the above calculated strength and are in M_1 of the form:

$$\text{Electrostatic field:} \quad E = q \left(\frac{1}{z + \Delta} - e^\alpha \frac{1}{z - \Delta} \right), \quad (\text{D.3})$$

$$\text{Magnetostatic field:} \quad B = q \left(\frac{1}{z + \Delta} + e^\alpha \frac{1}{z - \Delta} \right). \quad (\text{D.4})$$

In the limiting case were in the magnetic case $\mu_2 \gg \mu_1$ or $\varepsilon_2 \gg \varepsilon_1$ the strength of $|q'| \rightarrow 1$. $|q'|$ is idealized and can be seen as a perfect image. In the case the material is not perfect, the strength of $|q'| < |q|$, which means it's intensity is reduced.

D.1.2 The Problem of Two Infinite Plates

Now, knowing how to solve the continuity problem along an infinite plate, we use this framework to solve the problem of two parallel infinite plates.

Assuming conditions as given in many standard textbooks, we have two parallel infinite plates at a distance d , and we employ the method of images. Δ denotes the distance of the source to the symmetry axes in the middle of the two plates, so the

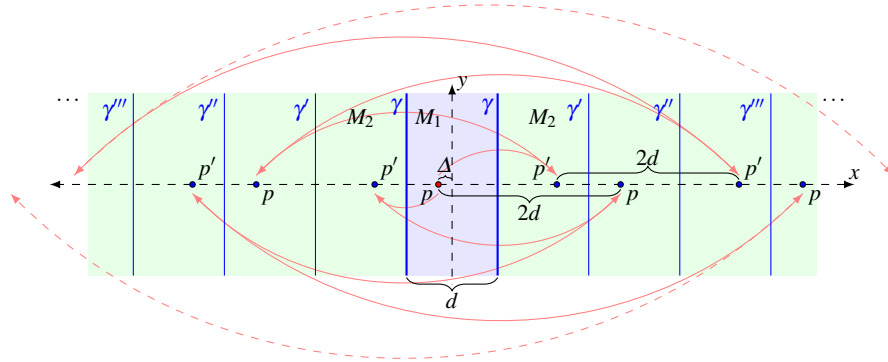


Fig. D.2: The situation of two infinite plates γ separating two regions M_1 (blue) and M_2 (green) with different material properties.

coordinates are $(-\Delta, 0)$. First, we reflect p on γ at the right side, as in the previous case with one plate and obtain the position $(d + \Delta, 0)$. Subsequently, we do the same on the left side, which yields: $(-d + \Delta, 0)$. In the electric case the sign of the image changes at each reflection eq. (D.3), the magnetic images show no sign change eq. (D.4), hence the sign alternates in the electric case for consecutive reflections. Now the new images on the opposite side disturb the solution, hence we reflect them again obtaining two new images at $(2d - \Delta, 0)$ for $(d + \Delta, 0)$ and $(2d + \Delta, 0)$ for $(-d + \Delta, 0)$. Again the sign for the electric case has changed. The infinite series of images is drawn in fig. D.2, which shows the source, reflected at the reflections of γ ($\gamma', \gamma'' \dots$). Due to symmetry we identified one series of ps and one of $p's$, where each element has a distance of twice the distance d of the plates.

Writing it down as a series expression, with the images placed at:

$$s_n = nd + (-1)^n \Delta, \quad (\text{D.5})$$

this yields for the field ($\delta = 1$, if we consider electric images and $\delta = 0$ if we consider magnetic images):

$$F(z) \propto \sum_{n \in \mathbb{Z}} \frac{(-1)^{\delta n}}{(z - s_n)^*}. \quad (\text{D.6})$$

A series $\sum_n a_n$ is called absolute convergent if $\sum_n |a_n|$ converges. In this step we show that the series eq. (D.6) is not absolute convergent. We have a look at the coefficients:

$$\begin{aligned} \left| \frac{(-1)^{\delta n}}{(z - s_n)} \right| &= \frac{1}{|z - s_n|} = \frac{1}{|z - (-1)^n \Delta + nd|} \\ &\geq \frac{1}{|z + \Delta + |n|d|} \geq \frac{1}{|n||z + \Delta + d|} \propto \frac{1}{|n|} \text{ (harmonic series!)} \end{aligned} \quad (\text{D.7})$$

The coefficient have as a lower boundary the harmonic series, which diverges, hence the series cannot be absolute convergent. It is conditional convergent. We state now a famous theorem of Riemann (see e.g. [136], p.513): *In absolutely convergent series rearrangement of the terms does not affect the convergence, and the value of the sum of the series is unchanged, exactly as in finite sums. In conditionally convergent series, on the other hand, the value of the sum of the series can be changed at will by a suitable rearrangement of the series, and the series can even be made to diverge if desired.* The question is why the method nevertheless works? The series can converge to any arbitrary number. It could be 0, so no effect would be measured, or it could be very large, so the effect would be extremely large and would make some well-established settings impossible. To understand this, we have to step back to the previous chapter, where we saw that the image of the source only reaches lossless strength in the limit of perfect material conditions. There is a physical origin of the effect, which restricts the rearrangement of the series. This is the initial point of the following argumentation.

The regarded point is reflected at both plates, which is reflected again and so on. In principle, this leads to an infinite series of sources. In reality, there is no material which reflects the source perfectly; the effective strength of the image always is smaller than of its origin. This was formulated as a damping factor $= e^{-\alpha}$ with $\alpha > 0$ in eqs. (D.1) and (D.2). The field eq. (D.6) can now be written as:

$$F_{\alpha_{\pm}}(z) = \sum_{n \in \mathbb{Z}} \left((\pm 1)^n \frac{e^{-\alpha|n|}}{z - s_n} \right)^* . \quad (\text{D.8})$$

This series is obviously absolute convergent and we can rearrange the terms and give a solution in terms of Hypergeometric Functions ${}_2F_1$ (details given in appendix C.1.5):

$$\begin{aligned} F_{\alpha_{\pm}} &= \sum_{n \in \mathbb{Z}} \left((\pm 1)^n \frac{e^{-\alpha|n|}}{z - s_n} \right)^* = \sum_{n \in \mathbb{Z}} \left(\frac{e^{-\alpha|2n|}}{z - \Delta - 2nd} \pm \frac{e^{-\alpha|2n+1|}}{z + \Delta - d - 2nd} \right)^* \\ &= \left[\frac{{}_2F_1\left(1, \frac{\Delta-z}{2d}; 1 + \frac{\Delta-z}{2d}; e^{-2\alpha}\right)}{z - \Delta} + \frac{{}_2F_1\left(1, 1 - \frac{\Delta-z}{2d}; 2 - \frac{\Delta-z}{2d}; e^{-2\alpha}\right)}{2d - \Delta + z} \right]^* \\ &\quad \pm 2e^{-\alpha} \left[\frac{{}_2F_1\left(1, \frac{1}{2} + \frac{(\Delta+z)}{2d}; \frac{3}{2} + \frac{(\Delta+z)}{2d}; e^{-2\alpha}\right)}{2d - 2(\Delta + z)} - \right. \\ &\quad \left. \frac{{}_2F_1\left(1, \frac{1}{2} - \frac{(\Delta+z)}{2d}; \frac{3}{2} - \frac{\Delta+z}{2d}; e^{-2\alpha}\right)}{2d + 2(\Delta + z)} \right]^* . \end{aligned}$$

In the limit $\alpha \rightarrow 0$ we obtain:

$$F_{\pm}(z) := \lim_{\alpha \rightarrow 0} F_{\alpha_{\pm}}(z) = - \left[\cot \frac{\pi}{2d} (\Delta - z) \pm \tan \frac{\pi}{2d} (\Delta + z) \right]^* . \quad (\text{D.9})$$

If $\alpha \rightarrow 0$ eq. (D.8) is conditional convergent, hence one has to be very careful with the limit. We can use the case of perfectly conducting boundaries (as the limiting case of imperfectly conducting boundaries) as an approximation for cases with ($\varepsilon_1 \ll \varepsilon_2$) in the electric case or ($\mu_1 \ll \mu_2$) in the magnetic case. Although after taking the limit $\alpha \rightarrow 0$ the obtained fields do not vanish at infinity (they are constant there), the result is locally usable as an approximative description of the physical fields. This point of view justifies the use of all results based on perfect electromagnetic properties. The fields for the strip S_d of width d are (here in complex notation):

$$E(z, z_0) = \frac{\pi\lambda}{d} \left[\tan \frac{\pi}{2d} (z^* + z_0^*) - \cot \frac{\pi}{2d} (z^* - z_0^*) \right], \quad (\text{D.10})$$

$$B(z, z_0) = -\frac{\pi i \lambda \beta_0}{d} \left[\cot \frac{\pi}{2d} (z_0^* - z^*) + \tan \frac{\pi}{2d} (z_0^* + z^*) \right], \quad (\text{D.11})$$

which is in agreement with the results of the in-dependent conformal mapping method sections 3.6.2.1 and 4.2.1 and the literature.

D.1.3 A New Representation of the Potential of a Rectangular Shape

From eq. (D.10) we formulate the Green function ($\bar{\partial}g =: E$) of two infinite planes using coordinates in real space (see also appendix C.1.3):

$$g_{S_{d_y}}(x, y; x_0, y_0) = \frac{1}{2\pi} \log \left(\frac{\cosh \left(\frac{\pi(x_0 - x)}{d_y} \right) - \cos \left(\frac{\pi(y - y_0)}{d_y} \right)}{\cosh \left(\frac{\pi(x_0 - x)}{d_y} \right) + \cos \left(\frac{\pi(y + y_0)}{d_y} \right)} \right) \quad (\text{D.12})$$

d_y denotes the distance between two planes, which are parallel to the x -axis located at $\pm \frac{d_y}{2}$ and (x_0, y_0) are the coordinates of the source point. To calculate the Green function of a rectangle, the idea is to use again the method of images. As shown, the Green function of the strip eq. (D.12) can be seen as the result of the infinite reflection of image charges with positive charge at the positions $2ind_y + y_0$, $n \in \mathbb{Z}$ and negative charges at $2ind_y + id_y - y_0$, $n \in \mathbb{Z}$. Now we reflect this solution again at two planes aligned parallel to the y -axis at a distance $\pm \frac{d_x}{2}$. As shown in fig. D.3 by the coloured dots in blue and green, reflected solutions appear with positive sign at the positions $2nd_x + x_0$, $n \in \mathbb{Z}$ and with negative sign at $2nd_x + d_x - x_0$, $n \in \mathbb{Z}$. The form of the Green function of the rectangle with side length d_x and d_y can be given as:

$$g_{R(d_x, d_y)}(x, y; x_0, y_0) =$$

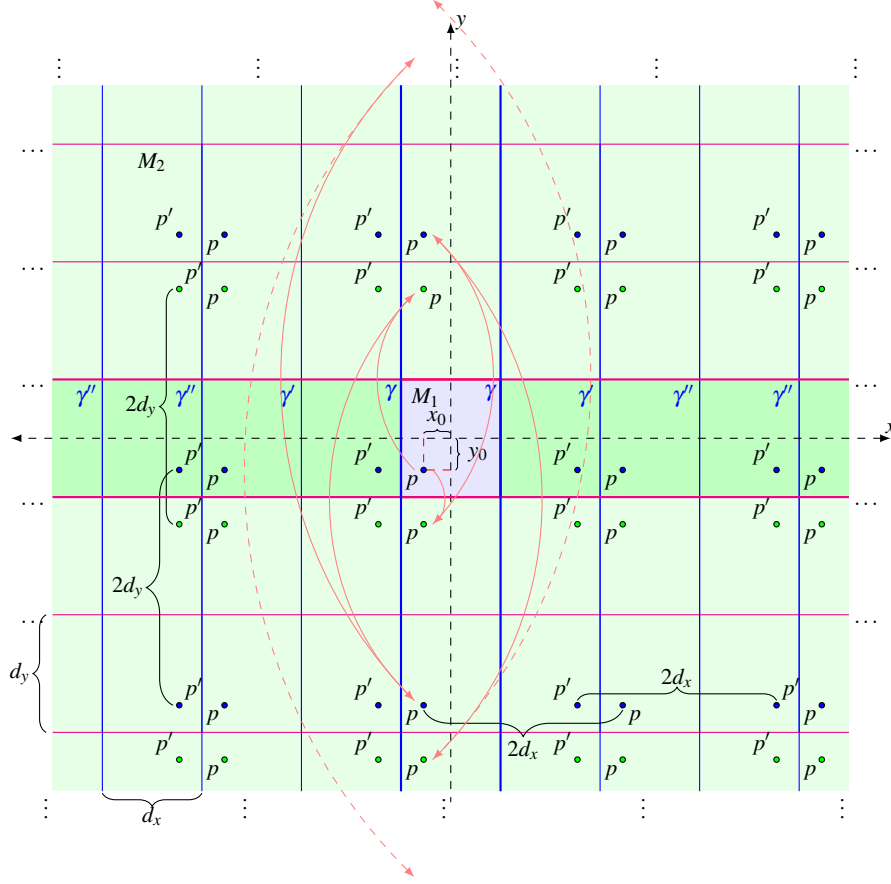


Fig. D.3: The situation of a rectangle γ separating two regions M_1 (blue) and M_2 (green) with different material properties.

$$\sum_{n \in \mathbb{Z}} \frac{1}{2\pi} \left(\log \left(\frac{\cosh \left(\frac{\pi(2nd_x + x_0 - x)}{d_y} \right) - \cos \left(\frac{\pi(y - y_0)}{d_y} \right)}{\cosh \left(\frac{\pi(2nd_x + x_0 - x)}{d_y} \right) + \cos \left(\frac{\pi(y + y_0)}{d_y} \right)} \right) \right. \\ \left. - \log \left(\frac{\cosh \left(\frac{\pi(2nd_x + d_x - x_0 - x)}{d_y} \right) - \cos \left(\frac{\pi(y - y_0)}{d_y} \right)}{\cosh \left(\frac{\pi(2nd_x + d_x - x_0 - x)}{d_y} \right) + \cos \left(\frac{\pi(y + y_0)}{d_y} \right)} \right) \right). \quad (\text{D.13})$$

In the following we proof the convergence of this representation.

Proof. Some algebraic manipulation of eq. (D.13) yields:

$$\begin{aligned}
g_R(d_x, d_y)(x, y; x_0, y_0) = & \\
& \frac{1}{2\pi} \log \left(\frac{\cosh \left(\frac{\pi(x_0-x)}{d_y} \right) - \cos \left(\frac{\pi(y-y_0)}{d_y} \right)}{\cosh \left(\frac{\pi(x_0-x)}{d_y} \right) + \cos \left(\frac{\pi(y+y_0)}{d_y} \right)} \right) - \\
& \frac{1}{2\pi} \log \left(\frac{\cosh \left(\frac{\pi(d_x-(x+x_0))}{d_y} \right) - \cos \left(\frac{\pi(y-y_0)}{d_y} \right)}{\cosh \left(\frac{\pi(d_x-(x+x_0))}{d_y} \right) + \cos \left(\frac{\pi(y+y_0)}{d_y} \right)} \right) - \\
& \frac{1}{2\pi} \log \left(\frac{\cosh \left(\frac{\pi(d_x+(x+x_0))}{d_y} \right) - \cos \left(\frac{\pi(y-y_0)}{d_y} \right)}{\cosh \left(\frac{\pi(d_x+(x+x_0))}{d_y} \right) + \cos \left(\frac{\pi(y+y_0)}{d_y} \right)} \right) - \\
& \frac{1}{2\pi} \log \left(\frac{\cosh \left(\frac{\pi(3d_x-(x+x_0))}{d_y} \right) - \cos \left(\frac{\pi(y-y_0)}{d_y} \right)}{\cosh \left(\frac{\pi(3d_x-(x+x_0))}{d_y} \right) + \cos \left(\frac{\pi(y+y_0)}{d_y} \right)} \right) + \\
& \sum_{n \in \mathbb{N}} \sum_{l \in \{-1, 1\}} \frac{1}{2\pi} \left[\log \left(1 - \frac{2 \cos \left(\frac{\pi y}{d_y} \right) \cos \left(\frac{\pi y_0}{d_y} \right)}{\cos \left(\frac{\pi(y+y_0)}{d_y} \right) + \cosh \left(\frac{\pi(2nd_x+l(x-x_0))}{d_y} \right)} \right) \right. \\
& \left. + \log \left(1 - \frac{2 \cos \left(\frac{\pi y}{d_y} \right) \cos \left(\frac{\pi y_0}{d_y} \right)}{\cos \left(\frac{\pi(y-y_0)}{d_y} \right) - \cosh \left(\frac{\pi(2(n+1)d_x+l(d_x-(x+x_0)))}{d_y} \right)} \right) \right]. \quad (\text{D.14})
\end{aligned}$$

The first four terms are extracted, because the first includes the source singularity and the second and the third the image singularities. The fourth term is extracted due to practical reasons only. To investigate the convergence of this representation we look at the first summand:

$$\begin{aligned}
& \left(1 - \frac{2 \cos \left(\frac{\pi y}{d_y} \right) \cos \left(\frac{\pi y_0}{d_y} \right)}{\cosh \left(\frac{\pi(2nd_x+l(x-x_0))}{d_y} \right) + \cos \left(\frac{\pi(y+y_0)}{d_y} \right)} \right) \\
& \leq \left(1 + \frac{2}{\cosh \left(\frac{\pi(2nd_x+l(x-x_0))}{d_y} \right) - 1} \right) \leq \left(1 + \frac{4e^{-\frac{\pi(2nd_x+l(x-x_0))}{d_y}}}{1 - \frac{1}{\cosh \left(\frac{\pi(2nd_x+l(x-x_0))}{d_y} \right)}} \right) \\
& \leq \left(1 + e^{-\frac{2n\pi d_x}{d_y}} \frac{4e^{\frac{\pi d_x}{d_y}}}{1 - \frac{1}{\cosh \left(\frac{(2n-1)\pi d_x}{d_y} \right)}} \right). \quad (\text{D.15})
\end{aligned}$$

The last step follows from the fact that $x - x_0 \in [-d_x, d_x]$. For the second summand we get:

$$\begin{aligned}
& \left(1 - \frac{2 \cos\left(\frac{\pi y}{d_y}\right) \cos\left(\frac{\pi y_0}{d_y}\right)}{\cos\left(\frac{\pi(y+y_0)}{d_y}\right) - \cosh\left(\frac{\pi(2(n+1)d_x + l(d_x - (x+x_0)))}{d_y}\right)} \right) \\
& \leq \left(1 + e^{-\frac{2n\pi d_x}{d_y}} \frac{4}{1 - \frac{1}{\cosh\left(\frac{2n\pi d_x}{d_y}\right)}} \right) \leq \left(1 + e^{-\frac{2n\pi d_x}{d_y}} \frac{4e^{\frac{\pi d_x}{d_y}}}{1 - \frac{1}{\cosh\left(\frac{(2n-1)\pi d_x}{d_y}\right)}} \right). \quad (D.16)
\end{aligned}$$

The reminder is:

$$\begin{aligned}
& |R_N(x, y, x_0, y_0)| = \\
& \left| \sum_{n \geq N+1} \sum_{l \in \{-1, 1\}} \frac{1}{2\pi} \left(\log \left(1 - \frac{2 \cos\left(\frac{\pi y}{d_y}\right) \cos\left(\frac{\pi y_0}{d_y}\right)}{\cos\left(\frac{\pi(y+y_0)}{d_y}\right) + \cosh\left(\frac{\pi(2nd_x + l(x-x_0))}{d_y}\right)} \right) \right. \right. \\
& \quad \left. \left. + \log \left(1 - \frac{2 \cos\left(\frac{\pi y}{d_y}\right) \cos\left(\frac{\pi y_0}{d_y}\right)}{\cos\left(\frac{\pi(y-y_0)}{d_y}\right) - \cosh\left(\frac{\pi(2(n+1)d_x + l(d_x - (x+x_0)))}{d_y}\right)} \right) \right) \right|. \quad (D.17)
\end{aligned}$$

It holds that:

$$|R_N(x, y, x_0, y_0)| \leq \frac{2}{\pi} \sum_{n \geq N+1} \log \left(1 + e^{-\frac{2n\pi d_x}{d_y}} \frac{4e^{\frac{\pi d_x}{d_y}}}{1 - \frac{1}{\cosh\left(\frac{(2n-1)\pi d_x}{d_y}\right)}} \right) \quad (D.18)$$

$$= \frac{2}{\pi} \log \prod_{n \geq N+1} \left(1 + e^{-\frac{2n\pi d_x}{d_y}} \frac{4e^{\frac{\pi d_x}{d_y}}}{1 - \frac{1}{\cosh\left(\frac{(2n-1)\pi d_x}{d_y}\right)}} \right) \quad (D.19)$$

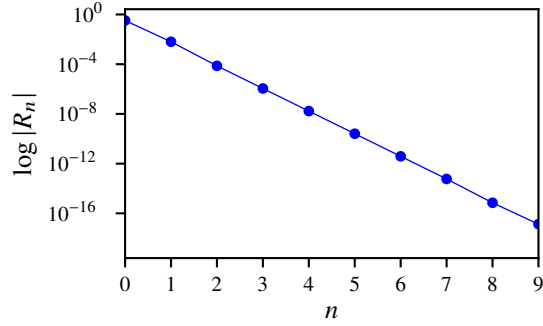
$$= \frac{2}{\pi} \log \frac{\left(\frac{4e^{\frac{\pi d_x}{d_y}}}{\frac{1}{\cosh\left(\frac{(2n-1)\pi d_x}{d_y}\right)} - 1}; e^{-\frac{2\pi d_x}{d_y}} \right)_{\infty}}{\left(\frac{4e^{\frac{\pi d_x}{d_y}}}{\frac{1}{\cosh\left(\frac{(2n-1)\pi d_x}{d_y}\right)} - 1}; e^{-\frac{2\pi d_x}{d_y}} \right)_{N+1}}. \quad (D.20)$$

□

The q -Pochhammer symbol $(a; q)_n$ is defined in appendix C.1.4 and this form is absolute convergent. In fig. D.4 a logarithmic plot of the reminder in the case $d_x/d_y = 2/3$ is shown. The N^{th} reminder decreases exponentially as $e^{-\frac{2\pi+2}{3}}$. The formula of the reminder as used in fig. D.4 is:

$$|R_N(x, y, x_0, y_0)| \leq \frac{2}{\pi} \log \left(\frac{\left(\frac{4e^{2\pi/3}}{\operatorname{sech}\left(\frac{4}{3}\left(n-\frac{1}{2}\right)\pi\right)-1}; e^{-4\pi/3} \right)_{\infty}}{\left(\frac{4e^{2\pi/3}}{\operatorname{sech}\left(\frac{4}{3}\left(n-\frac{1}{2}\right)\pi\right)-1}; e^{-4\pi/3} \right)_{n+1}} \right). \quad (\text{D.21})$$

Fig. D.4 The upper bound for the reminder in eq. (D.21).



Summary: After the introduction of the *method of images*, we highlighted *convergence problems* in the case of perfect boundaries at the example of the two infinite plate problem, which are usually neglected in presentations of the configuration. The proposed solution, as the limiting case of imperfect boundary conditions, yielding convergent series, was given. The result eqs. (D.10) and (D.11) is in agreement with the literature and the independently applied method of conformal mappings sections 3.6.2.1 and 4.2.1.

Moreover, the rectangular problem was solved with the method of images, revealing a *new representation of the Green function* of the rectangle (eq. (D.13)), including the proof of convergence.

We saw that the method is involved even for relatively simple configurations, and the advantage of working in the simple framework of conformal invariants is evident.

Appendix E

Closed-Form Potentials for the PS Combined Function Magnets

The novel closed-form for the PS combined function magnets are visualized in this chapter. There are global pictures of the Neumann function including the field directions as arrows depicted in figs. E.1 and E.3, for the closed and the open type, respectively as described in section 6.2. The red point indicates the position of the beam and the black ellipse shows the vacuum chamber of the PS. A close ups of these functions are shown in figs. E.2 and E.4, for the closed and the open type, respectively as described. The corresponding formulas were provided in eqs. (6.1) and (6.2).

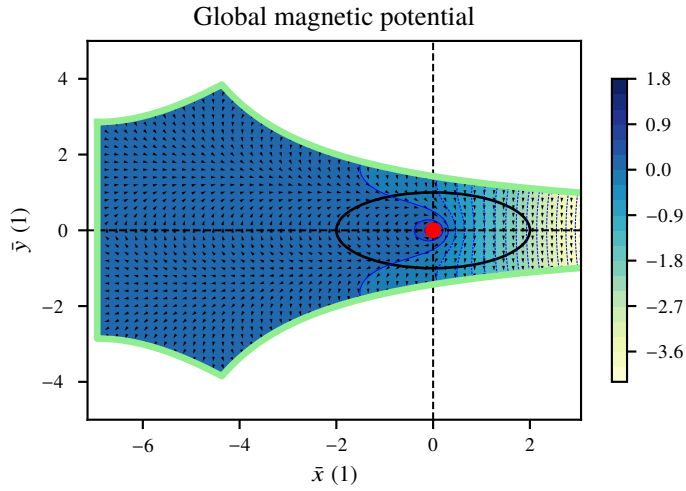


Fig. E.1: A potential plot including field lines of the closed PS magnet. The red point shows a centred beam.

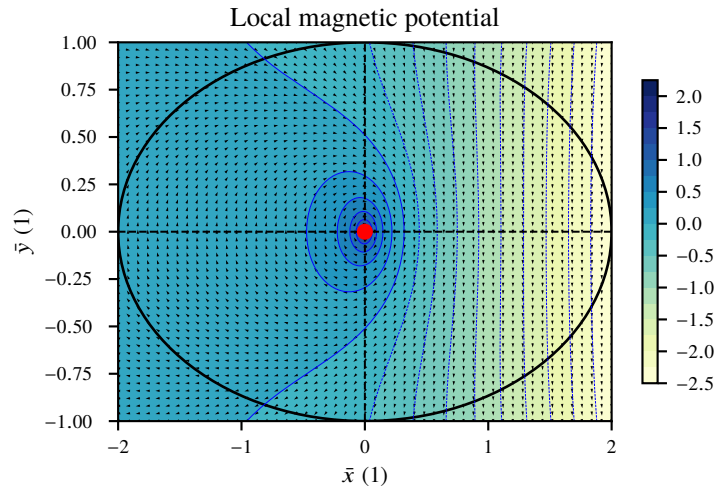


Fig. E.2: A potential plot including field lines of the open PS magnet. The red point shows a centred beam.

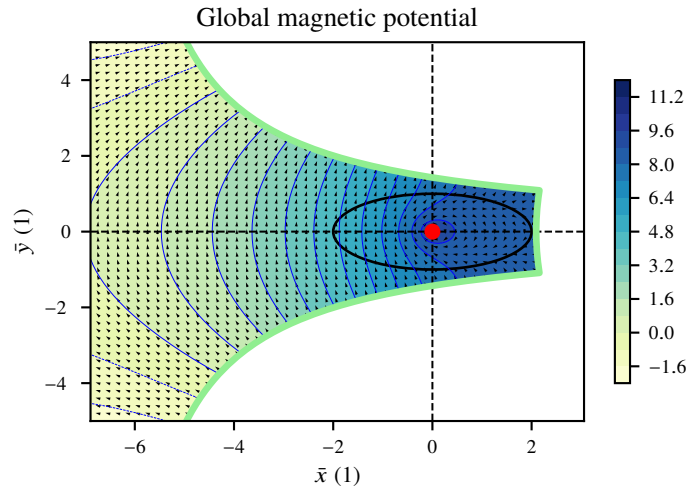


Fig. E.3: A potential plot including field lines of the PS closed magnet model.

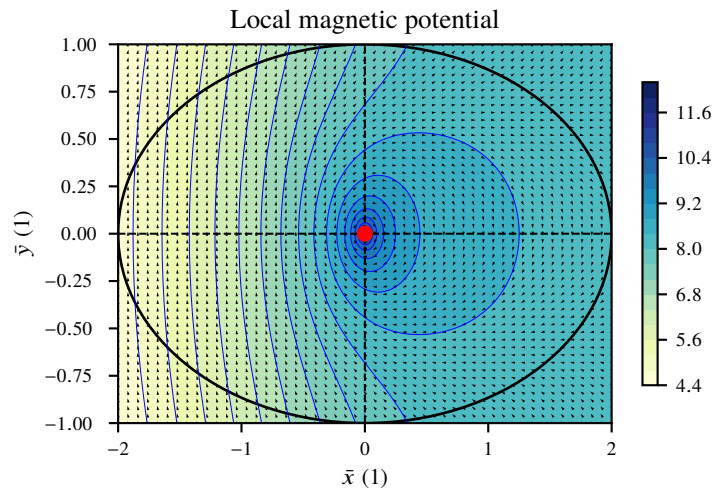


Fig. E.4: A potential plot including field lines of the open PS magnet. The red point shows a centred beam.

References

- [1] L. Laslett. On intensity limitations imposed by transverse space-charge effects in circular particle accelerators. *Summer Study on Storage Rings, BNL Report*, 7534:325–367, 1963.
- [2] S. Petracca. The Tensor nature of Laslett coefficients: Square versus circular cross-section liners. *Part. Accel.*, 48:181,192, 1994.
- [3] S. Petracca, I. M. Pinto, and F. Ruggiero. Normal mode Laslett coefficients. *Conf. Proc.*, C960610:1408–1410, 1996.
- [4] S. Petracca. Normal mode laslett coefficients for one (co-axial) and two (offset) liners in a circular magnetic yoke. *Part. Accel.*, 62(CERN-SL-97-003-AP):241–258, January 1997.
- [5] S. Petracca, V. Galdi, and I. M. Pinto. Efficient computation of electrical Laslett coefficients for rounded-rectangular pipes. *Particle Accelerators*, 63:37–55, January 1999.
- [6] K. Y. Ng. Exact Solutions for the Longitudinal and Transverse Impedances of an Off Centered Beam in a Rectangular Beam Pipe. *Part. Accel.*, 16:63, 1984.
- [7] K. Y. Ng. Betatron tune shifts and laslett image coefficients. Technical Report FERMILAB-TM-2152, FERMILAB, Batavia, IL, 2001.
- [8] B. Zotter. Coherrent Q shift of a relativistic particle beam in a mettalic vacuum chamber. 1972.
- [9] B. Zotter. The Q-shift of off-center particle beams in elliptic vacuum chambers. *Nuclear Instruments and Methods*, 129(2):377–395, 1975.
- [10] F. Ruggiero. Single-beam collective effects in the LHC. *Part. Accel.*, 50(CERN-SL-95-09-AP. LHC-NOTE-313):83–104, 1995.
- [11] M. Migliorati, N. Biancacci, M. R. Masullo, L. Palumbo, and V. G. Vaccaro. Space charge impedance and electromagnetic fields in elliptical vacuum chambers. *Phys. Rev. Accel. Beams*, 21:124201, Dec 2018.
- [12] P. Loïez. The Proton Synchrotron (PS) in its tunnel.. Le synchrotron à protons (PS) dans son tunnel. Feb 1996.
- [13] P. Muth. 3D printed Synchrotron Magnet replica. 2019.
- [14] M. Brice. Cross section of an LHC dipole in the tunnel. Jul 2011.

- [15] AC-Team. Diagram of an LHC dipole magnet. Schéma d'un aimant dipôle du LHC. Jun 1999.
- [16] M. Giovannozzi, D. Quatraro, and G. Turchetti. Generating unstable resonances for extraction schemes based on transverse splitting. *Physical Review Special Topics - Accelerators and Beams*, 12(2), February 2009.
- [17] S. Machida, C. Prior, S. Gilardoni, M. Giovannozzi, A. Huschauer, and S. Hirlaender. Numerical investigation of space charge effects on the positions of beamlets for transversely split beams. *Physical Review Accelerators and Beams*, 20(12), December 2017.
- [18] W. Schlag. *A course in complex analysis and Riemann surfaces*, volume 154. American Mathematical Society, 2014.
- [19] T. H. Wolff. *Lectures on harmonic analysis (university lecture series)*. American Mathematical Society, 2003.
- [20] T. Ransford. *Potential theory in the complex plane*. London Mathematical Society Student Texts. Cambridge University Press, 1995.
- [21] M. Tsuji. *Potential theory in modern function theory*. Chelsea Publishing Company, 1975.
- [22] B. Hague. *The principles of electromagnetism applied to electrical machines*. Dover, New York, NY, 1962.
- [23] S. Bell. *The Cauchy transform, potential theory and conformal mapping, 2nd Edition*. Chapman and Hall/CRC, 2015.
- [24] Y. A. Melnikov. *Green's functions and infinite products: Bridging the divide*. Springer, Boston, 2011.
- [25] S. G. Krantz, editor. *Geometric function theory*. Birkhäuser Boston, 2007.
- [26] V. N. Dubinin. *Condenser capacities and symmetrization in geometric function theory*. Springer, Basel, 2014.
- [27] T. A. Driscoll, P. G. Ciarlet, A. Iserles, R. V. Kohn, L. N. Trefethen, and M. H. Wright. *Schwarz-Christoffel mapping*. Cambridge Univ. Press, Cambridge, 2002.
- [28] T. W. Gamelin. *Complex analysis*. Undergraduate Texts in Mathematics. Springer, New York, NY, 2001.
- [29] R. Courant and D. Hilbert. *Methoden der mathematischen Physik*. Number 1 in Die Grundlehren der mathematischen Wissenschaften. Springer, 1968.
- [30] M. J. Ablowitz and A. S. Fokas. *Complex variables: introduction and applications; 2nd ed.* Cambridge Univ. Press, Cambridge, 2003.
- [31] J. Van Bladel. Magnetostatic fields at an iron-air boundary. *American Journal of Physics*, 29(11):732–736, 1961.
- [32] S. G. Mikhlin. *Mathematical physics, an advanced course*. North-Holl Ser. Appl. Math. Mech. North-Holland, Amsterdam, 1970. Trans. from the Russian.
- [33] P. Henrici. *Applied and computational complex analysis*. Pure and Applied Mathematics. Wiley, New York, NY, 1986.
- [34] J. Bruna and J. Cufi. *Complex analysis*. European Mathematical Society Publishing House, May 2013.

- [35] C. Pommerenke. *Boundary behaviour of conformal maps*. Grundlehren der mathematischen Wissenschaften A Series of Comprehensive Studies in Mathematics. Springer, Berlin, 1992.
- [36] R. C. Fernow. *Principles of magnetostatics*. Cambridge University Press, 2016.
- [37] S. G. Krantz. *The theory and practice of conformal geometry*. Aurora: Dover Modern Math Originals. Dover Publications, 2016.
- [38] P. K. Kythe. *Computational conformal mapping*. Springer, Boston, 1998.
- [39] N. Papamichael and N. Stylianopoulos. *Numerical conformal mapping: Domain decomposition and the mapping of quadrilaterals*. World Scientific, 2010.
- [40] R. Schinzinger and P. A. A. Laura. *Conformal mapping: Methods and applications*. Dover Books on Mathematics. Dover Publications, 2012.
- [41] Z. Nehari. *Conformal mapping*. International series in pure and applied mathematics. McGraw-Hill, New York, NY, 1952.
- [42] M. Schaumann, M. Solfaroli, and J. Wenninger. Tune and chromaticity: Decay and snapback. In *Proceedings, 6th Evian Workshop on LHC beam operation: Evian Les Bains, France, December 15-17, 2015*, pages 55,64, Geneva, 2016. CERN.
- [43] R. De Maria, M. Giovannozzi, T. Persson, R. Tomas, and Y. Wei. Analysis of intensity-dependent effects on LHC transverse tunes at injection energy. In *Proceedings, 6th International Particle Accelerator Conference (IPAC 2015): Richmond, Virginia, USA, May 3-8, 2015*, 2015.
- [44] M. Campanelli. *Inside CERN's large hadron collider*. WORLD SCIENTIFIC, January 2015.
- [45] P. Gagnon. *Who Cares about Particle Physics?: Making Sense of the Higgs Boson, the Large Hadron Collider and Cern*. OUP Oxford, 2018.
- [46] O. Brüning and L. Rossi. *The high luminosity large hadron collider*. WORLD SCIENTIFIC, February 2015.
- [47] R. Vanden Broeck. THE CERN ACCELERATOR COMPLEX. Complexe des accélérateurs du CERN. Sep 2019.
- [48] T. D. Frank. *Nonlinear Fokker-Planck equations: Fundamentals and applications*. Springer Series in Synergetics. Springer Berlin Heidelberg, 2005.
- [49] J. Rosenzweig. *Fundamentals of Beam Physics*. OUP Oxford, 2003.
- [50] A. Chao. *Physics of collective beam instabilities in high energy accelerators*. Wiley Interscience publication. Wiley, 1993.
- [51] K. Y. Ng. *Physics of intensity dependent beam instabilities*. World Scientific, Hoboken, NJ, 2006.
- [52] V. I. Arnold. *Mathematical methods of classical mechanics; 2nd ed*. Graduate Texts in Mathematics. Springer, New York, 1989.
- [53] H. Goldstein, C. Poole, and J. Safko. *Classical mechanics; 3rd ed*. Addison-Wesley, San Francisco, CA, 2002.
- [54] J. H. Argyris, G. Faust, M. Haase, and R. Friedrich. *An exploration of dynamical systems and chaos; 2nd ed*. Springer, Berlin, 2015.

- [55] T. Bountis and H. Skokos. *Complex Hamiltonian dynamics*. Springer Berlin Heidelberg, 2012.
- [56] S. Wimberger. *Nonlinear dynamics and quantum chaos*. Springer International Publishing, 2014.
- [57] M. Berz. Symplectic tracking through circular accelerators with high order maps. *Nonlinear Problems in Future Accelerators*, eds. W. Scandale and G. Turchetti (World Scientific, Singapore, 1991), pages 288–296, 1991.
- [58] S. Peggs and T. Satogata. *Introduction to accelerator dynamics*. Cambridge University Press, 2017.
- [59] E. D. Courant and H. S. Snyder. Theory of the alternating-gradient synchrotron. *Annals of Physics*, 281(1-2):360–408, April 2000.
- [60] G. Stupakov and G. Penn. Action-angle variables for betatron oscillations. In *Graduate Texts in Physics*, pages 87–94. Springer International Publishing, 2018.
- [61] E. Zeidler, editor. *Springer-Taschenbuch der Mathematik*. Springer Fachmedien Wiesbaden, 2013.
- [62] G. Servizi, G. Turchetti, A. Bazzani, and E. Todesco. A normal form approach to the theory of nonlinear betatronic motion, 1994.
- [63] A. Bazzani, P. Mazzanti, G. Servizi, and G. Turchetti. Normal forms for Hamiltonian maps and nonlinear effects in a particle accelerator. *Il Nuovo Cimento B (1971-1996)*, 102(1):51–80, July 1988.
- [64] A. Bazzani, M. Giovannozzi, G. Servizi, E. Todesco, and G. Turchetti. Resonant normal forms, interpolating Hamiltonians and stability analysis of area preserving maps. *Physica D: Nonlinear Phenomena*, 64(1):66–97, 1993.
- [65] D. T. Abell. *Analytic properties and cremona approximation of transfer maps for Hamiltonian systems*. PhD thesis, UNIVERSITY OF MARYLAND COLLEGE PARK., January 1995.
- [66] A. J. Dragt and J. M. Finn. Lie series and invariant functions for analytic symplectic maps. *Journal of Mathematical Physics*, 17(12):2215–2227, 1976.
- [67] A. J. Dragt. *Lie methods for nonlinear dynamics with applications to accelerator physics*. 2019.
- [68] E. Forest and R. D. Ruth. Fourth-order symplectic integration. *Phys. D*, 43(1):105–117, April 1990.
- [69] H. Yoshida. Construction of higher order symplectic integrators. *Physics Letters A*, 150(5-7):262–268, November 1990.
- [70] W. Scandale and G. Turchetti. Nonlinear problems in future particle accelerators. In *Nonlinear Problems in Future Particle Accelerators*. WORLD SCIENTIFIC, March 1991.
- [71] M. Berz. *Modern map methods in particle beam physics*. *Advances in electronics and electron physics*. Advances in Imaging and Electron Physics. Academic Press, San Diego, CA, 1999.
- [72] Y. Yan, P. J. Channell, and M. J. Syphers. An algorithm for symplectic implicit Taylor-map tracking. Technical Report SSCL-P-157. SSCL-Preprint-157, SSC, Dallas, TX, October 1992.

- [73] J. S. Berg, R. L. Warnock, R. D. Ruth, and E. Forest. Construction of symplectic maps for nonlinear motion of particles in accelerators. *Physical Review*, 49:722–739, 1994.
- [74] J. D. Jackson. *Classical electrodynamics; 1st ed.* Wiley, New York, NY, 1962.
- [75] G. Wendt, F. E. Bordnis, H. Bremmer, L. Hartshorn, R. W. P. King, Charles H. Papas, and J. A. Saxton. *Handbuch der Physik.* Springer, Berlin, 1958.
- [76] A. Wolski. *Beam dynamics in high energy particle accelerators.* IMPERIAL COLLEGE PRESS, April 2013.
- [77] P. J. Bryant and K. Johnsen. *The principles of circular accelerators and storage rings.* Cambridge University Press, 1993.
- [78] P. K. Kythe. *Green's functions and linear differential equations: Theory, applications, and computation.* Chapman & Hall/CRC Applied Mathematics & Nonlinear Science. CRC Press, January 2011.
- [79] Y. A. Melnikov and M. Y. Melnikov. *Green's functions.* DE GRUYTER, January 2012.
- [80] H. Amann and J. Escher. Integration auf Mannigfaltigkeiten. In *Analysis III.* Springer, January 2008.
- [81] P. Blanchard and E. Brüning. *Mathematical methods in physics: distributions, Hilbert space operators, variational methods, and applications in quantum physics; 2nd ed.* Progress in Mathematical Physics. Springer, Cham, 2015.
- [82] E. Wegert. *Visual complex functions: An introduction with phase portraits.* SpringerLink : Bücher. Springer Basel, 2012.
- [83] W. Rudin. *Real and complex analysis.* McGraw-Hill international editions. McGraw-Hill, third. ed., international ed. edition, 2013. S. 405 - 406.
- [84] B. Riemann. *Gesammelte mathematische Werke und wissenschaftlicher Nachlass: Herausgegeben unter Mitwirkung von R. Dedekind von H. Weber.* BG Teubner, 1876.
- [85] M. I. Monastyrsky. *Riemann, topology, and physics.* Modern Birkhäuser Classics. Birkhäuser Boston, 2008.
- [86] P. K. Kythe. *Complex analysis: Conformal inequalities and the Bieberbach conjecture.* Chapman & Hall/CRC Monographs and Research Notes in Mathematics. CRC Press, 2016.
- [87] E. Zeidler. *Springer-Handbuch der Mathematik I: Begründet von I. N. Bronstein und K. A. Semendjaew Weitergeführt von G. Grosche, V. Ziegler und D. Ziegler Herausgegeben von E. Zeidler.* Springer-Handbuch der Mathematik. Springer Fachmedien Wiesbaden, 2012.
- [88] S. G. Krantz. *Complex variables: A physical approach with applications and MATLAB.* Textbooks in Mathematics. Taylor & Francis, 2007.
- [89] C. F. Matta, L. Massa, A. V. Gubskaya, and E. Knoll. Can One Take the Logarithm or the Sine of a Dimensioned Quantity or a Unit? Dimensional Analysis Involving Transcendental Functions. *Journal of Chemical Education*, 88(1):67–70, jan 2011.
- [90] S. Russenschuck. *Field computation for accelerator magnets.* Wiley VCH Verlag GmbH, 2010.

- [91] A. L. Davalos and D. H. Zanette. *Fundamentals of electromagnetism: vacuum electrodynamics, media, and relativity*. Springer, Berlin, 1999.
- [92] J. Van Bladel. *Singular electromagnetic fields and sources*. Oxford Eng. Sci. Clarendon Press, Oxford, 1991.
- [93] J. Van Bladel. *Electromagnetic Fields; 2nd ed.* Wiley, Newark, 2007.
- [94] M. Fujimoto. *Physics of classical electromagnetism*. Springer, Berlin, 2007.
- [95] S. Axler, P. Bourdon, and W. Ramey. *Harmonic Function Theory*. Springer New York, 2001.
- [96] S. Lang. *Complex analysis—4th ed.* Graduate Texts in Mathematics. Springer, New York, 1999.
- [97] L. Ahlfors. *Complex analysis: an introduction to the theory of analytic functions of one complex variable; 3rd ed.* International series in pure and applied mathematics. McGraw-Hill, New York, NY, 1979.
- [98] P. Byrd and M. Friedman. *Handbook of elliptic integrals for engineers and physicists*. Grundlehren Math. Wiss. Springer, Berlin, 1954.
- [99] P. Curie. Sur la symétrie dans les phénomènes physiques, symétrie d'un champ électrique et d'un champ magnétique. *Journal de Physique Théorique et Appliquée*, 3(1):393–415, 1894.
- [100] S. G. Krantz. *Handbook of complex variables*. Birkhäuser, Basel, 1999.
- [101] P. K. Kythe. *Handbook of Conformal Mappings and Applications*. Chapman and Hall/CRC, 2019.
- [102] Bruce P. Palka. *An Introduction to Complex Function Theory*. Springer New York, 2012.
- [103] H. Weyl, S. J. Patterson, and R. Meyer. *Einführung in die Funktionentheorie*. Birkhäuser Basel, 2008.
- [104] B. Schnizer. *Analytische Methoden der theoretischen Physik*. University Lecture, 2013.
- [105] T. A. Driscoll and S. A. Vavasis. Numerical conformal mapping using cross-ratios and delaunay triangulation. *SIAM Journal on Scientific Computing*, 19(6):1783–1803, November 1998.
- [106] M. A. Brilleslyper, M. J. Dorff, J. M. McDougall, J. S. Rolf, L. E. Schaubroek, R. L. Stankewitz, and K. Stephenson. *Explorations in complex analysis*. Classroom Resource Materials. Mathematical Association of America, 2012.
- [107] K. Aomoto, M. Kita, T. Kohno, and K. Iohara. *Theory of hypergeometric functions*. Springer Monographs in Mathematics. Springer, Dordrecht, 2011.
- [108] Website of the MATLAB - based SC Toolbox by Driscoll and Trefethen. Online. <http://www.math.udel.edu/~driscoll/SC/>.
- [109] L. N. Trefethen. Numerical computation of the Schwarz–Christoffel transformation. *SIAM Journal on Scientific and Statistical Computing*, 1(1):82–102, March 1980.
- [110] K. Fritzsche. *Grundkurs Funktionentheorie: Eine Einführung in die komplexe Analysis und ihre Anwendungen*. Für Bachelor und Diplom. Spektrum Akademischer Verlag, 2008.
- [111] I. Hofmann. *Space charge physics for particle accelerators*. Springer International Publishing, 2017.

- [112] L. Laslett and L. Resegotti. The space-charge intensity limit imposed by coherent oscillation of a bunched synchrotron beam. In *Proceedings, 6th International Conference on High-Energy Accelerators, HEACC 1967: Cambridge, Massachusetts, September 11-15, 1967*, pages 150–152, 1967.
- [113] W. Fischer and I. Lieb. *Einführung in die Komplexe Analysis: Elemente der Funktionentheorie*. Bachelorkurs Mathematik. Vieweg+Teubner Verlag, 2011.
- [114] D. A. Edwards and L. C. Teng. Parametrization of linear coupled motion in periodic systems. *IEEE Transactions on Nuclear Science*, 20(3):885–888, June 1973.
- [115] H. Wiedemann. *Particle accelerator physics; 4th ed.* Springer, Berlin, 2015.
- [116] J. P. Burnet, C. Carli, M. Chanele, R. Garoby, S. Gilardoni, M. Giovannozzi, S. Hancock, H. Haseroth and K. Hübner, D. Küchler, J. Lewis, A. Lombardi, D. Manglunki, M. Martini, S. Maury, E. Metral, D. Möhl, G. Plass, L. Rinolfi, R. Scrivens, R. Steerenberg, C. Steinbach, M. Vretenar, and T. Zickler. *Fifty years of the CERN Proton Synchrotron: Volume 1*. CERN, Geneva, 2011.
- [117] M. Brice. PS facility - view of the magnets in the Proton Synchrotron. May 2014. General Photo.
- [118] R. Capi and M. Giovannozzi. Adiabatic capture of charged particles in stable islands: A novel approach to multi-turn extraction.
- [119] Website of the MAD project (Methodical Accelerator Design). Online. <http://mad.web.cern.ch/mad>.
- [120] The CERN layout database. Online. <https://layout.web.cern.ch>.
- [121] F. Schmidt, E. Forest, and E. McIntosh. Introduction to the polymorphic tracking code: fibre bundles, polymorphic taylor types and "exact tracking". Technical Report CERN-SL-2002-044-AP. KEK-REPORT-2002-3, CERN, Geneva, July 2002.
- [122] S. Gilardoni, M. Giovannozzi, and C. Hernalsteens. First observations of intensity-dependent effects for transversely split beams during multiturn extraction studies at the CERN proton synchrotron. *Phys. Rev. ST Accel. Beams*, 16:051001, May 2013.
- [123] S. Machida and M. Ikegami. Simulation of space charge effects in a synchrotron. *AIP Conference Proceedings*, 448(1):73–84, 1998.
- [124] S. Machida, S. S. Gilardoni, M. Giovannozzi, S. Hirlander, A. Huschauer, and C. R. Prior. Intensity effects in the formation of stable islands in phase space during the multi-turn extraction process at the CERN PS. In *Proc. of ICFA Advanced Beam Dynamics Workshop on High-Intensity and High-Brightness Hadron Beams (HB'16), Malmö, Sweden, July 3-8, 2016*, number 57 in ICFA Advanced Beam Dynamics Workshop on High-Intensity and High-Brightness Hadron Beams, pages 283–287, Geneva, Switzerland, August 2016. JACoW.
- [125] L. R. Evans and P. Bryant. LHC Machine. *JINST*, 3, 2008. This report is an abridged version of the LHC Design Report (CERN-2004-003).
- [126] M. Brice. Cross section of an LHC dipole in the tunnel. <https://cds.cern.ch/record/1365795>, July 2011.

- [127] P. Loiez, P. Rakosy, and L. Guiraud. Beam screens for the LHC beam pipes. Les conduits pour les faisceaux LHC. <https://cds.cern.ch/record/39110>, May 1997.
- [128] N. Svartholm. Die Loesung der Fuchsschen Differentialgleichung zweiter Ordnung durch hypergeometrische Polynome. *Mathematische Annalen*, 116(3):413–421, 1939.
- [129] A. Ronveaux. *Heun's differential equations*. Clarendon Press, 1995.
- [130] L. H. Howell. Numerical conformal mapping of circular arc polygons. *Journal of Computational and Applied Mathematics*, 46(1):7–28, 1993.
- [131] W. Gautschi. *Numerical Analysis*. Birkhäuser Boston, 1997.
- [132] S. Wright J. Nocedal. *Numerical Optimization*. Springer-Verlag GmbH, 2006.
- [133] M. Schaumann, M. Juchno, M. Lamont, M. Solfaroli Camillocci, E. Todesco, and J. Wenninger. Tune and Chromaticity Control During Snapback and Ramp in 2015 LHC Operation. In *Proc. of International Particle Accelerator Conference (IPAC'16), Busan, Korea, May 8-13, 2016*, number 7 in International Particle Accelerator Conference, pages 1501–1504, Geneva, Switzerland, June 2016. JACoW.
- [134] M. Morrone, C. Garion, and M. Aurisicchio. Thermomechanical study of complex structures in the aperture of superconducting magnets: Application to the design of the High-Luminosity LHC shielded beam screen, January 2018. Presented 23 Mar 2018.
- [135] G. Grubb. *Distributions and operators*. Graduate Texts in Mathematics. Springer, New York, 2009.
- [136] R. Courant and F. John. *Introduction to calculus and analysis*. Classics in Mathematics. Springer, Berlin, 1999.