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# Local beating corrections with orbit bumps in the LHC

Lokale Strahlkorrektur im LHC mittels Orbitmodulierungen

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## Local beating corrections with orbit bumps in the LHC

The purpose of this Master thesis is to explore further possibilities to increase the Luminosity of the Large Hadron Collider (LHC) by improving the optics correction. As a basic principle, turn-by-turn data is measured with BPMs (beam position monitors) and the local phase of the betatron oscillation is obtained by a harmonic analysis. The beta function describes the envelope of the beam motion if multiplied by the emittance and can be extracted from the turn-by-turn data by utilising the N-BPM method [29] after a spectral analysis. If there are lattice imperfections, then they will perturb the beam motion and the distortion oscillates around the ring. This phenomenon is generally called beating. In order to increase the Luminosity, the beta function at the interaction point (IP) has to be reduced to a minimum (Hourglass effect limits  $\mathcal{L}$  if  $\beta^* \approx \sigma_s$ ). This is realised by the achromatic telescopic squeezing scheme (ATS-optics) for the High Luminosity Large Hadron Collider (HL-LHC). The disadvantage of a small beta function at the IP is a very large beta function in the triplets. Hence, any error of the magnets in the triplet causes a significant beating of the optics functions and perturbs the beta function at the IP as well. Fortunately, the triplets are powered individually, wherefore the field strengths can be adjusted separately. After applying corrections based on measurements, the distortions within the lattice are equalized as much as possible by varying the knobs. The special optics configuration of the ATS sequence also enlarges the beta function in the arcs, where the quadrupoles are powered in series. A possible correction strategy for the arcs is to use feed-downs from sextupoles by applying orbit bumps to the beam. The feasibility of this approach has been demonstrated [27]. This Master thesis focuses on the validation of the correction method and investigates potentially degenerated orbit bumps. In addition, the measurement principle for the lattice functions is tested at the ELENA synchrotron.

## Lokale Strahlkorrektur im LHC mittels Orbitmodulierungen

Das Ziel dieser Masterarbeit ist es, die Luminosität des LHC durch Optimierung der Optik zu erhöhen. Die betatron Oszillationen werden durch BPMs (beam position monitors) induktiv erfasst. Die Betafunktion beschreibt die Einhüllende der Strahlbewegung, wenn sie mit der Emittanz multipliziert wird und kann mit der N-BPM Methode [29], im Anschluss an eine Spektralanalyse, bestimmt werden. Lokale Gitterimperfectionen stören die Strahlbewegung und erzeugen globale Variationen des Orbits. Dieses Phänomen wird allgemein als Schwebung bezeichnet. Um die Luminosität zu erhöhen, muss die Beta-Funktion am IP auf ein Minimum reduziert werden (Sanduhr-Effekt begrenzt  $\mathcal{L}$ , wenn  $\beta^* \approx \sigma_s$ ). Dies wird durch das achromatische Teleskop-Squeezing-Schema (ATS-Optik) für den HL-LHC realisiert. Der Nachteil einer kleinen Betafunktion am IP ist eine sehr große Betafunktion in den Triplets. Daher verursacht jeder Fehler der Magneten in den Triplets eine signifikante Abweichung der Optikfunktionen und stört auch die Betafunktion am IP. Die Triplets werden individuell gespeist, sodass die Feldstärken separat eingestellt werden können. Nach der Anwendung von Korrekturen, die auf Messungen beruhen, werden die Verzerrungen innerhalb des Gitters durch Variation der Stromstärke so weit wie möglich ausgeglichen. Die spezielle Optikkonfiguration der ATS-Sequenz vergrößert auch die Beta-Funktion in der Kurvatur, wo die Quadrupole in Reihe gespeist werden. Eine mögliche Korrekturstrategie für die Bögen ist die Verwendung von Feed-downs von Sextupolen, indem der Orbit lokal elongiert wird. Die Machbarkeit des Ansatzes wurde bereits für das ATS-layout des LHC's demonstriert [27]. Diese Masterarbeit dient der Validierung der Korrekturmethode und untersucht potenziell entartete Orbit-bumps. Darüber hinaus wird das Messprinzip bzgl. der Gitterfunktionen am ELENA Synchrotron getestet.

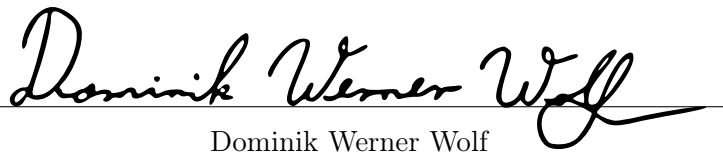
**Declaration of Authorship**

I hereby declare that the thesis submitted is my own unaided work. All direct or indirect sources used are acknowledged as references. I observed the current version of Heidelberg University's statutes for ensuring good scientific practice.

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The overall structure of the thesis reflects the expenditure of time. Chapter 1 to 4 give a theoretical introduction into the field of accelerator physics and lays the foundation for the subsequent chapters. Chapter 5 to 7 are of special importance for the implementation of the correction method. In detail, several sophisticated derivations were elaborated, which lead to the beating equations used for the analysis of the measurement data at ELENA in Chapter 8 and for the simulations in Chapter 9. In addition, Chapter 8 provides detailed insight into the data acquisition system and the measurement principle for the lattice functions. It will be shown that a chirp can be used to excite the beam at ELENA at a single frequency. The correction method of using orbit bumps in sextupoles to correct for local quadrupolar imperfections is validated by the simulations presented in Chapter 9. Furthermore, the scope of the correction method is specified and previous measurement results [27] are examined for a degeneracy of the orbit bump corrections.

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# 1 Introduction

In order to explore physics beyond the standard model and to validate our understanding of the subatomic world, rare processes have to be investigated wherefore the event rate  $\dot{N}$  needs to be maximised. The event rate (1) depends on the Luminosity  $\mathcal{L}$  as well as the cross section  $\sigma$ .

$$\frac{dN}{dt} = \mathcal{L} \cdot \sigma \quad (1)$$

The underlying physics characterises the observation probability of a specific subatomic process. As a consequence, the cross section is fully predetermined by quantum field theory and only the luminosity can be affected by adjusting the lattice design. Neglecting higher order effects and assuming head-on collisions the luminosity is specified by Equation (2).

$$\mathcal{L} = \frac{f \cdot N_1 \cdot N_2 \cdot N_b}{4\pi \cdot \sigma_x^* \cdot \sigma_y^*} \quad (2)$$

From Equation (2) it becomes apparent, that the beam size  $\sigma_x^*$  at the interaction point has to be controlled very accurately in order to increase the luminosity. The beam width is determined by the standard deviation of the displacement coordinate  $x$ . In general, the particle motion is described in the Frenet–Serret coordinate system  $(s, x, y)$ , which is defined in Chapter 2.1. The relative coordinates with regard to the reference particle are labeled by  $x$  and  $y$  (normal vector of plane  $(x, y)$  aligns with the momentum vector of the reference particle). Since it is expected that the particle ensemble follows the reference orbit, the expectation value of the first moment in  $x$  is zero and the beam width results as the second moment in  $x$  (3).

$$\sigma_x^2 \equiv \langle (x - \langle x \rangle)^2 \rangle = \langle x^2 \rangle \quad , \quad \text{since:} \quad \langle x \rangle = 0 \quad (3)$$

In Section 3, the Courant-Snyder Ansatz [7] will be presented to describe single particle trajectories  $x(s)$  by the lattice functions  $\alpha(s)$ ,  $\beta(s)$ ,  $\gamma(s)$ . Hence, the second moment in  $x$  is given by Equation (4).

$$\langle x^2 \rangle = \langle 2J_x \cdot \beta_x \cdot \cos^2(\psi_x + \phi_0) \rangle = \langle J_x \rangle \cdot \beta_x \quad (4)$$

The expectation value of the action  $J_x$  corresponds to the emittance  $\epsilon_x$ . The action  $J_x$  characterises the phase space volume of the ellipse spanned by  $x$  and  $x'$  for a single particle, as elaborated in detail in Chapter 3.2. The emittance  $\epsilon_x$  defines the contour ellipse on 68% confidence level for the particle bunch. The volume of the contour ellipse in phase space is statistically specified by the determinant of the phase space covariance matrix  $\Sigma$  (5). In other words, the state vector in phase space is mapped within the area of the contour ellipse with a probability of  $\sim 68\%$ . Consequently,  $\sqrt{2J_x\beta_x}$  denotes the envelope trajectory for a single particle and  $\sqrt{\epsilon_x\beta_x}$  characterizes the envelope trajectory enclosing 68% of the beam particles if the action  $J_x$  is Gaussian distributed.

$$\epsilon_x \equiv \langle J_x \rangle = \sqrt{|\Sigma|} = \sqrt{\langle x^2 \rangle \cdot \langle [x']^2 \rangle - [\langle xx' \rangle]^2} \quad (5)$$

It is important to emphasise, that the emittance  $\epsilon_x$  (as well as the action) is invariant under mapping. Consequently, the emittance is constant around the ring and does not depend on the longitudinal coordinate  $s$ . Thus, the beam width  $\sigma_x^*$  at the interaction point is given by Equation (6).

$$\sigma_x^* = \sqrt{\epsilon_x \cdot \beta_x^*} \quad (6)$$

Here,  $\beta_x^*$  labels the  $\beta$ -function at the interaction point. Finally, the luminosity  $\mathcal{L}$  can be rewritten in terms of  $\beta_x^*$  as presented in Equation (7).

$$\mathcal{L} = \frac{f \cdot N_1 \cdot N_2 \cdot N_b}{4\pi \cdot \sqrt{\epsilon_x \cdot \beta_x^*} \cdot \sqrt{\epsilon_y \cdot \beta_y^*}} \quad (7)$$

For increasing the Luminosity, the beta function at the interaction point (IP) has to be reduced to a minimum (Hourglass effect limits  $\mathcal{L}$  if  $\beta^* \approx \sigma_s$ ). For the high luminosity upgrade of the LHC the achromatic telescopic squeezing scheme (ATS-optics) will be used to squeeze the beam at the IP to 15 cm and even beyond in the most pushed scenario (flat-optics) for which  $\beta^*$  will approach the hourglass limitations at 7.5 cm [34].

## 1.1 Physics perspective

In general, there are two different perspectives on how to describe the dynamics in an accelerator:

- Particle dynamics: Analysis of single particle trajectories  $\rightarrow$  particle by particle approach
- Optics layout: Analysis of lattice functions  $\alpha(s)$ ,  $\beta(s)$ ,  $\gamma(s) \rightarrow$  particle ensemble study

In circular machines there exists a stable solution of the equation of motion under periodic boundary conditions (closed orbit) and individual particles oscillate around it. The lattice layout of the machine predetermines the periodic envelope trajectory for the beam. In contrast, individual particle trajectories are not enforced to obey periodic boundary conditions. Furthermore, the particle motion is symplectic in hadron machines and nearly symplectic in electron machines (but also slightly stochastic due to synchrotron radiation). Since this thesis focuses on the LHC, symplecticity of the particle motion can be assumed, wherefore all transfer maps are supposed to obey Relation (142).

## 1.2 Physics framework

In principle, there are three formulations to describe the dynamics: Newtonian dynamics, Lagrangian dynamics and Hamiltonian dynamics. Newton's formulation involves relating a force  $\vec{F}$  to the resulting motion by Newton's second law (8).

$$\frac{d\vec{p}}{dt} = \vec{F}(\vec{x}, \dot{\vec{x}}, t) \quad (8)$$

On the contrary, Hamiltonian mechanics is characterised by the Hamilton function  $H$ , which depends on the canonical variables  $\{q_i, p_i\}$ . The natural choice for the canonical variables are the  $n$  independent generalized coordinates  $q_i$  from Lagrangian mechanics and the  $n$  independent conjugate momenta  $p_i$ , defined by the derivative of the Lagrangian  $\mathcal{L}$  with respect to  $\dot{q}_i$ . The  $2n$  canonical variables define the state vector  $\vec{\eta}$  in phase space. The Hamiltonian  $H(q_i, p_i)$  is obtained by a Legendre transformation of the Lagrangian  $\mathcal{L}(q_i, \dot{q}_i)$  [42]. The physics is completely embedded in the Hamiltonian as a function of the canonical variables and the dynamics of the particles are described by the  $2n$  Hamilton equations (9). The benefit of Hamiltonian dynamics is the reduction of the equation of motion to  $2n$  first order differential equations. In comparison, the Lagrange equation is of second order but comprises only  $n$  differential equations for the generalized coordinates. Hence, the complexity of the problem is not reduced but a first order differential equation is generally easier solvable.

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \quad , \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \quad (9)$$

Furthermore, the advantage of the Hamilton formalism lies in the handling of invariant quantities, which is related to Noether's theorem [4]. According to Noether's theorem, each invariance can be identified as a conservation law.

### 1.3 Guiding force

The guiding force in a Synchrotron accelerator is the Lorentz force (10).

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \quad (10)$$

Since the velocities of the particles are close to the speed of light, magnetic fields are highly efficient for guiding particles in accelerators. On the contrary, for acceleration, electric fields are needed due to the fact that the Lorentz force acts always perpendicular to the velocity vector. Hence, magnetic fields do not change the absolute momentum of a particle. The elements in an accelerator which are employed to modify the particle energy are called RF-cavities.

### 1.4 Design orbit

The design orbit of an accelerator is determined by the magnetic flux density  $B$  of the bending dipoles. In the picture of Newtonian dynamics, the radius  $\rho$  of the design orbit is given by the equilibrium between Lorentz force and centripetal force (11).

$$\frac{\gamma m v^2}{\rho} \stackrel{!}{=} q v B_0 \quad \Leftrightarrow \quad \rho B_0 = \frac{p}{q} \quad (11)$$

Here,  $q$  denotes the charge of the particle and  $p$  labels the absolute value of the relativistic particle momentum. The product of  $B_0$  times  $\rho$  is known as the beam rigidity. Generally, the magnetic field strengths are normalised by the beam momentum in order to obtain energy independent values.

### 1.5 Betatron oscillation and natural tune

Consider the reference particle on the design orbit. If a second, off-axis particle, is assumed which carries the same energy it will also undergo a synchrotron motion with the same radius as the reference particle. In the rest frame of the reference particle, this looks like an oscillation around it (geometric focusing). This oscillation is commonly known as the betatron oscillation.

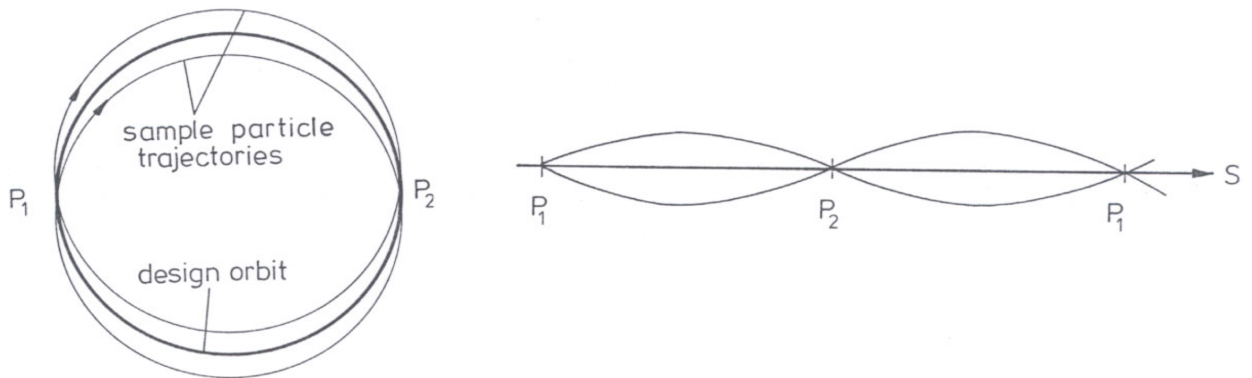


Figure 1: Two transversely displaced but synchronous particles are shown. The displacement is related to the design orbit and the particles are of equal energy with reference momentum  $p_0$ . Both plots depict equivalent physical trajectories but in different rest frames: (left) laboratory system and (right) reference particle. The transversal displacement results in oscillations around the design orbit, which are known as betatron oscillations [3].

Generally, betatron oscillations are generated by the weak focusing of the dipoles as well as the strong focusing of the quadrupoles for transversely displaced particles. The number of oscillations per turn is defined as the natural tune  $Q_o$  of the accelerator. The horizontal tune  $Q_x$  will be influenced by the dipoles and quadrupoles whereas  $Q_y$  is entirely determined by the strong focusing of the quadrupoles. In the illustrative example of Figure (1) the natural tune is equal to one. In Chapter 3.5 the natural tune  $Q_o$  will be mathematically defined as an integral over the inverse of the  $\beta$ -function.

## 1.6 Multipole expansion

In free space, e.g. the interior of an accelerator vacuum chamber, the static Maxwell equations are given by Expression (12). Implying the requirement of zero charged particle flux ( $\vec{J} = \vec{0}$ ) through the reference surface.

$$\vec{\nabla} \cdot \vec{B} = 0 \quad , \quad \vec{\nabla} \times \vec{B} = \vec{0} \quad (12)$$

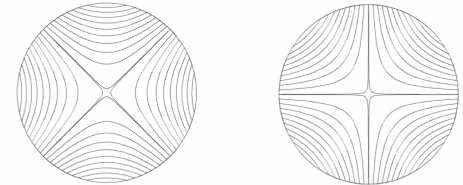
Furthermore, in free space the relative permeability is equal to the identity matrix. Hence, the magnetic flux points into the same direction as the magnetic field vector ( $\vec{B} = \mu_0 \vec{H}$ ). If a constant longitudinal magnetic field (e.g.  $B_z = 0$ ) is considered the solution of the set of differential equations from Expression (12) can be written as presented in Equation (13) [43]. The result in Equation (13) implies an additional degree of freedom which represents the order  $n$  of the multipole expansion.

$$B_y + iB_x = \sum_{n=0}^{\infty} c_n \cdot [x + iy]^n \quad , \quad \text{with: } c_n \in \mathbb{C} \quad (13)$$

The magnetic fields for quadrupole and sextupole are visualized in Figure (2). In addition, there are combined function magnets which assign a field gradient to a dipole by tilting the magnetic pole faces.

In order to derive the equation of motion for a particle in an accelerator it is required to determine the motion of the particle with respect to the closed orbit. Furthermore, small deviations from the closed orbit will be considered, wherefore  $x, y \ll \rho$  holds. Consequently, an arbitrary guiding field can be Taylor expanded with respect to the relative coordinates  $x$  and  $y$  around the design orbit (14). In order to get an energy independent field gradient  $k$  the magnetic flux density  $B$  has been normalized by the reference rigidity of the reference particle with momentum  $p_0$ . It is reasonable to take only linear terms of the Taylor expansion into account since the perturbations are assumed to be sufficiently small.

$C_2 = \text{real, normal quadrupole}$     $C_2 = \text{imaginary, skew quadrupole}$



$C_3 = \text{real, normal sextupole}$     $C_3 = \text{imaginary, skew sextupole}$

Figure 2:  $C_n$  represents the expansion coefficient of the multipole expansion.  $n = 1$  is known as a dipole field.  $n = 2$  is known as a quadrupole field.  $n = 3$  is known as a sextupole field [1].

$$\begin{aligned} \frac{q}{p} B_y(x) &= \frac{q}{p} B_y \Big|_{x=0} + x \cdot \frac{q}{p} \left[ \frac{dB_y}{dx} \right] \Big|_{x=0} + x^2 \cdot \frac{q}{2p} \left[ \frac{d^2 B_y}{dx^2} \right] \Big|_{x=0} + \mathcal{O}(x^3) \equiv \sum_{n=0}^{\infty} \frac{k_n}{n!} \cdot x^n \\ \Rightarrow \frac{q}{p} \vec{B} &\equiv \begin{pmatrix} g \cdot y + \mathcal{O}(y^2) \\ \rho^{-1} + g \cdot x + \mathcal{O}(x^2) \end{pmatrix} \quad , \quad \text{with: } 0 \stackrel{!}{=} \frac{q}{p} B_x \Big|_{y=0} \end{aligned} \quad (14)$$

## 2 Hills equations

### 2.1 Frenet–Serret coordinate system

For a flat machine, the Frenet–Serret coordinate system can be expressed as a co-moving cylindrical coordinate system. Generally, a cylindrical coordinate system references to the laboratory rest frame and is stationary. Stationary signifies that the origin is a fixpoint of the system. In contrast, the Frenet–Serret coordinate system refers to the rest frame of the reference particle, which is non-stationary since the particle is moving. Consequently, the Frenet–Serret coordinate system is moving with the reference particle around the design orbit and is composed by three orthonormal vectors: unit tangential vector with regard to the trajectory (direction of motion  $\hat{s}$ ), derivative of the tangential vector with respect to the longitudinal position  $s$  (unit normal vector  $\hat{x}$ ), complementary unit vector which spans the right handed coordinate system (binormal vector  $\hat{y}$  formed by the cross product). The design orbit is characterised by the magnetic field strength of the dipoles. For a reference particle without torsion it holds that:  $(\hat{s}, \hat{x}, \hat{y}) = (\hat{\theta}, \hat{\rho}, \hat{z})$ . In other words, in the special case of a flat machine the Frenet–Serret vector space is spanned by the cylindrical basis vectors but referred to the rest frame of the reference particle on the design orbit. In transverse beam dynamics the deviation from the reference particle is studied, wherefore the physical description is entirely embedded in the Frenet–Serret coordinate system. For the purpose of this thesis, the longitudinal displacement  $\tau$  is generally neglected and set to zero. The position vector  $\vec{R}$  of an arbitrary particle is described in Equation (15). Since the Frenet–Serret coordinates quantify deviations from the design orbit, they are typically in the range of millimetres. The transversal phase space of an off-design particle is spanned by the coordinates  $(x, y)$  and by the corresponding derivatives in the longitudinal coordinate  $s$  ( $x', y'$ ). The fourth coordinate  $\theta = \rho_0^{-1}s$  defines the basis vectors and characterizes the evolution of the particle motion in time.

|                                                                                                        |  |                                                                                                                                                                                                                                                                    |
|--------------------------------------------------------------------------------------------------------|--|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <p style="text-align: center;"><b>Laboratory rest frame</b><br/>(Rest frame of reference particle)</p> |  | <p style="text-align: center;"><b>Co-moving Frenet–Serret frame</b><br/>(Rest frame of real particle)</p>                                                                                                                                                          |
| $\vec{R}_{\text{Lab}} = (\rho_0 + x) \cdot \hat{\rho} + y \cdot \hat{z} + \tau \cdot \hat{\theta}$     |  | $\begin{aligned} \vec{R}_{\text{FS}} &= \vec{R}_{\text{Lab}} \{\text{real particle}\} - \vec{R}_{\text{Lab}} \{\text{reference particle}\} \\ \Leftrightarrow \vec{R}_{\text{FS}} &= x \cdot \hat{\rho} + y \cdot \hat{z} + \tau \cdot \hat{\theta} \end{aligned}$ |
| (15)                                                                                                   |  |                                                                                                                                                                                                                                                                    |

The kinematics of an arbitrary particle with position vector  $\vec{R}$  is determined by Expression (16). For the derivation of the equation of motion the real particle is described in the laboratory rest frame.

$$\begin{aligned}
\dot{\vec{R}}(x, y, \theta | \tau = 0) &\stackrel{(15)}{=} \dot{x} \cdot \hat{\rho} + (\rho_0 + x) \cdot \dot{\hat{\rho}} + \dot{y} \cdot \hat{z} \\
\Leftrightarrow \dot{\vec{R}}(x, y, \theta | \tau = 0) &= \dot{x} \cdot \hat{\rho} + (\rho_0 + x) \dot{\hat{\theta}} \cdot \hat{\theta} + \dot{y} \cdot \hat{z} \\
\Leftrightarrow \dot{\vec{R}}(x, y, s | \tau = 0) &\equiv v_x \cdot \hat{\rho} + v_s \cdot \hat{\theta} + v_y \cdot \hat{z} \\
\ddot{\vec{R}}(x, y, \theta | \tau = 0) &\stackrel{(15)}{=} \ddot{x} \cdot \hat{\rho} + \dot{x} \cdot \dot{\hat{\rho}} + \dot{x} \dot{\hat{\theta}} \cdot \hat{\theta} + (\rho_0 + x) \ddot{\hat{\theta}} \cdot \hat{\theta} + (\rho_0 + x) \dot{\hat{\theta}} \cdot \dot{\hat{\theta}} + \ddot{y} \cdot \hat{z} \\
\Leftrightarrow \ddot{\vec{R}}(x, y, \theta | \tau = 0) &= \ddot{x} \cdot \hat{\rho} + 2\dot{x}\dot{\hat{\theta}} \cdot \hat{\theta} + (\rho_0 + x) \ddot{\hat{\theta}} \cdot \hat{\theta} - (\rho_0 + x) \dot{\hat{\theta}}^2 \cdot \hat{\rho} + \ddot{y} \cdot \hat{z} \\
\Leftrightarrow \ddot{\vec{R}}(x, y, \theta | \tau = 0) &= \left[ \ddot{x} - (\rho_0 + x) \dot{\hat{\theta}}^2 \right] \cdot \hat{\rho} + \left[ 2\dot{x}\dot{\hat{\theta}} + (\rho_0 + x) \ddot{\hat{\theta}} \right] \cdot \hat{\theta} + \ddot{y} \cdot \hat{z} \\
\Leftrightarrow \ddot{\vec{R}}(x, y, s | \tau = 0) &\equiv a_x \cdot \hat{\rho} + a_s \cdot \hat{\theta} + a_y \cdot \hat{z}
\end{aligned} \tag{16}$$

## 2.2 Equation of motion

The trajectory of a particle with velocity  $\vec{v}$  is determined by the Lorentz force according to Equation (17).

$$\begin{aligned}
\frac{d\vec{p}}{dt} &\stackrel{!}{=} q \cdot \vec{v} \times \vec{B} \\
\Leftrightarrow \frac{d}{dt} \left[ \gamma m \dot{\vec{R}}(x, y, \tau) \right] &= q \cdot \begin{vmatrix} \hat{\rho} & \hat{z} & \hat{\theta} \\ v_x & v_y & v_s \\ B_x & B_y & 0 \end{vmatrix} \\
\Leftrightarrow \gamma m \ddot{\vec{R}}(x, y, \tau) &= -qv_s B_y \cdot \hat{\rho} + qv_s B_x \cdot \hat{z} + q(v_x B_y - v_y B_x) \cdot \hat{\theta} \\
\Leftrightarrow \vec{0} &\stackrel{(16)}{=} \begin{pmatrix} \gamma m \left[ \ddot{x} - (\rho_0 + x) \dot{\theta}^2 \right] + qv_s B_y \\ \gamma m \ddot{y} - qv_s B_x \\ \gamma m \left[ 2\dot{x}\dot{\theta} + (\rho_0 + x) \ddot{\theta} \right] - q[v_x B_y - v_y B_x] \end{pmatrix}
\end{aligned} \tag{17}$$

At this point, the longitudinal equation of motion will be discarded. Note, that the equation of motions in the horizontal and vertical plane are not symmetric due to the weak focusing of the dipoles. It is desirable to transform the second derivative in time  $t$  into a second derivative with respect to the path length  $s$ , since  $s$  is the independent variable which determines the order of the lattice. For performing the transformation, it is of paramount importance to reconsider the origin of the Frenet-Serret coordinate system. The longitudinal velocity  $v_s$  of an off-design orbit particle needs to be scaled-back to the reference orbit, since the origin is defined to coincide with the reference particle. Hence, the path length  $s$  is inherently connected to the reference orbit. The rescaling rule for the velocity  $v_s$  of an off-design particle is presented in Equation (18).

$$\begin{aligned}
\frac{ds}{dt} &\equiv \rho_0 \dot{\theta} \hat{=} v_s(x=0) \quad , \quad \text{with: } v_s(x) \stackrel{(16)}{=} (\rho_0 + x) \dot{\theta} \\
\Rightarrow \frac{ds}{dt} &= \frac{\rho_0}{\rho_0 + x} \cdot v_s(x)
\end{aligned} \tag{18}$$

The resulting transformation rule, from time  $t$  to path length  $s$ , is derived in Expression (19). At this stage, it is assumed that all differentials are exact differentials so that the commutator of different derivatives vanishes.

$$\begin{aligned}
\dot{x} &= \frac{dx}{dt} = \frac{dx}{ds} \cdot \frac{ds}{dt} \stackrel{(18)}{=} \frac{\rho_0 v_s}{\rho_0 + x} \cdot x' \\
\ddot{x} &= \frac{d^2 x}{dt^2} = \frac{d}{dt} \left[ \frac{\rho_0 v_s}{\rho_0 + x} \cdot x' \right] \stackrel{(18)}{=} \left[ \frac{\rho_0 v_s}{\rho_0 + x} \right] \cdot \frac{d}{dt} [x'] \quad , \quad \text{with: } \frac{d}{dt} \frac{ds}{dt} = 0 \\
\Leftrightarrow \ddot{x} &= \left[ \frac{\rho_0 v_s}{\rho_0 + x} \right] \cdot \frac{d}{ds} [x'] \frac{ds}{dt} = \left[ \frac{\rho_0 v_s}{\rho_0 + x} \right]^2 \cdot x'' \\
\Rightarrow \ddot{y} &= \left[ \frac{\rho_0 v_s}{\rho_0 + x} \right]^2 \cdot y''
\end{aligned} \tag{19}$$

In the second step of derivation (19), the fact was used, that the second time derivative of the path length  $s$  with respect to time vanishes. This holds only if there are no dissipating forces and no cavity units in the accelerator. Hence, the transformation implicitly assumes no longitudinal forces, which is in contradiction to the equation of motion (17) for the longitudinal direction. Applying the transforma-



tion (19) to the equation of motion (17) for the horizontal and vertical direction results in Expression (20), which is now a function of the independent variable  $s$ .

| horizontal                                                                                                                     | vertical                                                                                                |
|--------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------|
| $0 = \gamma m \left[ \left( \frac{\rho_0 v_s}{\rho_0 + x} \right)^2 \cdot x'' - \frac{v_s^2}{\rho_0 + x} \right] + q v_s B_y$  | $0 = \gamma m \left( \frac{\rho_0 v_s}{\rho_0 + x} \right)^2 \cdot y'' - q v_s B_x$                     |
| $\Leftrightarrow 0 = \left( \frac{\rho_0}{\rho_0 + x} \right)^2 \cdot x'' - \frac{1}{\rho_0 + x} + \frac{q B_y}{\gamma m v_s}$ | $\Leftrightarrow 0 = \left( \frac{\rho_0}{\rho_0 + x} \right)^2 \cdot y'' - \frac{q B_x}{\gamma m v_s}$ |

(20)

In general, the horizontal displacement  $x$  is a perturbation from the reference trajectory. Since the perturbation is assumed to be small, a Taylor expansion of the non-polynomial parts in the equation of motion (20) is expedient. The corresponding Taylor expansion of the rational functions is presented in Equation (21).

$$\begin{aligned} \frac{1}{\rho_0 + x} &= \frac{1}{\rho_0} - \frac{x}{\rho_0^2} + \mathcal{O} \left[ \left( \frac{x}{\rho_0} \right)^2 \right] \\ \left( \frac{\rho_0}{\rho_0 + x} \right)^2 &= 1 - \frac{2x}{\rho_0} + \mathcal{O} \left[ \left( \frac{x}{\rho_0} \right)^2 \right] \end{aligned} \quad (21)$$

Approximating the equation of motion (20) by using the Taylor expansions of Equation (21) yields the final version for the equation of motion presented in Expression (22). Note, only first order terms in  $x$  and  $y$  are considered to contribute significantly.

| horizontal                                                                                                                                     | vertical                                                                                |
|------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------|
| $0 \approx \left( 1 - \frac{2x}{\rho_0} \right) \cdot x'' - \left( \frac{1}{\rho_0} - \frac{x}{\rho_0^2} \right) + \frac{q B_y}{\gamma m v_s}$ | $0 \approx \left( 1 - \frac{2x}{\rho_0} \right) \cdot y'' - \frac{q B_x}{\gamma m v_s}$ |
| $\Rightarrow 0 \stackrel{(11)}{\approx} x'' + \frac{x}{\rho_0^2} + \frac{q}{p} (B_y - B_0) \quad , \quad \text{with: } p \approx p_s$          | $\Rightarrow 0 \approx y'' - \frac{q}{p} B_x \quad , \quad \text{with: } p \approx p_s$ |

(22)

In the second step of Expression (22) the total momentum  $p$  of the particle was approximated by the longitudinal momentum  $p_s$ . This is valid, since the amplitude of the perturbations are small in comparison to the synchrotron motion of ultra-relativistic particles. Note, the magnetic field components in Equation (22) are intrinsically normalised by the rigidity. Furthermore, the field components can be arbitrary functions, where  $B_0$  denotes the guiding field component of the bending dipoles.

### 2.3 Equation of motion restricted to linear orders

Magnetic fields of linear order are composed by dipole and quadrupole components. The Taylor expansion of the magnetic flux density up to the first order was presented in Equation (14). Restricting the equation of motion (22) to linear order terms results in Expression (23).

| horizontal                                                                                                                           | vertical                                                                                     |
|--------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------|
| $0 = x'' + \frac{x}{\rho_0^2} + g \cdot x \quad , \text{ with: } \frac{q}{p} B_y(x) \stackrel{(14)}{=} \frac{1}{\rho_0} + g \cdot x$ | $0 = y'' - g \cdot y \quad , \text{ with: } \frac{q}{p} B_x(y) \stackrel{(14)}{=} g \cdot y$ |
| $\Leftrightarrow 0 = \frac{\partial^2 x}{\partial s^2} + \left( \frac{1}{\rho_0^2} + g \right) \cdot x$                              | $\Leftrightarrow 0 \equiv \frac{\partial^2 y}{\partial s^2} - k_y(s) \cdot y$                |
| $\Leftrightarrow 0 \equiv \frac{\partial^2 x}{\partial s^2} + k_x(s) \cdot x$                                                        | $(23)$                                                                                       |

It is important to emphasise that the field gradient  $g$  is a periodic function of the independent variable  $s$ . This periodicity is either the machine circumference or the length of a repeating magnet sequence. Due to the periodicity of  $g$ , the equation of transverse motion (23) can be interpreted as a harmonic oscillator with periodic spring constant  $k(s)$ . As a result, the equations of transverse motion (23) for linear order fields describe a focusing behaviour in one particular direction and a defocusing effect in the perpendicular direction. The bottom line of Expression (23) is commonly known as the homogeneous Hill's differential equation.

In typical applications of accelerator physics it holds that the squared inverse of the reference radius is very small compared to the absolute value of the magnetic field gradient ( $\rho^{-2} \ll |g|$ ). Hence, the gradient  $g$  determines the sign of  $k_x(s)$ . Furthermore, the focusing effect caused by the dipole component is called weak focusing and the focusing effect by the quadrupole component is known by implication as strong focusing. The magnetic field of a quadrupole magnet is characterized by the field gradient  $g$ . For  $k(s) > 0$  the solution of Hill's differential equation is given by trigonometric functions and for  $k(s) < 0$  the solution of (23) is given by hyperbolic functions. With this knowledge in mind, it is possible to summarize the physical impact of a quadrupole magnet on the particle beam as shown in the case study (24).

$$\left. \begin{array}{l}
 \text{For } g > 0 : \left\{ \begin{array}{ll} \text{horizontal:} & \text{focusing} \\ \text{vertical:} & \text{defocusing} \end{array} \right\} \\
 \text{For } g < 0 : \left\{ \begin{array}{ll} \text{horizontal:} & \text{defocusing} \\ \text{vertical:} & \text{focusing} \end{array} \right\}
 \end{array} \right\} \equiv \text{Quadrupole magnet} \quad (24)$$

Without loss of generality, the field gradient  $g$  can be chosen to be positive ( $g > 0$ ). The solution of Hill's differential Equation (23), for the horizontal and the vertical component, is composed by a superposition of orthogonal functions as presented in Expression (25). For solving Hill's differential equation it is assumed that  $k(s)$  is piecewise constant, which corresponds to the physical length of the single magnetic structure.

$$\begin{aligned}
 x(s) &= \alpha_1 \cdot \cos(\sqrt{k}s) + \alpha_2 \cdot \sin(\sqrt{k}s) \\
 y(s) &= \beta_1 \cdot \cosh(\sqrt{k}s) + \beta_2 \cdot \sinh(\sqrt{k}s)
 \end{aligned} \quad (25)$$

The initial conditions for Equation (25) are given by the initial state vector  $\vec{\eta}_i$  in phase space (26).

$$\vec{\eta}_i = \left( x_i, \quad x'_i, \quad y_i, \quad y'_i \right)^T, \quad \text{with: } x'_i \equiv \frac{\partial x_i}{\partial s} \quad (26)$$

The piecewise solution of Hill's equation, with the initial conditions of (26), are given by (27). Notice

that  $x'$  receives a negative kick (focusing) and  $y'$  experiences a positive kick (defocusing).

$$\left. \begin{aligned} x(s) &= x_i \cdot \cos(\sqrt{k}s) + \frac{x'_i}{\sqrt{k}} \cdot \sin(\sqrt{k}s) \\ x'(s) &= -x_i \sqrt{k} \cdot \sin(\sqrt{k}s) + x'_i \cdot \cos(\sqrt{k}s) \\ y(s) &= y_i \cdot \cosh(\sqrt{k}s) + \frac{y'_i}{\sqrt{k}} \cdot \sinh(\sqrt{k}s) \\ y'(s) &= y_i \sqrt{k} \cdot \sinh(\sqrt{k}s) + y'_i \cdot \cosh(\sqrt{k}s) \end{aligned} \right\} \Leftrightarrow \vec{\eta}_f = \mathcal{M}_{fi} \cdot \vec{\eta}_i \quad (27)$$

$$\begin{bmatrix} \cos(\sqrt{k}s) & \frac{1}{\sqrt{k}} \cdot \sin(\sqrt{k}s) & 0 & 0 \\ -\sqrt{k} \cdot \sin(\sqrt{k}s) & \cos(\sqrt{k}s) & 0 & 0 \\ 0 & 0 & \cosh(\sqrt{k}s) & \frac{1}{\sqrt{k}} \cdot \sinh(\sqrt{k}s) \\ 0 & 0 & \sqrt{k} \cdot \sinh(\sqrt{k}s) & \cosh(\sqrt{k}s) \end{bmatrix} \equiv \mathcal{M}_{fi}$$

The effective transfer matrix  $\mathcal{M}_{fi}$  covers all physical properties of a quadrupole magnet. It is important to emphasise, that the determinant of  $\mathcal{M}_{fi}$  is equal to unity for any arbitrary beamline [2]. The physical reason for that is Liouville's theorem of phase space volume conservation or equivalently the invariance of the action  $J$ . However, there are some cases where the general statement of  $|\mathcal{M}_{fi}| = 1$  does not hold. Consequently, there are additional accelerating or decelerating forces in play which the equation of motion (23) does not account for. This gives rise to a case differentiation as shown in expression (28).

$$|\mathcal{M}_{fi}| = \begin{cases} < 1 & , \text{Damping} \\ = 1 & , \text{negligible dissipating forces} \\ > 1 & , \text{Anti-Damping} \end{cases} \quad (28)$$

## 2.4 Effective magnet lengths

In reality, magnets tend to have fields which extend beyond the physical length  $L_{\text{mag}}$  of the magnet. In order to account for this phenomena an effective length  $L_{\text{eff}}$  is defined by Equation (29), which is the length of the magnetic elements as seen by the beam.

$$B(s=0) \cdot L_{\text{eff}} \equiv \int_{-L_{\text{mag}}/2}^{L_{\text{mag}}/2} B(s) \, ds \quad (29)$$

## 2.5 Drift space

A drift space is a region of the beam line characterised by the absence of electromagnetic fields. The transformation rule for the phase space vector in a drift space can be derived from the quadrupole matrix (27) by considering the limit of  $k$  goes to zero. This corresponds to the force free Hill's equation. Drift space is parameterized by a certain length  $L$ . Hence, the quadrupole matrix in the limit of vanishing forces is evaluated at  $s \stackrel{!}{=} L$ . In order to obtain the limit of  $\mathcal{M}_{fi}$  when  $k$  goes to zero, it is necessary to Taylor expand the entries  $\mathcal{M}_{fi}(1,2)$  and  $\mathcal{M}_{fi}(3,4)$ . The Taylor expansions of

the corresponding entries are given by Expression (30).

$$\begin{aligned}\mathcal{M}_{fi}(1,2) &= \lim_{k \rightarrow 0} \frac{\sin(\sqrt{k} \cdot L)}{\sqrt{k}} = \lim_{k \rightarrow 0} [L + \mathcal{O}(k)] = L \\ \mathcal{M}_{fi}(3,4) &= \lim_{k \rightarrow 0} \frac{\sinh(\sqrt{k} \cdot L)}{\sqrt{k}} = \lim_{k \rightarrow 0} [L + \mathcal{O}(k)] = L\end{aligned}\tag{30}$$

As a result, the free drift matrix (31) can be written down explicitly by applying the Taylor expansions for the entries governed by rational functions.

$$\mathcal{M}_{\text{drift}}(L) = \begin{bmatrix} 1 & L & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L \\ 0 & 0 & 0 & 1 \end{bmatrix}\tag{31}$$

## 2.6 Thin lens approximation

If the length  $L$  of a quadrupole is short in comparison to the focal length  $f$  it is expedient to take the limit  $L \rightarrow 0$ , whilst keeping the focal length constant. In order to perform the thin lens approximation it is necessary to define the focal length properly. The focal length  $f$  is defined as the distance between the midpoint of the magnetic structure and the intersection point of the particle trajectory with the design orbit. The physical properties of the focal length are derived in Expression (32).

$$\begin{aligned}0 &\stackrel{!}{=} x_f + x'_f \cdot f \\ \Leftrightarrow 0 &\stackrel{(27)}{=} x_i \cos(\sqrt{k}L) - x_i \sqrt{k} \sin(\sqrt{k}L) \cdot f \quad , \quad \text{with: } x'_i \stackrel{!}{=} 0 \\ \Leftrightarrow f &= \frac{1}{\sqrt{k} \tan(\sqrt{k}L)} \stackrel{L \rightarrow 0}{\approx} \frac{1}{kL}\end{aligned}\tag{32}$$

Consequently, keeping the focal length  $f$  constant induces the restriction of  $kL \stackrel{!}{=} \text{const}$ . This additional constraint is fulfilled in the double limit of  $k \rightarrow \infty$  while  $L \rightarrow 0$ . Evaluating the double limit of the quadrupole matrix (27) results in Expression (33).

$$\begin{aligned}\mathcal{M}_{\text{thin}}|_{(x,x')} &\stackrel{(32)}{=} \lim_{k \rightarrow \infty} \lim_{L \rightarrow 0} \begin{bmatrix} \cos[(\sqrt{k}f)^{-1}] & \frac{1}{\sqrt{k}} \cdot \sin[(\sqrt{k}f)^{-1}] \\ -\sqrt{k} \cdot \sin[(\sqrt{k}f)^{-1}] & \cos[(\sqrt{k}f)^{-1}] \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -f^{-1} & 1 \end{bmatrix} \\ \mathcal{M}_{\text{thin}}|_{(y,y')} &\stackrel{(32)}{=} \lim_{k \rightarrow \infty} \lim_{L \rightarrow 0} \begin{bmatrix} \cosh[(\sqrt{k}f)^{-1}] & \frac{1}{\sqrt{k}} \cdot \sinh[(\sqrt{k}f)^{-1}] \\ \sqrt{k} \cdot \sinh[(\sqrt{k}f)^{-1}] & \cosh[(\sqrt{k}f)^{-1}] \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ f^{-1} & 1 \end{bmatrix} \\ \Rightarrow \mathcal{M}_{\text{thin}}(f) &= \begin{bmatrix} \mathcal{M}_{\text{thin}}|_{(x,x')} & \mathbf{0}_{2 \times 2} \\ \mathbf{0}_{2 \times 2} & \mathcal{M}_{\text{thin}}|_{(y,y')} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -f^{-1} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & f^{-1} & 1 \end{bmatrix}\end{aligned}\tag{33}$$

## 2.7 From piecewise elements to the one-turn matrix

In an accelerator there are a lot of elements in the beam line. Each element can be represented by a matrix. Multiplying all single matrices together results in an overall transfer matrix for the entire lattice. In summary, in the first step, Hill's differential equation was solved assuming piecewise constant magnetic properties  $k(s)$  and the equation has been linearized. In the second step, the matrix representation of the single elements is used to determine the one-turn effect of the lattice. Hence, the one-turn transfer matrix can be calculated according to Equation (34).

$$\mathcal{M}_{\text{turn}} = \prod_{i=1}^n \mathcal{M}_{i,(i-1)} = \mathcal{M}_{n,(n-1)} \cdot \prod_{i=2}^{n-1} [\mathcal{M}_{i,(i-1)}] \cdot \mathcal{M}_{1,0} \quad (34)$$

Here,  $n$  denotes the number of beam elements in the accelerator. The one-turn transfer matrix  $\mathcal{M}_{\text{turn}}$  reproduces the entire physics of the lattice. It is important to highlight that the matrices of the elements do not commute. Consequently, the order of the matrix product needs to be obeyed. Furthermore, in order to evaluate the particle motion it is required to know the fields and the magnet positions very accurate.

## 2.8 Dipole transfer matrix

Due to the weak focusing effect there is a non-trivial matrix representation for dipole magnets. If the quadrupolar field gradient  $g$  is set to zero in Hill's differential equation (23) sinusoidal solutions for the horizontal plane result due to the remaining  $\rho_0^{-2}$  term from the weak focusing of the dipole. In contrast, the vertical plane is characterised by  $k_y(s) = g = 0$  and acts as a drift space. Hence, the dipole matrix is composed by a horizontal component from the quadrupole matrix (27) with  $k_x(s) = \rho_0^{-2}$  and a vertical component from the drift space matrix (31). The resulting dipole matrix is presented in formula (35). If a full turn ( $s = 2\pi\rho_0$ ) is considered the dipole transfer matrix reduces to the identity matrix. It is important to emphasize, that the argument of the sinusoidal functions corresponds to the bending angle of the dipole. This results from the fact that  $s$  denotes the length of the beamline path within the dipole and not the overall longitudinal position in the ring.

$$\mathcal{M}_{fi} \equiv \begin{bmatrix} \cos\left(\frac{s}{\rho_0}\right) & \rho_0 \cdot \sin\left(\frac{s}{\rho_0}\right) & 0 & 0 \\ -\frac{1}{\rho_0} \cdot \sin\left(\frac{s}{\rho_0}\right) & \cos\left(\frac{s}{\rho_0}\right) & 0 & 0 \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (35)$$

## 2.9 FODO-cell structure

If two quadrupoles are placed sequentially in the beamline with opposite field gradient  $g$  the beam will be focused effectively in both planes at the same time. For proving this statement, the state vector needs to be propagated through the entire magnetic structure by using the piecewise solutions of Hill's equation. Assuming the thin lens approximation, the transfer matrix for the magnetic structure is given by the matrix product of the quadrupole transfer matrix  $\mathcal{M}_1$  (33) with the drift space matrix  $\mathcal{M}_{\text{drift}}$  (31) and the quadrupole transfer matrix  $\mathcal{M}_2$  (33). The result is given by Equation (36). Here,

$L$  denotes the length of the drift section and  $f_1$  labels the focal length of the first quadrupole and similarly  $f_2$  assigns the focal length to the final quadrupole.

$$\begin{aligned} \mathcal{M}_{\text{FODO}} &\equiv \mathcal{M}_2 \cdot \mathcal{M}_{\text{drift}} \cdot \mathcal{M}_1 \\ \Leftrightarrow \mathcal{M}_{\text{FODO}} &= \begin{bmatrix} 1 - \frac{L}{f_1} & L & 0 & 0 \\ -\frac{1}{f_2} - \frac{1}{f_1} + \frac{L}{f_1 f_2} & 1 - \frac{L}{f_2} & 0 & 0 \\ 0 & 0 & 1 + \frac{L}{f_1} & L \\ 0 & 0 & \frac{1}{f_2} + \frac{1}{f_1} + \frac{L}{f_1 f_2} & 1 + \frac{L}{f_2} \end{bmatrix} \end{aligned} \quad (36)$$

$$\begin{aligned} \text{with: } f_1 &\stackrel{!}{=} -f_2 \equiv f \Rightarrow \mathcal{M}_{\text{FODO}}[2, 1] = \mathcal{M}_{\text{FODO}}[4, 3] = -\frac{L}{f^2} \hat{=} -\frac{1}{f_{\text{FODO}}} \\ &\Rightarrow f_{\text{FODO}} > 0 \quad \blacksquare \end{aligned}$$

## 2.10 Mapping

The underlying concept of propagating an initial state vector through a magnetic structure in the beamline by applying a transfer matrix is called mapping. In general, the transfer matrix  $\mathcal{M}$  is a map that transforms an initial state vector to a final state vector. Mathematically, the concept is illustrated by Equation (37). For the linear case, the map can be represented as a matrix. In contrast, for non-linear systems, matrices do not work any more and new representations of the maps need to be found (for example Taylor maps or Lie maps). In the linear case, the associative nature of matrix algebra can be exploited in order to define the one-turn matrix. The one-turn matrix evolves the state vector in revolutions of the ring.

$$\vec{\eta}(s_f) = \mathcal{M}(s_f | s_i) \cdot \vec{\eta}(s_i) \quad (37)$$

### 3 Lattice functions

The mathematical structure of Hill's Equation (23) is similar to the equation of motion of a harmonic oscillator, except from the restoring force which depends on the position itself. Hence, a quasi-harmonic oscillation, where the frequency and the amplitude depend on the location in the ring and show periodicity similar to that of  $k(s)$ , is expected.

#### 3.1 The Courant-Snyder formalism

The proposed Ansatz (38) of a quasi-harmonic oscillation as a solution of Hill's Equation (23) has been elaborated by Courant and Snyder [7].

$$x(s) = \sqrt{2J \cdot \beta(s)} \cdot \cos(\psi(s) + \phi_0) \quad (38)$$

In the Courant-Snyder formalism the beta function  $\beta(s)$  has the physical meaning of an amplitude which depends on the position around the accelerator.  $\psi(s)$  is a position dependent phase. The action  $J$  and the initial phase  $\phi_0$  are constants specifying the two degrees of freedom of a second order differential equation. Since the transversal deviation  $x(s)$  from the reference trajectory is a real quantity, the beta function and the action are positive definite. The beta function is directly intertwined with the focusing properties of the lattice. A small beta function indicates a highly focused beam and vice versa. The periodicity of the beta function is subject to the periodicity of the lattice. In order to proof that the Courant-Snyder Ansatz (38) is a solution to Hill's differential equation (23), it is necessary to substitute the Ansatz into the equation of motion as elaborated in Expression (39).

$$\begin{aligned}
x'(s) &= \frac{1}{2} \sqrt{\frac{2J}{\beta(s)}} \cdot \beta'(s) \cdot \cos(\psi(s) + \phi_0) - \sqrt{2J\beta(s)} \cdot \sin(\psi(s) + \phi_0) \cdot \psi'(s) \\
x''(s) &= -\frac{1}{4} \sqrt{\frac{2J}{\beta^3(s)}} \cdot [\beta'(s)]^2 \cdot \cos(\psi(s) + \phi_0) + \frac{1}{2} \sqrt{\frac{2J}{\beta(s)}} \cdot \beta''(s) \cdot \cos(\psi(s) + \phi_0) \\
&\quad - \frac{1}{2} \sqrt{\frac{2J}{\beta(s)}} \cdot \beta'(s) \cdot \sin(\psi(s) + \phi_0) \cdot \psi'(s) - \frac{1}{2} \sqrt{\frac{2J}{\beta(s)}} \cdot \beta'(s) \cdot \sin(\psi(s) + \phi_0) \cdot \psi'(s) \\
&\quad - \sqrt{2J\beta(s)} \cdot \cos(\psi(s) + \phi_0) \cdot [\psi'(s)]^2 - \sqrt{2J\beta(s)} \cdot \sin(\psi(s) + \phi_0) \cdot \psi''(s) \\
0 &\stackrel{(23)}{=} x'' + k(s) \cdot x \\
&\Leftrightarrow 0 = \sqrt{2J} \cdot \left[ \begin{array}{c} -\frac{\beta'(s) \cdot \psi'(s)}{\sqrt{\beta(s)}} - \sqrt{\beta(s)} \cdot \psi''(s) \\ -\frac{[\beta'(s)]^2}{4\sqrt{\beta^3(s)}} + \frac{\beta''(s)}{2\sqrt{\beta(s)}} - \sqrt{\beta(s)} \cdot [\psi'(s)]^2 + \sqrt{\beta(s)} \cdot k(s) \end{array} \right] \cdot \left[ \begin{array}{c} \sin(\psi(s) + \phi_0) \\ \cos(\psi(s) + \phi_0) \end{array} \right] \\
&\Rightarrow \vec{0} = \left[ \begin{array}{c} \beta'(s) \cdot \psi'(s) + \beta(s) \cdot \psi''(s) \\ \frac{1}{2} \left( \beta(s) \cdot \beta''(s) - \frac{1}{2} [\beta'(s)]^2 \right) - \beta^2(s) \cdot [\psi'(s)]^2 + \beta^2(s) \cdot k(s) \end{array} \right] \\
&\Rightarrow 0 = [\beta(s) \cdot \psi'(s)]' \quad \mapsto \quad \beta(s) \cdot \psi'(s) \stackrel{!}{=} 1
\end{aligned} \quad (39)$$

In (39), there are two orthogonal components which need to vanish separately in order to satisfy Hill's equation. Hence, two differential equations are obtained for the coefficients of the trigonometric functions. The first differential equation for the beta function can be integrated directly. Here, the upcoming degree of freedom in terms of a integration constant is set to one. This is an arbitrary choice of the initial condition. The phase function  $\psi(s)$  can be immediately computed by integrating over the beta function along the beamline. Therefore, the phase as a function of position is given by Equation (40) and is known as the betatron phase advance.

$$\psi(s) = \int_0^s \frac{d\xi}{\beta(\xi)} \quad (40)$$

If the phase function is substituted into the second last line of (39) a differential equation for the beta function results (41), which only depends on the lattice structure  $k(s)$ .

$$\frac{1}{2} \left( \beta(s) \cdot \beta''(s) - \frac{1}{2} [\beta'(s)]^2 \right) + \beta^2(s) \cdot k(s) = 1 \quad (41)$$

Consequently, knowing the distribution of focusing strengths along the beamline determines the beta function. Since Equation (41) is a second order, non-linear, ordinary differential equation it is not possible to solve this equation analytically.

### 3.2 The phase space ellipse

It is reasonable to aim for an equation for the state vector in phase space. In order to obtain such an equation it is necessary to combine  $x(s)$  with  $x'(s)$ . This can be done by trying to eliminate the trigonometric dependencies. Combining Expression (38) with (39) by exploiting the Relation (40) leads to Equation (42).

$$\begin{aligned} x'(s) &\stackrel{(40)}{=} \sqrt{\frac{2J}{\beta(s)}} \cdot \left[ \frac{\beta'(s)}{2} \cdot \cos(\psi(s) + \phi_0) - \sin(\psi(s) + \phi_0) \right] \\ \Leftrightarrow x'(s) &\stackrel{(38)}{=} \sqrt{\frac{2J}{\beta(s)}} \cdot \left[ \frac{\beta'(s)}{2} \cdot \frac{x(s)}{\sqrt{2J\beta(s)}} - \sin(\psi(s) + \phi_0) \right] \\ \Leftrightarrow \sin(\psi(s) + \phi_0) &= \frac{\beta'(s)}{2\sqrt{2J\beta(s)}} \cdot x(s) - \sqrt{\frac{\beta(s)}{2J}} \cdot x'(s) \\ \text{with: } 1 &= \sin^2(\psi(s) + \phi_0) + \cos^2(\psi(s) + \phi_0)^2 \\ \Leftrightarrow 1 &\stackrel{(38)}{=} \left[ \frac{\beta'(s)}{2\sqrt{2J\beta(s)}} \cdot x(s) - \sqrt{\frac{\beta(s)}{2J}} \cdot x'(s) \right]^2 + \left[ \frac{x(s)}{\sqrt{2J\beta(s)}} \right]^2 \\ \Leftrightarrow 2J &= \left[ \frac{4 + [\beta'(s)]^2}{4\beta(s)} \right] \cdot x^2(s) - \beta'(s) \cdot x(s) \cdot x'(s) + \beta(s) \cdot [x'(s)]^2 \\ \Leftrightarrow 2J &\stackrel{!}{=} \gamma(s) \cdot x^2(s) + 2\alpha(s) \cdot x(s) \cdot x'(s) + \beta(s) \cdot [x'(s)]^2 \\ \Rightarrow \alpha(s) &= -\frac{\beta'}{2} \quad , \quad \gamma(s) = \frac{4 + [\beta'(s)]^2}{4\beta(s)} \hat{=} \frac{1 + \alpha^2(s)}{\beta(s)} \end{aligned} \quad (42)$$

As a result, the phase space is characterised by three lattice functions  $\alpha(s)$ ,  $\beta(s)$  and  $\gamma(s)$  as well as the



action  $J$ . The phase space Equation (42) represents an ellipse with an area equal to  $2\pi J$ . Consequently, the area of the ellipse is conserved since the action  $J$  is an invariant quantity. This is expected in the light of Liouville's theorem. The magnetic properties of the lattice and the initial conditions for the action  $J$  and the initial phase  $\phi_0$  fully determine the single particle trajectory.

In a nutshell, the turn-by-turn particle motion in phase space follows an ellipse and each point of the ellipse is a possible manifestation for the state vector  $\vec{\eta}$  at position  $s$ . As illustrated in Figure (3), the beta function is related to the off-design position  $x$  and the gamma function indicates the off-design momentum  $x'$ . The alpha function scales the cross-term in the ellipse equation and quantifies the skewness of the ellipse in phase space. The orientation of the ellipse will rotate as a function of the longitudinal beamline position  $s$  because the alpha function, as well as the beta and gamma function, depends on the local focusing properties of the lattice.

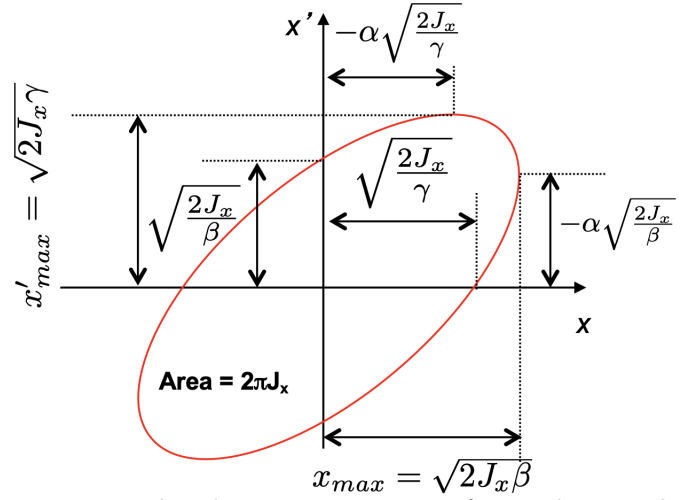


Figure 3: The phase space motion of a single particle is indicated. The single particle motion follows an ellipse which is characterised by the lattice functions  $\alpha(s)$ ,  $\beta(s)$  and  $\gamma(s)$  as well as the action  $J$  [22].

### 3.3 Transfer matrix in terms of the Courant-Snyder functions

The main objective of this section is to find a mapping of the phase space state vector without using the magnetic structure function  $k(s)$  explicitly. This is in contrast to the quadrupole transfer matrix (27), where Hill's differential equation has been exploited for piecewise constant  $k(s)$ . In the Courant-Snyder formalism all magnetic lattice properties are covered by the lattice functions. The Courant-Snyder Ansatz (38) can be expressed in terms of two orthogonal solutions (43) by absorbing the initial conditions  $J$  and  $\phi_0$ .

$$x(s) = a_1 \sqrt{\beta(s)} \cdot \cos \psi(s) + a_2 \sqrt{\beta(s)} \cdot \sin \psi(s) \quad (43)$$

The coefficients  $a_1$ ,  $a_2$  of the trigonometric functions in Equation (43) can be determined by specifying the initial values of the lattice functions for a certain position. Without loss of generality, the phase advance  $\psi$  evaluated at  $s = 0$  is set to zero and the lattice functions are given by  $\alpha(0) = \alpha_0$ ,  $\beta(0) = \beta_0$  and  $\gamma(0) = \gamma_0$ . As a consequence, the Courant-Snyder Ansatz can be rewritten in terms of the initial values for the lattice functions (44).

$$x(s) = \sqrt{\frac{\beta(s)}{\beta_0}} \cdot [\cos \psi(s) + \alpha_0 \sin \psi(s)] \cdot x_0 + \sqrt{\beta_0 \beta(s)} \cdot \sin(\psi(s)) \cdot x'_0 \quad (44)$$

Out of academic interest, the derivative of Expression (44) is evaluated explicitly and is given by Equation (45).

$$\begin{aligned} x'(s) &= \sqrt{\beta_0 \beta(s)}^{-1} \cdot [(\alpha_0 - \alpha(s)) \cdot \cos \psi(s) - (1 + \alpha_0 \alpha(s)) \cdot \sin \psi(s)] \cdot x_0 \\ &+ \sqrt{\frac{\beta_0}{\beta(s)}} \cdot [\cos \psi(s) - \alpha(s) \cdot \sin \psi(s)] \cdot x'_0 \end{aligned} \quad (45)$$

Equation (44) and (45) can be cast into a convenient matrix form as the equations are linear in terms of  $x_0$  and  $x'_0$ . As a consequence, the transfer matrix  $\mathcal{M}_{fi}$  (46) between two locations in the beamline is purely determined by the lattice functions at each location and the phase advance  $\psi$  between them.

$$\vec{\eta}(s_f) = \mathcal{M}(s_f|s_i = 0) \cdot \vec{\eta}(s_i = 0) \quad , \quad \text{with:}$$

$$\mathcal{M}_{fi} = \begin{bmatrix} \sqrt{\frac{\beta(s)}{\beta_0}} \cdot [\cos \psi(s) + \alpha_0 \sin \psi(s)] & \sqrt{\beta_0 \beta(s)} \cdot \sin \psi(s) \\ \left[ \frac{\alpha_0 - \alpha(s)}{\sqrt{\beta_0 \beta(s)}} \cdot \cos \psi(s) - \frac{1 + \alpha_0 \alpha(s)}{\sqrt{\beta_0 \beta(s)}} \cdot \sin \psi(s) \right] & \sqrt{\frac{\beta_0}{\beta(s)}} \cdot [\cos \psi(s) - \alpha(s) \cdot \sin \psi(s)] \end{bmatrix} \quad (46)$$

The Courant-Snyder Ansatz for the transfer matrix (46) can be exploited to redefine the one-turn map in terms of the lattice functions. The one-turn map evolves the phase space vector in revolutions. Since the lattice functions have the same periodicity as the magnetic structure of the accelerator, the lattice functions will be mapped to the same value after one revolution. Consequently, it holds that:  $\alpha(L) = \alpha_0$ ,  $\beta(L) = \beta_0$  and  $\gamma(L) = \gamma_0$ . Hence, the one-turn map in terms of the lattice functions can be simplified as presented in Equation (47).

$$\vec{\eta}(s_i + L) = \mathcal{M}(s_i + L|s_i) \cdot \vec{\eta}(s_i) \quad , \quad \text{with:}$$

$$\mathcal{M}(s + L|s) = \begin{bmatrix} \cos \psi(s) + \alpha(s) \cdot \sin \psi(s) & \beta(s) \cdot \sin \psi(s) \\ -\gamma(s) \cdot \sin \psi(s) & \cos \psi(s) - \alpha(s) \cdot \sin \psi(s) \end{bmatrix} \quad (47)$$

It is important to emphasize that the determinant of  $\mathcal{M}(s_i + L|s_i)$  has to be equal to one in order to satisfy Liouville's theorem. Hence, the one-turn matrix has three degrees of freedom but is parameterized by four functions. Nonetheless, overdeterminacy is avoided since the lattice functions are not orthogonal to each other. As a result, the lattice functions carry two degrees of freedom. The one-turn map is particularly useful in order to determine the lattice functions. On the one hand, the one-turn map  $\mathcal{M}_1$  can be determined if all transfer matrices of the lattice elements are multiplied together according to Equation (34). On the other hand, the one-turn map  $\mathcal{M}_2$  is characterised by the lattice functions according to Equation (47).

$$\begin{aligned} \mathcal{M}_1(s_i + L|s_i) &\stackrel{!}{=} \mathcal{M}_2(s_i + L|s_i) \\ \Leftrightarrow \begin{bmatrix} m_{11}(s) & m_{12}(s) \\ m_{21}(s) & m_{22}(s) \end{bmatrix} &\stackrel{!}{=} \begin{bmatrix} \cos \psi(s) + \alpha(s) \cdot \sin \psi(s) & \beta(s) \cdot \sin \psi(s) \\ -\gamma(s) \cdot \sin \psi(s) & \cos \psi(s) - \alpha(s) \cdot \sin \psi(s) \end{bmatrix} \end{aligned} \quad (48)$$

Hence, by comparing the elements of the two matrices (48) it is possible to explicitly determine the lattice functions at some beamline position  $s$  (49).

$$\begin{aligned} \begin{pmatrix} \alpha(s) \\ \beta(s) \\ \gamma(s) \end{pmatrix} &= \left( \frac{m_{11}(s) - m_{22}(s)}{2 \sin \psi_0}, \frac{m_{12}(s)}{\sin \psi_0}, -\frac{m_{21}(s)}{\sin \psi_0} \right)^T \\ \psi_0 &= \arccos \left( \frac{m_{11}(s) + m_{22}(s)}{2} \right) \end{aligned} \quad (49)$$

In general, the determined values of the lattice functions correspond to the starting position  $s_i$  of the

one-turn map. The phase advance equation in (49) imposes a restriction on the trace of the one-turn matrix  $\mathcal{M}_1(s_i + L|s_i)$ . In order to restrict the phase advance to real values, the trace needs to fulfill the constraint:  $|\text{Tr}[\mathcal{M}_1]| \leq 2$ . Real phase values are mandatory for the purpose of bounded solutions for Hill's equation. Consequently, the constraint on the trace of the one-turn map  $\mathcal{M}_1$  is a key requirement for stable particle motion.

### 3.4 Mapping of the Courant-Snyder functions

The goal of this subsection is to find a mapping for the lattice functions. In order to succeed, the one-turn map (47) at a different location is considered. Imagine that the mapping  $\mathcal{M}_{21}(s_2|s_1)$  of the state vector  $\vec{\eta}$  from  $s_1$  to  $s_2$  has already been determined. For that reason, it is required to know the magnetic structure  $k(s)$  of the lattice segment. The one-turn map at position  $s_2$  results from a similarity transformation of the one-turn map at position  $s_1$  by applying the transfer matrix  $\mathcal{M}_{21}(s_2|s_1)$ . The mapping representation of the similarity transformation is given by Equation (50).

$$\begin{aligned} \mathcal{M}(s_2 + L|s_2) &= \mathcal{M}_{21}(s_2|s_1) \cdot \mathcal{M}(s_1 + L|s_1) \cdot \mathcal{M}_{12}(s_1|s_2) \\ \Leftrightarrow \mathcal{M}(s_2 + L|s_2) &= \mathcal{M}_{21}(s_2|s_1) \cdot \mathcal{M}(s_1 + L|s_1) \cdot \mathcal{M}_{21}^{-1}(s_2|s_1) \end{aligned} \quad (50)$$

The one-turn map at position  $s_2$  can be calculated explicitly, given an arbitrary transfer matrix  $\mathcal{M}_{21}(s_2|s_1)$  of the form of Equation (51).

$$\mathcal{M}_{21}(s_2|s_1) \equiv \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \quad (51)$$

In order to execute the calculation of Equation (50) it is required to determine the inverse of the transfer matrix  $\mathcal{M}_{21}^{-1}(s_2|s_1)$  (52).

$$\begin{aligned} \mathcal{M}_{21}(s_2|s_1) \cdot \mathcal{M}_{12}(s_1|s_2) &\stackrel{!}{=} 1 \\ \Leftrightarrow \mathcal{M}_{21}(s_2|s_1) \cdot \mathcal{M}_{21}^{-1}(s_2|s_1) &\stackrel{!}{=} 1 \\ \Rightarrow \mathcal{M}_{21}^{-1}(s_2|s_1) &= \frac{1}{m_{11}m_{22} - m_{12}m_{21}} \cdot \begin{bmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{bmatrix} \end{aligned} \quad (52)$$

If the fact is considered that the determinant of the transfer matrix is equal to one, in order to satisfy Liouville's theorem, the inverse transfer matrix simplifies to Equation (53).

$$\mathcal{M}_{21}^{-1}(s_2|s_1) = \begin{bmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{bmatrix} \quad (53)$$

As a consequence, the matrix products of Equation (50) can be worked out in detail. The algebraic result is presented in Equation (54).

$$\begin{aligned}
\begin{bmatrix} \textcircled{1} & \textcircled{2} \\ \textcircled{3} & \textcircled{4} \end{bmatrix} &\hat{=} \mathcal{M}(s_2 + L|s_2) = \mathcal{M}_{21}(s_2|s_1) \cdot \mathcal{M}(s_1 + L|s_1) \cdot \mathcal{M}_{21}^{-1}(s_2|s_1) \\
\textcircled{1} : \cos \psi_2 + \alpha_2 \cdot \sin \psi_2 &\stackrel{!}{=} \cos \psi_1 + \sin \psi_1 \cdot [\alpha_1 (m_{12}m_{21} + m_{11}m_{22}) - m_{11}m_{21}\beta_1 - m_{12}m_{22}\gamma_1] \\
\textcircled{2} : \beta_2 \cdot \sin \psi_2 &\stackrel{!}{=} \sin \psi_1 \cdot [m_{11}^2\beta_1 + m_{12}^2\gamma_1 - 2m_{11}m_{12}\alpha_1] \\
\textcircled{3} : -\gamma_2 \cdot \sin \psi_2 &\stackrel{!}{=} \sin \psi_1 \cdot [2m_{21}m_{22}\alpha_1 - m_{21}^2\beta_1 - m_{22}^2\gamma_1] \\
\textcircled{4} : \cos \psi_2 - \alpha_2 \cdot \sin \psi_2 &\stackrel{!}{=} \cos \psi_1 + \sin \psi_1 \cdot [m_{11}m_{21}\beta_1 + m_{12}m_{22}\gamma_1 - \alpha_1 (m_{12}m_{21} + m_{11}m_{22})] \quad (54) \\
\Rightarrow \psi_2 = \psi_1 &\hat{=} \textcircled{5} \quad \text{using: } \arccos\left(\frac{1}{2}[\textcircled{1} + \textcircled{4}]\right) \\
\Rightarrow \alpha_2 = (m_{12}m_{21} + m_{11}m_{22}) \cdot \alpha_1 - m_{11}m_{21}\beta_1 - m_{12}m_{22}\gamma_1 &\quad \text{using: } \textcircled{5} \circ \frac{1}{2}[\textcircled{1} - \textcircled{4}] \\
\Rightarrow \beta_2 = m_{11}^2\beta_1 + m_{12}^2\gamma_1 - 2m_{11}m_{12}\alpha_1 &\quad \text{using: } \textcircled{5} \circ \textcircled{2} \\
\Rightarrow \gamma_2 = m_{21}^2\beta_1 + m_{22}^2\gamma_1 - 2m_{21}m_{22}\alpha_1 &\quad \text{using: } \textcircled{5} \circ [-\textcircled{3}]
\end{aligned}$$

It is important to emphasize the result, that the one-turn phase advance is position invariant. This seems to be reasonable considering the one-turn effect of the lattice structure. The evolution rules for  $\alpha(s)$ ,  $\beta(s)$  and  $\gamma(s)$  from Equation (54) can be cast into a transfer matrix for the Courant-Snyder functions. The matrix representation for mapping the lattice functions is given by Equation (55).

$$\begin{bmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} (m_{12}m_{21} + m_{11}m_{22}) & -m_{11}m_{21} & -m_{12}m_{22} \\ -2m_{11}m_{12} & m_{11}^2 & m_{12}^2 \\ -2m_{21}m_{22} & m_{21}^2 & m_{22}^2 \end{bmatrix} \cdot \begin{bmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{bmatrix} \quad (55)$$

### 3.5 Relation between phase advance and natural tune

The natural tune  $Q$  is a characteristic quantity of the betatron oscillations. The natural tune describes the phase advance over one turn in the accelerator in radian units of  $2\pi$ . The physical concept of the natural tune has been visualized in Figure (1) and can be calculated by the phase advance of the Courant-Snyder formalism as presented in Equation (56).

$$Q \equiv \frac{\psi(s+L) - \psi(s)}{2\pi} \stackrel{(40)}{=} \frac{1}{2\pi} \cdot \int_s^{s+L} \frac{d\xi}{\beta(\xi)} \quad (56)$$

Since the betatron oscillations occur in all three planes of the Frenet-Serret coordinate system, there is one tune value for each plane (horizontal, vertical and longitudinal tune). It is important to highlight, that the natural tune is beam position invariant since the overall one-turn impact of the lattice remains constant. Equation (54) underpins this fact explicitly and shows that the natural tune is an accelerator quantity.

## 4 Optics and lattice design

In this section it is the goal to introduce the principle ideas of lattice design and to provide illustrative examples. In order to standardize the results for the lattice quantities, the software tool MAD-X developed by CERN will be utilised for comparison.

### 4.1 Symmetry points

The concepts of symmetry points is closely related to the Courant-Snyder formalism. Reconsider the FODO-cell transfer matrix which has been derived in Equation (36). The general idea was to propagate the phase space vector through the cell by applying the individual transfer matrices of the elements. The starting point has been chosen to be in front of the focusing quadrupole but this was an arbitrary choice rather than an explicit requirement of the FODO-cell structure. Alternatively, one could start in the middle of the cell (middle of a quadrupole magnet) and derive the transfer matrix for the FODO-cell similarly. In this way, a symmetry point has been introduced implicitly. As a consequence of the symmetry point in the middle of the lattice segment, the alpha function vanishes ( $\alpha = 0$ ). Therefore, the phase space ellipse at the FODO-cell center does not show a skewness anymore. This symmetry property is independent of the physical properties of the elements, such as the drift space length or the quadrupole strength. In a nutshell, if the transfer matrix of a lattice segment can be symmetrized then the value of the alpha function in the middle of the segment will vanish. Since the alpha function is inherently linked to the derivative of the beta function with respect to the beamline coordinate  $s$ , a vanishing alpha value implies an extremum of the beta function.

In order to demonstrate the idea of a symmetry point, the beam dynamics of a FODO-cell sequence is revisited. In Expression (36) it was proven that an alternating structure of focusing and defocusing quadrupoles focuses the beam in both planes effectively. If a symmetry point is implemented in the middle of the focusing quadrupole the one-period map of the FODO-cell is given by Equation (57).

$$\mathcal{M}_{\text{FODO}}(s_i + L|s_i) = \mathcal{M}_{\text{FQ}}(2f) \cdot \mathcal{M}_{\text{drift}} \cdot \mathcal{M}_{\text{DQ}}(-f) \cdot \mathcal{M}_{\text{drift}} \cdot \mathcal{M}_{\text{FQ}}(2f) \quad (57)$$

If the thin lens approximation of Equation (33) for the quadrupoles is applied in order to calculate the one-period map of the symmetrized FODO-cell, Expression (58) results.

$$\mathcal{M}_{\text{FODO}}(s_i + L|s_i) = \begin{bmatrix} \left(1 - \frac{L^2}{2f^2}\right) & \left(2L + \frac{L^2}{f}\right) \\ \left(\frac{L^2}{4f^3} - \frac{L}{2f^2}\right) & \left(1 - \frac{L^2}{2f^2}\right) \end{bmatrix} \quad (58)$$

From the one-period map of the symmetrized FODO-cell it is possible to extract the Courant-Snyder functions at the symmetry point  $s_i$  by exploiting the relations from Expression (49). First of all, the condition for stable beam motion can be related to the design parameters  $L$  and  $f$  of the FODO-cell as presented in Equation (59).

$$|\text{Tr}[\mathcal{M}_{\text{FODO}}]| \leq 2 \quad \Rightarrow \quad |f| \geq \frac{L}{2} \quad (59)$$

As a consequence of the stability condition of a FODO-cell, the beam motion is stable if the focusing is weak and the focal length large respectively. In general, trying too hard to speed up only slows you down [6]. In addition to the stability condition, the Courant-Snyder functions at the symmetry point

$s_i$  are given by Expression (60).

$$\begin{aligned}
\psi(s_i) &= \arccos\left(\frac{\mathcal{M}_{\text{FODO}}[1, 1] + \mathcal{M}_{\text{FODO}}[2, 2]}{2}\right) \Rightarrow \sin\left(\frac{\psi(s_i)}{2}\right) = \frac{L}{2|f|} \\
\alpha(s_i) &= \frac{\mathcal{M}_{\text{FODO}}[1, 1] - \mathcal{M}_{\text{FODO}}[2, 2]}{2 \sin \psi(s_i)} \Rightarrow \alpha(s_i) = 0 \\
\beta(s_i) &= \frac{\mathcal{M}_{\text{FODO}}[1, 2]}{\sin \psi(s_i)} \Rightarrow \beta(s_i) = \frac{L}{f} \cdot \frac{2f + L}{\sin \psi(s_i)} \\
\gamma(s_i) &= -\frac{\mathcal{M}_{\text{FODO}}[2, 1]}{\sin \psi(s)} \Rightarrow \gamma(s_i) = \frac{1}{\beta(s_i)} = \frac{f}{L} \cdot \frac{\sin \psi(s_i)}{2f + L}
\end{aligned} \tag{60}$$

The Courant-Snyder functions in the middle of the defocusing quadrupole of the FODO-cell are obtained by inverting the sign of the focal length ( $f_{\text{DQ}} = -f_{\text{FQ}}$ ) in Expression (60).

## 4.2 Beam waist

A sequence of FODO-cells can stabilise the motion of the beam by focusing in both planes at the same time. Consequently, the FODO-cells generate stable and periodic realisations of the beta function in the lattice. Unfortunately, it is not possible to fill the entire lattice with FODO-cells because the purpose of an accelerator requires free beamline sections for experiments. From the beam dynamics perspective, experiments or interaction points (IPs) are drift spaces without guiding fields. The beta function at the interaction point, labelled as  $\beta^*$ , can be evolved in the drift space region by applying the transfer matrix (55). The beta function around the interaction point is given by Equation (61).

$$\begin{aligned}
\beta(s) &\stackrel{(55)}{=} -2m_{11}m_{12} \cdot \alpha^* + m_{11}^2 \cdot \beta^* + m_{12}^2 \cdot \gamma^* \\
\Rightarrow \beta(s) &\stackrel{(31)}{=} -2s \cdot \alpha^* + \beta^* + s^2 \cdot \gamma^* \quad \text{with: } s \hat{=} L \\
\Rightarrow \beta(s) &\stackrel{(60)}{=} \beta^* + s^2 \cdot \gamma^* \\
\Leftrightarrow \beta(s) &\stackrel{(42)}{=} \beta^* + \frac{s^2}{\beta^*}
\end{aligned} \tag{61}$$

The mapping of the beta function in the drift section or detector section gives rise to large beta values in the arcs. Consequently, a large aperture is required in the arcs due to the quadratic increasing beam width. In general, in order to maximize the luminosity and the interaction rate respectively the beta function has to be minimized at the interaction point, see Equation (7). This is commonly known as the mini-beta principle and is illustrated by the graph in Figure (4a).

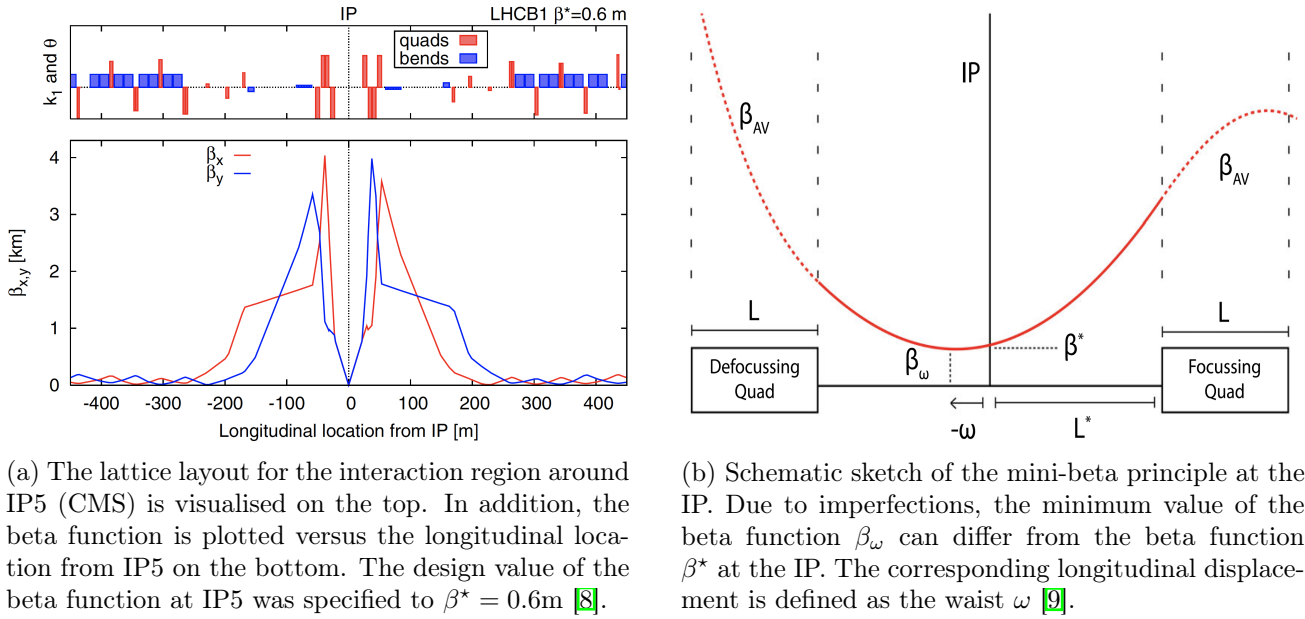


Figure 4: Behaviour of the beta function around the interaction point (IP).

The behaviour of the beta function in Figure (4a) is a textbook example. Far away from the IP the beta function is characterised by periodic solutions enforced by the FODO-cells. In the drift space of the interaction region the quadratic rise of the beta function according to Equation (61) is clearly recognisable. In order to ensure the continuity of the beta function, large beta values in the arcs need to be matched back to the periodic solutions of the FODO-cells by quadrupole triplets. The quadratic behaviour of the beta function around the IP can be understood in terms of the phase space ellipse. The area of the phase space ellipse has to be conserved according to Liouville's theorem. Consequently, squeezing the beta function at the IP increases the gamma function and the beam divergence respectively. As a result, the beam rapidly diverges after it leaves the symmetry point and causes large beta values in the arcs.

In real life imperfections of the lattice come into play and cause a difference between the minimum beta value  $\beta_\omega$  and the value of the beta function  $\beta^*$  at the IP. Hence, Equation (61) has to be modified if real accelerators are considered. The longitudinal shift of the minimum beta value  $\beta_\omega$  from the IP is called the waist  $\omega$  and is taken into account in Equation (62). Furthermore, Figure (4b) visualizes the waist  $\omega$  of the beta function in the interaction region.

$$\beta(s) \stackrel{(61)}{=} \beta_\omega + \frac{(s - \omega)^2}{\beta_\omega} \quad (62)$$

### 4.3 Collimator

A collimator is used in accelerators in order to remove beam particles with too large offset from the reference orbit. As a consequence, the aperture of the collimator defines the maximum beam width at given  $s$ .

### 4.4 Designing the lattice

In Equation (34) the one-turn map for an accelerator has been defined. This concept assumes a piecewise-constant representation of the accelerator structure and exploits the idea of linear transfer

maps. Theoretically, it is expected to have as much transfer matrices as elements in the lattice but because of edge focusing the number of matrices is larger as the number of elements.

#### 4.5 Example: Designing a storage ring model with MAD-X

The methodology of the previous sections shall be illustrated by a generic example. For this reason a storage ring has been designed within this work, which is composed by 10 bending FODO-cells. The detailed structure of the lattice is defined by Table (1). The lattice functions were evaluated at different locations by using transfer matrices, segment by segment. In order to forecast the phase space ellipses the formalism of transfer matrices was used to propagate the lattice functions through the ring. The lattice functions at  $s_0$  have been determined by the one-turn map and the action was implicitly assigned by the initial phase space vector  $\eta_i = [+0.1, -0.1]$ . The first section of the storage ring is visualized in Figure (5). The beta function, in combination with the action, indicates the envelope of the ensemble of possible particle trajectories. Hence, the horizontal elongation from the reference orbit is maximal at focusing quadrupoles and minimal at defocusing quadrupoles. Since dispersion is driven by dipoles, there is no dispersion in the vertical plane. The primary idea of this example is to demonstrate the validity of the results obtained by MAD-X. Figure (6) illustrates the phases space ellipses at different locations within the first FODO-cell. Furthermore, the ellipses at the exit of every FODO-cell are plotted. Due to the symmetrical design of the lattice, the phase space ellipses at the exit of every FODO-cell are congruent. In comparison, Figure (7) shows the turn-by-turn tracking data of the particle with initial phase space vector  $\eta_i$ , obtained by MAD-X. Obviously, the ellipses are identical, which underpins the fact that MAD-X is also exploiting transfer matrices to determine the lattice functions. Consequently, this illustrative example demonstrates the reliability of MAD-X.

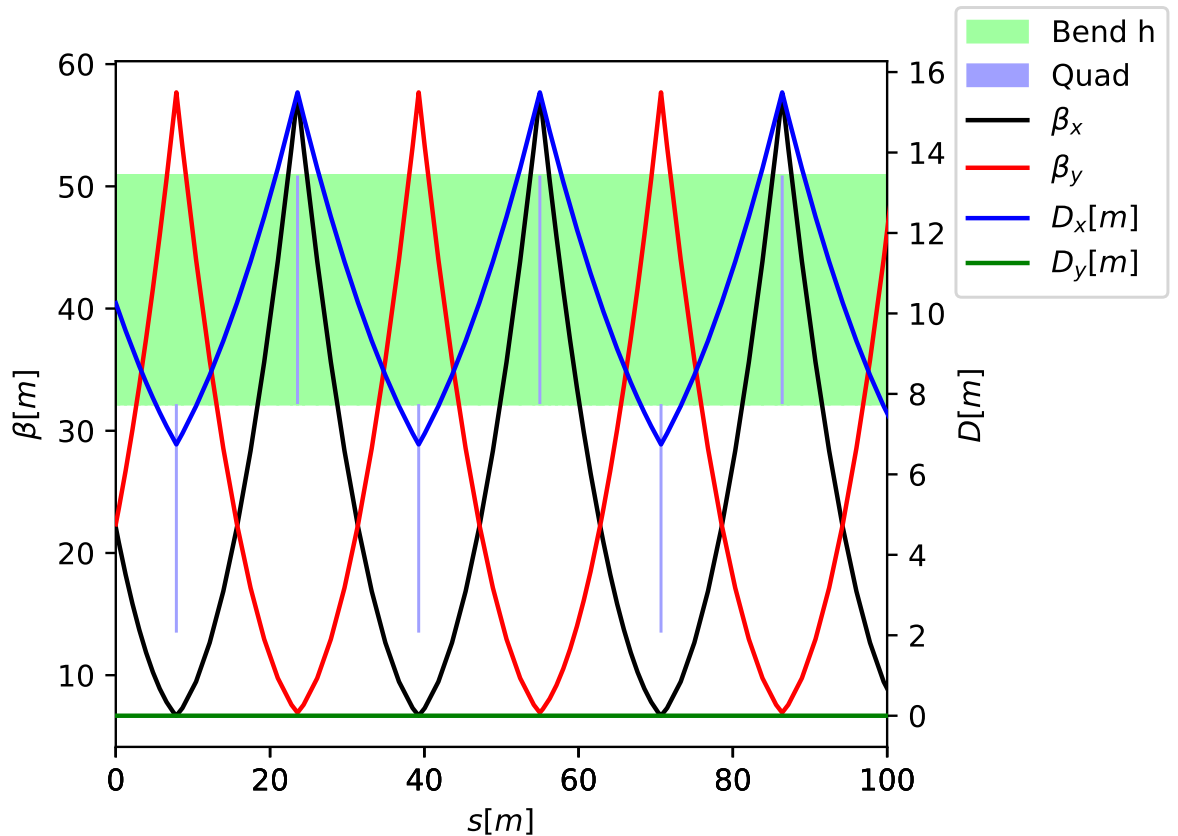


Figure 5: The first bending FODO-cell of the storage ring is indicated. In total, the ring comprises 10 bending FODO-cells which bend the beam by  $36^\circ$  per cell.



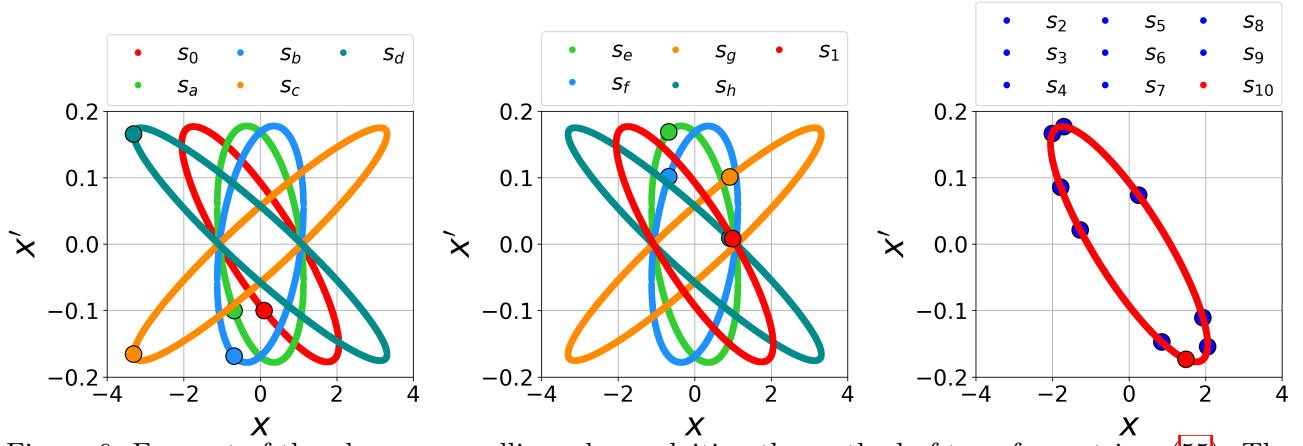


Figure 6: Forecast of the phase space ellipses by exploiting the method of transfer matrices (55). The Courant-Snyder functions have been determined by the one-turn map of the lattice. The dots indicate the phase space vector during the first turn. The locations are given by Table (1).

Table 1: Lattice structure of the model storage ring with radius  $\rho_0 = 100$  m.

| label | element         | property             | position                   | label                | element                                   | position                            |
|-------|-----------------|----------------------|----------------------------|----------------------|-------------------------------------------|-------------------------------------|
| $s_0$ | Start marker    | ---                  | $s = 0$                    | $s_2$                | Marker at 2 <sup>nd</sup> FODO-cell exit  | $s = 2 \cdot 8\theta \cdot \rho_0$  |
| $s_a$ | thick dipole    | $\theta = 4.5^\circ$ | $s = \theta \cdot \rho_0$  | $s_3$                | Marker at 3 <sup>rd</sup> FODO-cell exit  | $s = 3 \cdot 8\theta \cdot \rho_0$  |
| $s_b$ | thin quadrupole | $k_d \cdot l = -0.1$ | $s = \theta \cdot \rho_0$  | $s_4$                | Marker at 4 <sup>th</sup> FODO-cell exit  | $s = 4 \cdot 8\theta \cdot \rho_0$  |
| $s_c$ | thick dipole    | $\phi = 2\theta$     | $s = 3\theta \cdot \rho_0$ | $s_5$                | Marker at 5 <sup>th</sup> FODO-cell exit  | $s = 5 \cdot 8\theta \cdot \rho_0$  |
| $s_d$ | thin quadrupole | $k_f \cdot l = +0.1$ | $s = 3\theta \cdot \rho_0$ | $s_6$                | Marker at 6 <sup>th</sup> FODO-cell exit  | $s = 6 \cdot 8\theta \cdot \rho_0$  |
| $s_e$ | thick dipole    | $\phi = 2\theta$     | $s = 5\theta \cdot \rho_0$ | $s_7$                | Marker at 7 <sup>th</sup> FODO-cell exit  | $s = 7 \cdot 8\theta \cdot \rho_0$  |
| $s_f$ | thin quadrupole | $k_d \cdot l = -0.1$ | $s = 5\theta \cdot \rho_0$ | $s_8$                | Marker at 8 <sup>th</sup> FODO-cell exit  | $s = 8 \cdot 8\theta \cdot \rho_0$  |
| $s_g$ | thick dipole    | $\phi = 2\theta$     | $s = 7\theta \cdot \rho_0$ | $s_9$                | Marker at 9 <sup>th</sup> FODO-cell exit  | $s = 9 \cdot 8\theta \cdot \rho_0$  |
| $s_h$ | thin quadrupole | $k_f \cdot l = +0.1$ | $s = 7\theta \cdot \rho_0$ | $s_{10} \hat{=} s_0$ | Marker at 10 <sup>th</sup> FODO-cell exit | $s = 10 \cdot 8\theta \cdot \rho_0$ |
| $s_1$ | thick dipole    | $\theta = 4.5^\circ$ | $s = 8\theta \cdot \rho_0$ |                      |                                           |                                     |

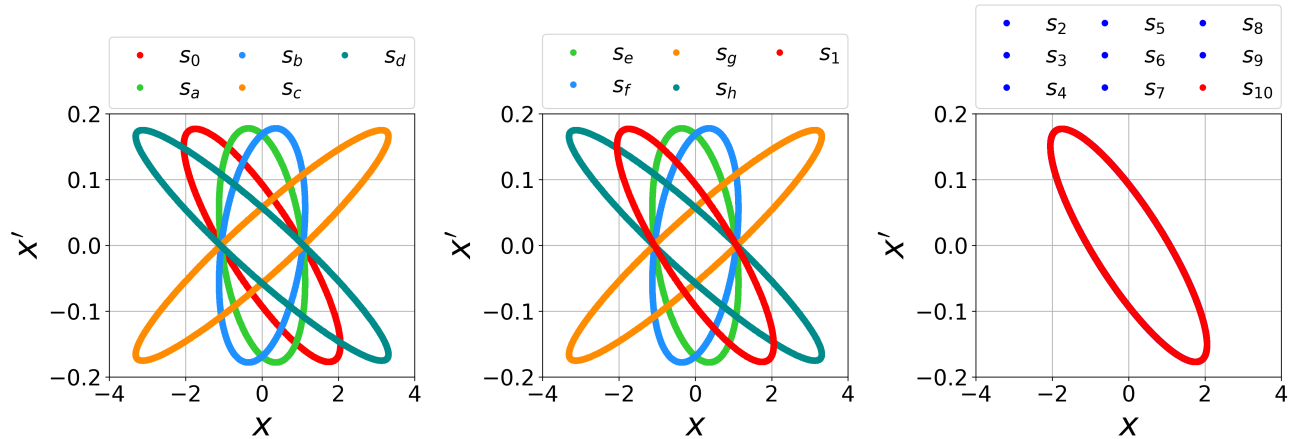


Figure 7: Scatter plot of the phase space ellipses from the tracking module of MAD-X. The locations are given by Table (1). The phase space vector  $\vec{\eta}$  has been initialized as  $\eta_i = [+0.1, -0.1]$ .

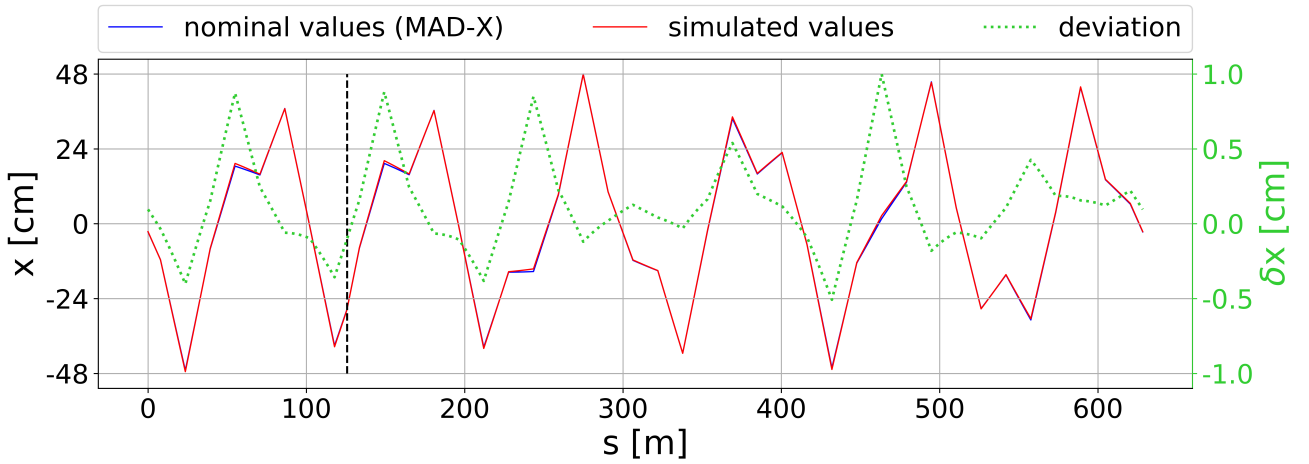


Figure 8: Closed orbit distortion simulated for the storage ring example from Table (1). A dipole distortion with a kicking strength of  $d\theta = +0.01$  has been considered. The location of the distortion is indicated as a black dashed line. As expected, Equation (68) quantifies the closed orbit offset accurately, which is demonstrated by the coherence of the red and the blue line.

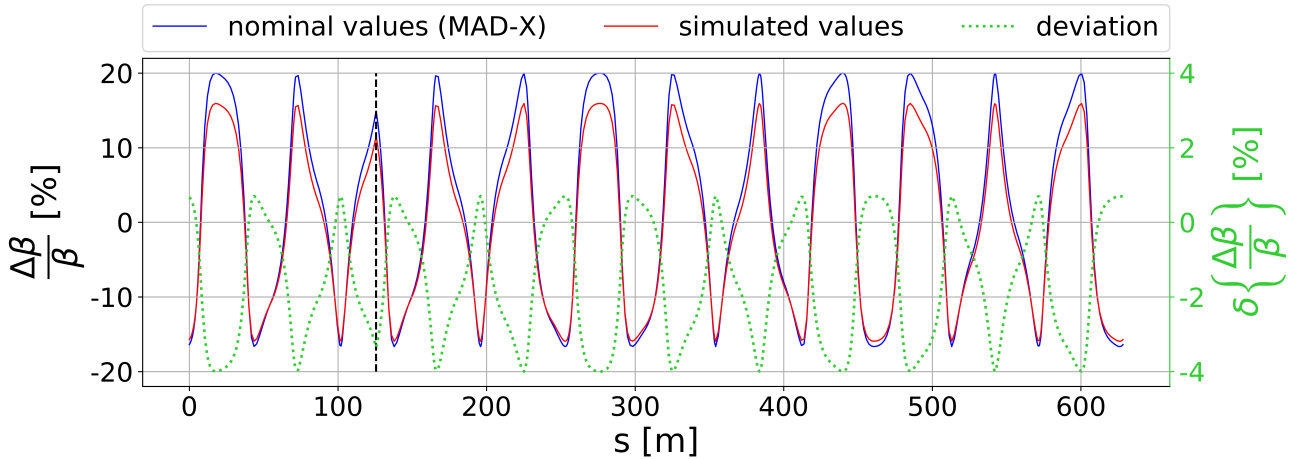


Figure 9: Effect of beta beating illustrated for the storage ring example from Table (1). A quadrupole perturbation with an integrated strength of  $\Delta k \cdot l = +0.01$  has been considered. The location of the distortion is indicated as a black dashed line. As expected, Equation (82) reproduces the effect of beta beating almost entirely, which is demonstrated by the coherence of the red and the blue line. It is clearly noticeable, that the amplitude of the beta beating oscillation remains constant over the entire domain of the storage ring. As a consequence, one can deduce that there is only a single quadrupolar perturbation in the lattice. In contrast, if there were more than one quadrupolar distortions present in the lattice an interference of several beta beating waves should be observable. As a result, a non-constant amplitude behaviour of the overall beta beating oscillation is expected for the case of several quadrupolar distortions.

#### 4.6 LHC Layout

The LHC consists of eight arcs which subdivide the ring into eight octants. The arcs are the bending sections of the LHC which are composed by bending FODO-cells of 110 m length [4]. In the middle of each octant there is a Long Straight Section (LSS), where the experiments are placed. The Interaction Region (IR) is the centre of the LSS and indicates the location of  $\beta^*$  at the Interaction Point (IP). The LSS subsection is sandwiched by two Dispersion Suppression (DS) modules which minimize the dispersion at the IP.

The DS modules are composed by a FODO-cell like structure but they are arranged slightly non-periodic in order to suppress dispersion. In the framework of the IP there is a DS on the right (DSR) and on the left (DSL) side respectively. In other words, there is a downstream (DSR) and an upstream (DSL) dispersion suppressor referencing on the flow direction of beam 1. Beam 1 and beam 2 cross each other at the IPs, where the experiments are placed. At LSS1 is ATLAS domiciled, at LSS2 ALICE, at LSS5 CMS and at LSS8 is LHC-b resident. All other LSS sections have a different purpose. At LSS4 there are RF cavities to accelerate the particle beam. At LSS6 it is possible to extract the beam in order to dump it by a fixed target collision. The particle beams are injected close to the experiments ALICE (beam 1) and LHC-b (beam 2). Figure (10) sketches the top view onto the LHC. In general, the longitudinal coordinate  $s$  is referenced to the location of the beam injection close to LSS2.

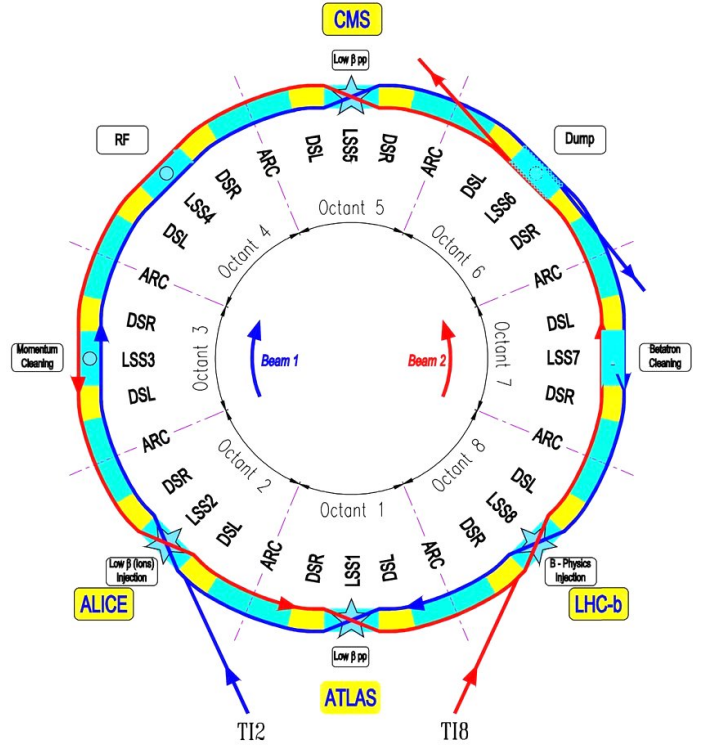


Figure 10: Lattice layout of the Large Hadron Collider (LHC) at CERN. In general, there are eight IPs centered in each octant. The octants are connected by the arcs, which are composed by FODO-cells to impose a periodic solution. The labelling convention for the arcs includes the names of the corresponding octants as a suffix, for example arc45 terms the arc between octant 4 and octant 5 [14].

At this point, it is of paramount importance to elaborate on the power supply of the lattice elements. In general, the elements in the arcs are powered in series and the elements in the IR are electrified individually. Consequently, the elements in the arcs can not be parametrised independently. As a result, quadrupole perturbations in the arcs can not be corrected on a element by element basis. Instead, it can only be accounted for an element distortion in the arcs by an ensemble averaging over the entire section. In other words, by slightly adjusting the strength of all elements in the arcs it might be possible to equalize a single perturbation. This Master thesis picks up on the approach of using orbit bumps in sextupoles in order to correct for local quadrupole distortions in the arcs. For perturbations localized within the interaction region it is possible to adjust the magnet powering individually, so that the model strengths can be reestablished based on measurements.

#### 4.6.1 Achromatic Telescopic Squeezing scheme (ATS-optics)

In graph (11a) the locations of the bending dipoles and the quadrupoles as well as the positioning of the sextupoles are indicated. There are other elements too, but for this thesis only those elements will be relevant. In general, there are eight interaction regions and two of them are matter of special importance. At IR5, the horizontal beta-function in the triplets is very large and the vertical beta-function in red is roughly four times smaller. The difference originates from the flat optics design, which maximizes the geometrical reduction factor in comparison to a round beam configuration [35].

The CMS and ATLAS experiments are of main importance for proton beam collisions. At both locations very large beta values in the triplets are noticeable as well as significant beta-beating in the neighboring arcs due to the telescopic squeezing scheme of the ATS-layout [34]. Plot (11b) enlarges the ATS-structure around IP5. It is clearly visible, that the ATS-design minimizes the beta-function at the interaction point in both planes. In order to squeeze the beam, very large beta-values in the triplets must be accepted. This phenomenon is caused by the quadratic rise of the beta-function (61) within the drift space around the IP. Since the beta-function serves as a sensitivity coefficient for magnetic field imperfections, large beta-values will amplify local distortions which leads to a global beating of the beam characteristics. Hence, quadrupole imperfections in the triplets and in the arcs have to be corrected, when the ATS scheme is intended to be applied.

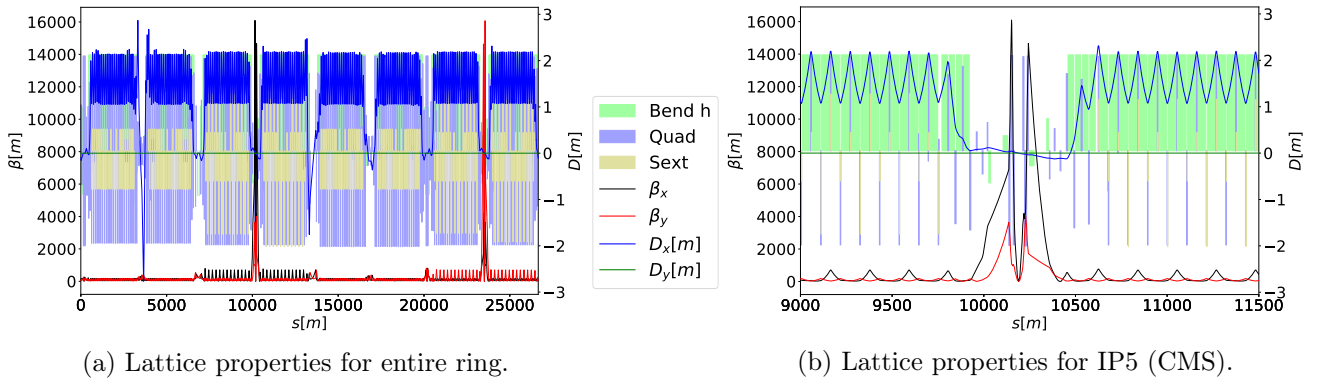


Figure 11: Visualisation of the lattice layout for the ATS-optics.

#### 4.6.1.1 Labelling convention of the lattice elements

The naming convention for a subsample of ATS lattice elements is presented in Table (2) as a demonstration example. This list is not intended to be exhaustive. In general, the label of a lattice element is composed of five criteria. First of all, a shortcut notation of the magnetic element type is used as a prefix. The middle part comprises information about the location of the lattice element. In detail, the number of the FODO-cell (numbered in half-cell units) within the arc is combined with the number of the corresponding octant. Due to the fact that a FODO-cell is constructed by more than one element of the same type, an alphabetic numbering complements the position labelling within the FODO-cell. Since the alternating lattice structure in the arcs is symmetric with respect to the interaction point (IP), the orientation with respect to the IP is added. Finally, the beam number is attached as a suffix.

| Magnetic element          | Shortcut | FODO-cell number<br>within arc                                                                                                                                                                                                                                                                                                                                                    | Orientation<br>left (L) / right (R) | Octant | Beam<br>number | Label                                                                                                                                                       |
|---------------------------|----------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------|--------|----------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Quadrupole                | MQ       | 14                                                                                                                                                                                                                                                                                                                                                                                | R                                   | 1      | B1             | MQ.14R1.B1                                                                                                                                                  |
| Sextupole                 | MS       | 31                                                                                                                                                                                                                                                                                                                                                                                | L                                   | 2      | B2             | MS.31L2.B2                                                                                                                                                  |
| Dipole - orbit kicker (H) | MCBH     | 13                                                                                                                                                                                                                                                                                                                                                                                | R                                   | 4      | B2             | MCBH.13R4.B2                                                                                                                                                |
| Dipole - orbit kicker (V) | MCBV     | 27                                                                                                                                                                                                                                                                                                                                                                                | L                                   | 3      | B1             | MCBV.27L3.B1                                                                                                                                                |
| Beam Position Monitor     | BPM      | 26                                                                                                                                                                                                                                                                                                                                                                                | L                                   | 8      | B1             | BPM.26L8.B1                                                                                                                                                 |
| Dipole - bending magnet   | MB       | <div style="display: flex; align-items: center;"> <div style="margin-right: 10px;"> <div style="display: flex; flex-direction: column; align-items: center;"> <span>A</span> <span>B</span> <span>C</span> </div> <div style="font-size: 2em; margin: 0 10px;">}</div> <div style="display: flex; align-items: center;"> <math>\oplus</math> <span>32</span> </div> </div> </div> | L                                   | 2      | B1             | <div style="display: flex; flex-direction: column; align-items: center;"> <span>MB.A32L2.B1</span> <span>MB.B32L2.B1</span> <span>MB.C32L2.B1</span> </div> |

Table 2: Random selection of ATS lattice elements to demonstrate the naming convention.

## 5 Errors and misalignments of the lattice

### 5.1 Design orbit versus closed orbit

In general, a lattice model can never be realized perfectly in practice due to manufacturing accuracy limitations. Since real life has to deal with fluctuations and expectation values, the lattice built will have elements which are slightly misaligned and the magnetic fields of the magnets are not fully congruent to the desired field distributions. Hence, the reference orbit is not a fixpoint in real life. As a consequence, the particle motion on the reference orbit becomes unstable and converges to a new fixpoint. Even if there is only a small dipole kick the orbit will change entirely, since the distortion will run around the ring. The resulting, stable trajectory forms a closed orbit which satisfies periodic boundary conditions. Furthermore, the closed orbit defines a new reference trajectory. Consequently, the betatron oscillations of the particle beam will occur around the closed orbit and not around the design orbit. It is important to emphasize, that the beam particle coordinates in the Frenet-Serret coordinate system are still related to the design orbit. Hence, the horizontal displacement  $x(s)$  from the design orbit has two contributions. On the one hand, there is a periodic displacement by the betatron oscillations  $x_\beta(s)$  around the closed orbit with zero expectation value over the ring structure. On the other hand, there is a position dependent closed orbit offset  $x_o(s)$  from the design orbit with non-zero expectation value over the entire ring. Equation (63) puts both contributions on record.

$$x(s) = x_\beta(s) + x_o(s) \quad (63)$$

### 5.2 Dipole error

Suppose that the reference orbit is distorted by an infinitesimal dipole error. The error of the magnetic field strength  $\delta B(s)$  will directly translate into an additional bending angle  $d\theta$ . The influence of  $\delta B(s)$  is weighted by the impact length  $ds$  of the distortion in order to receive the integrated strength of the dipole kick error. The ratio between the integrated strength of the field error and the nominal rigidity of the dipole determines the kick angle  $d\theta$ . If the field perturbation  $\delta B(s)$  is larger than zero the particle beam will be bent towards the center, wherefore the kick angle is smaller than zero and vice versa. In a nutshell, a dipole distortion results in an additional bending angle  $d\theta$  according to Equation (64). Here,  $\delta B(s)$  denotes the field error distribution, which is assumed to be a single dipole kick error following a Dirac-Delta-distribution around  $s_d$ .

$$d\theta = -\frac{\delta B(s) ds}{\rho \langle B_d \rangle} \Rightarrow d\theta = -\frac{\Delta B \cdot \delta(s - s_d) ds}{\rho \langle B_d \rangle} \quad (64)$$

The position dependent offset  $x_o(s)$  from the design orbit, caused by a closed orbit distortion, is derived in Equation (68) by executing the Ansatz [13] presented in Expression (65) and (66) respectively. Here,  $L$  characterises the circumference of the accelerator or the periodicity length respectively. Furthermore, the dipole distortion occurs at the beamline position  $s_d$ , where the derivative of the particle trajectory experiences a kick of  $d\theta$ . A stable orbit is characterised by a closed reference trajectory. For that reason, the distorted orbit must obey the most eminent constraint for a closed orbit, which is  $\vec{\eta}_o(s+L) \stackrel{!}{=} \vec{\eta}_o(s)$ . The phase advance  $\Delta\psi$  between the origin of the distortion at  $s_d$  and the point of interest at  $s$  has to be positive since the field distortion can only propagate forward in time and beamline position respectively. Hence, in order to satisfy causality it must hold that  $\Delta\psi \geq 0$ . As a consequence, it is necessary to consider two cases [12]. First of all, there is the downstream case with the condition  $s \geq s_d$ .

Secondly, there is the upstream case with the restriction  $s < s_d$ . In the upstream case, the distortion becomes effective at the point of interest after the first full turn because causality requires the order of action and reaction. Consequently, a case differentiation has to be undertaken for the upstream (66) and downstream (65) scenario. The matrix products are worked out in detail in Expression (67).

$$\text{Downstream: } s \geq s_d \left\{ \begin{array}{l} \vec{\eta}_o(s+L) = \mathcal{M}(s+L|s_d+L) \cdot \left[ \mathcal{M}(s_d+L|s) \cdot \vec{\eta}_o(s) + \begin{pmatrix} 0 \\ d\theta \end{pmatrix} \right] \stackrel{!}{=} \vec{\eta}_o(s) \\ \Leftrightarrow \vec{\eta}_o(s) = \mathcal{M}(s+L|s) \cdot \vec{\eta}_o(s) + \mathcal{M}(s|s_d) \cdot \begin{pmatrix} 0 \\ d\theta \end{pmatrix} \\ \Leftrightarrow \vec{\eta}_o(s) = [\mathbb{1} - \mathcal{M}(s+L|s)]^{-1} \cdot \mathcal{M}(s|s_d) \cdot \begin{pmatrix} 0 \\ d\theta \end{pmatrix} \end{array} \right. \quad (65)$$

$$\text{Upstream: } s < s_d \left\{ \begin{array}{l} \vec{\eta}_o(s+L) = \mathcal{M}(s+L|s_d) \cdot \left[ \mathcal{M}(s_d|s) \cdot \vec{\eta}_o(s) + \begin{pmatrix} 0 \\ d\theta \end{pmatrix} \right] \stackrel{!}{=} \vec{\eta}_o(s) \\ \Leftrightarrow \vec{\eta}_o(s) = \mathcal{M}(s+L|s) \cdot \vec{\eta}_o(s) + \mathcal{M}(s+L|s_d) \cdot \begin{pmatrix} 0 \\ d\theta \end{pmatrix} \\ \Leftrightarrow \vec{\eta}_o(s) = [\mathbb{1} - \mathcal{M}(s+L|s)]^{-1} \cdot \mathcal{M}(s+L|s_d) \cdot \begin{pmatrix} 0 \\ d\theta \end{pmatrix} \end{array} \right. \quad (66)$$

$$\vec{\eta}_o(s) \stackrel{(52)}{=} \frac{1}{4 \sin^2 \left( \frac{\psi_o}{2} \right)} \cdot \begin{bmatrix} 1 - (\cos \psi_o - \alpha \sin \psi_o) & \beta \sin \psi_o \\ -\gamma \sin \psi_o & 1 - (\cos \psi_o + \alpha \sin \psi_o) \end{bmatrix} \cdot \begin{cases} s \geq s_d : \mathcal{M}(s|s_d) \cdot \begin{pmatrix} 0 \\ d\theta \end{pmatrix} \\ s < s_d : \mathcal{M}(s+L|s_d) \cdot \begin{pmatrix} 0 \\ d\theta \end{pmatrix} \end{cases}$$

$$\Leftrightarrow \vec{\eta}_o(s) \stackrel{(46)}{=} \frac{d\theta}{4 \sin^2 \left( \frac{\psi_o}{2} \right)} \cdot \begin{bmatrix} 1 - (\cos \psi_o - \alpha \sin \psi_o) & \beta \sin \psi_o \\ -\gamma \sin \psi_o & 1 - (\cos \psi_o + \alpha \sin \psi_o) \end{bmatrix} \cdot \begin{pmatrix} \sqrt{\beta_d \beta} \cdot \sin \Delta\psi \\ \sqrt{\frac{\beta_d}{\beta}} \cdot (\cos \Delta\psi - \alpha \sin \Delta\psi) \end{pmatrix}$$

$$\hookrightarrow \text{with: } \begin{cases} \text{Downstream: } s \geq s_d, & \Delta\psi \equiv \psi(s) - \psi(s_d) \\ \text{Upstream: } s < s_d, & \Delta\psi \equiv \psi(s+L) - \psi(s_d) = \psi(s) - \psi(s_d) + 2\pi Q \end{cases}$$

$$\Leftrightarrow \vec{\eta}_o(s) = \frac{d\theta}{4 \sin^2 \left( \frac{\psi_o}{2} \right)} \cdot \begin{bmatrix} \sqrt{\beta_d \beta} \cdot \sin \Delta\psi \cdot (1 - (\cos \psi_o - \alpha \sin \psi_o)) + \sqrt{\beta_d \beta} \cdot \sin \psi_o \cdot (\cos \Delta\psi - \alpha \sin \Delta\psi) \\ -\gamma \sqrt{\beta_d \beta} \cdot \sin \Delta\psi \cdot \sin \psi_o + \sqrt{\frac{\beta_d}{\beta}} \cdot (\cos \Delta\psi - \alpha \sin \Delta\psi) \cdot (1 - (\cos \psi_o + \alpha \sin \psi_o)) \end{bmatrix}$$

$$\Leftrightarrow \vec{\eta}_o(s) \stackrel{(42)}{=} \frac{d\theta}{4 \sin^2 \left( \frac{\psi_o}{2} \right)} \cdot \begin{bmatrix} \sqrt{\beta_d \beta} \cdot (\sin \Delta\psi + \sin(\psi_o - \Delta\psi)) \\ \sqrt{\frac{\beta_d}{\beta}} \cdot (\alpha \sin(\Delta\psi - \psi_o) - \cos(\psi_o - \Delta\psi) + \cos \Delta\psi - \alpha \sin \Delta\psi) \end{bmatrix}$$

$$\Leftrightarrow \vec{\eta}_o(s) = \frac{\sqrt{\beta_d \beta} \cdot d\theta}{4 \sin^2 \left( \frac{\psi_o}{2} \right)} \cdot \begin{bmatrix} 2 \sin \left( \frac{\psi_o}{2} \right) \cdot \cos \left( \Delta\psi - \frac{\psi_o}{2} \right) \\ -\frac{2}{\beta} \cdot \left( \alpha \sin \left( \frac{\psi_o}{2} \right) \cdot \cos \left( \Delta\psi - \frac{\psi_o}{2} \right) + \sin \left( \frac{\psi_o}{2} \right) \cdot \sin \left( \Delta\psi - \frac{\psi_o}{2} \right) \right) \end{bmatrix}$$

$$\Leftrightarrow \vec{\eta}_o(s) = \frac{\sqrt{\beta_d \beta} \cdot d\theta}{2 \sin \left( \frac{\psi_o}{2} \right)} \cdot \begin{bmatrix} \cos \left( \Delta\psi - \frac{\psi_o}{2} \right) \\ -\frac{1}{\beta} \cdot \left( \alpha \cos \left( \Delta\psi - \frac{\psi_o}{2} \right) + \sin \left( \Delta\psi - \frac{\psi_o}{2} \right) \right) \end{bmatrix}$$

(67)



In general, the position dependent closed orbit offset  $x_o(s)$  underlies the same magnetic structure of the lattice as the unperturbed beam. Consequently, the offset itself has to obey Hill's differential Equation (23), wherefore the principle of transfer matrices is applicable. The phase advance between the dipole distortion and the observation point has been labelled as  $\Delta\psi$ . For simplicity, the case differentiation between up- and downstream has been incorporated into the declaration of the phase advance  $\Delta\psi$ , as presented in Expression (67). In addition, the phase advance over one-turn, which is determined by the natural tune (56), is termed as  $\psi_o$ . The first component of the perturbed phase space vector contains the information about the closed orbit offset  $x_o(s)$ . As a result, the case differentiation between up- and downstream can be merged together by taking the absolute value of the phase difference [10], which yields to a general Expression (68) for the closed orbit offset of an infinitesimal kick  $d\theta$ .

$$x_o(s) \stackrel{(67)}{=} \frac{\sqrt{\beta_d(s_d) \cdot \beta(s)} \cdot d\theta}{2 \sin(\pi Q)} \cdot \cos(\Delta\psi - \pi Q) = \frac{\sqrt{\beta_d(s_d) \cdot \beta(s)} \cdot d\theta}{2 \sin(\pi Q)} \cdot \begin{cases} s \geq s_d : & \cos(\psi(s) - \psi(s_d) - \pi Q) \\ s < s_d : & \cos(\psi(s) - \psi(s_d) + \pi Q) \end{cases} \quad (68)$$

$$\Leftrightarrow x_o(s) = \frac{\sqrt{\beta_d(s_d) \cdot \beta(s)} \cdot d\theta}{2 \sin(\pi Q)} \cdot \cos\left(\left|\psi(s) - \psi(s_d)\right| - \pi Q\right)$$

The closed orbit offset  $x_o(s)$  quantifies the impact of a single dipole distortion on the design orbit. In order to account for several dipole distortions, which are distributed around the ring, it is required to integrate over the closed domain starting from the observation point  $s$ . Consequently, the total horizontal displacement  $x(s)$  from the design orbit, in the picture of the reference particle, is given by Equation (69).

$$x(s) \stackrel{(64)}{=} \sqrt{2J\beta(s)} \cdot \cos(\psi(s) + \phi_0) - \frac{\sqrt{\beta(s)}}{2\rho \langle B_d \rangle \sin(\pi Q)} \cdot \int_s^{s+L} \sqrt{\beta_d(\xi)} \cdot \delta B(\xi) \cdot \cos\left(\left|\psi(s) - \psi(\xi)\right| - \pi Q\right) d\xi \quad (69)$$

The closed orbit offset  $x_o(s)$  depends on the strength  $\delta B(s)$  of the distortion as well as the beta function  $\beta_d$  at the origin of the perturbation and the natural tune  $Q$  of the accelerator. Hence, the beta function acts as a sensitivity coefficient for lattice distortions. For demonstrating the coherence of Equation (68) and (69), a dipole field distortion of the shape  $\delta B(\xi) = \Delta B \cdot \delta(\xi - s_d)$  is considered, where  $\Delta B$  quantifies the strength of the perturbation and  $\delta(\xi)$  corresponds to the Dirac-Delta-distribution. As required for consistency, the integrated closed orbit offset (70) for an infinitesimal dipole kick results in Expression (68).

$$x_o(s) = -\frac{\sqrt{\beta(s)}}{2\rho \langle B_d \rangle \sin(\pi Q)} \cdot \int_s^{s+L} \sqrt{\beta_d(\xi)} \cdot \Delta B \cdot \delta(\xi - s_d) \cdot \cos\left(\left|\psi(s) - \psi(\xi)\right| - \pi Q\right) d\xi$$

$$\Leftrightarrow x_o(s) = -\frac{\sqrt{\beta(s)} \cdot \Delta B}{2\rho \langle B_d \rangle \sin(\pi Q)} \cdot \begin{cases} \text{downstream: } s \geq s_d, & \int_L^{s+L} \sqrt{\beta_d(\xi)} \cdot \delta(\xi - s_d) \cdot \cos\left(\left|\psi(s) - \psi(\xi)\right| - \pi Q\right) d\xi \\ \text{upstream: } s < s_d, & \int_s^L \sqrt{\beta_d(\xi)} \cdot \delta(\xi - s_d) \cdot \cos\left(\left|\psi(s) - \psi(\xi)\right| - \pi Q\right) d\xi \end{cases}$$

$$\Leftrightarrow x_o(s) = -\frac{\sqrt{\beta(s)} \cdot \Delta B}{2\rho \langle B_d \rangle \sin(\pi Q)} \cdot \begin{cases} \text{downstream: } s \geq s_d, & \int_0^s \sqrt{\beta_d(\xi)} \cdot \delta(\xi - s_d) \cdot \cos\left(\left|\psi(s) - (\psi(\xi) + 2\pi Q)\right| - \pi Q\right) d\xi \\ \text{upstream: } s < s_d, & \int_s^L \sqrt{\beta_d(\xi)} \cdot \delta(\xi - s_d) \cdot \cos\left(\left|\psi(s) - \psi(\xi)\right| - \pi Q\right) d\xi \end{cases}$$

$$\Leftrightarrow x_o(s) = -\frac{\sqrt{\beta_d(s_d) \cdot \beta(s)} \cdot \Delta B}{2\rho \langle B_d \rangle \sin(\pi Q)} \cdot \cos\left(\left|\psi(s) - \psi(s_d)\right| - \pi Q\right) \quad \blacksquare \quad (70)$$

### 5.3 Tune resonances

The denominator of the overall effect of a closed orbit distortion from Equation (68) gives rise for a resonance behaviour if the tune is equal to an integer value. In the case of a resonance, the effect of a dipole error adds up turn after turn and there is no stable fixpoint orbit for the beam trajectory to converge to. In general, tune resonances occur in both planes and they are coupled by the resonance condition presented in Expression (71). As a result, resonances can only be avoided completely if the natural tune equals an irrational number.

$$m \cdot Q_x + n \cdot Q_y \stackrel{!}{=} k \quad \text{with:} \quad m, n, k \in \mathbb{Z} \quad (71)$$

From the resonance condition in Equation (71) the question arises, how it is possible to find a combination of the tunes which does not trigger a resonance. The answer might be terrifying, but it is not possible to prevent resonances completely. The reason why accelerators work despite this conflict lies in the fact, that the impact of resonances decreases with increasing order. If the tunes are chosen to be integers the resonance is labeled as first order. Hence, if the tunes are half integers the resonance is termed second order. At the Large Hadron Collider (LHC) at CERN the working point prevents for resonances up to  $10^{11}$ -th order [6].

### 5.4 Quadrupole error

#### 5.4.1 Misalignment error

If a quadrupole magnet is misaligned by some offset  $\Delta x$  an additional dipole field  $\Delta B$  is generated which kicks the beam. According to the multipole expansion of the magnetic field, presented in Equation (14), a position offset leads to an increased B-field in the order of  $\Delta B = g \cdot \Delta x$ .

#### 5.4.2 Magnetic field error

If a quadrupole field error is considered then the beam will experience a different focusing in the lattice. Consequently, the one-turn map of the lattice will change according to the additional quadrupole contribution. For the quadrupole field error the transfer map of a quadrupole in thin lens approximation can be used in order to determine the impact of the perturbation on the one-turn focusing properties of the lattice. The thin lens assumption is valid as long as the spatial extent of the perturbative quadrupole error is short in comparison to the focal length  $f_e$  of the distortion. The integrated strength of an infinitesimal quadrupole field error determines the focal length of the distortion according to Equation (72).

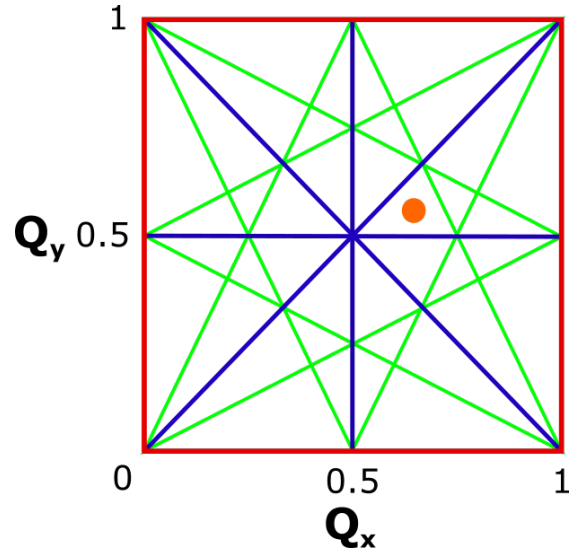


Figure 12: The first three orders of tune resonances are visualized. The abscissa denotes the natural tune  $Q_x$  in horizontal direction and the ordinate labels the vertical natural tune  $Q_y$ . The red lines indicate the tuning for first order resonance, the blue lines illustrate second order resonance and the green lines mark third order tune resonances. The working point of the accelerator, marked as a orange dot, has to be chosen somewhere between the resonance lines.



$$\frac{1}{f_\epsilon} = \delta k_1(s) \, ds \quad \Rightarrow \quad \frac{1}{f_\epsilon} = \Delta k_1 \cdot \delta(s - s_d) \, ds \quad (72)$$

Changing the focusing properties of the lattice will influence the natural tune of the machine as well as the beta function. The effect of a quadrupole field error on the tune can be quantified by calculating the distorted one-turn map  $\mathcal{M}_\epsilon(s_d + L|s_d)$  (73) at the location  $s_d$  of the perturbation.

$$\begin{aligned} \mathcal{M}_\epsilon(s_d + L|s_d) &= \mathcal{M}_o(s_d + L|s_d) \cdot \begin{bmatrix} 1 & 0 \\ -f_\epsilon^{-1} & 1 \end{bmatrix} \\ &\Leftrightarrow \mathcal{M}_\epsilon \stackrel{(47)}{=} \begin{bmatrix} \cos(2\pi Q_o) + \alpha_d \sin(2\pi Q_o) & \beta_d \sin(2\pi Q_o) \\ -\gamma_d \sin(2\pi Q_o) & \cos(2\pi Q_o) - \alpha_d \sin(2\pi Q_o) \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -f_\epsilon^{-1} & 1 \end{bmatrix} \\ &\Leftrightarrow \mathcal{M}_\epsilon = \begin{bmatrix} \cos(2\pi Q_o) + \alpha_d \sin(2\pi Q_o) - \frac{\beta_d \sin(2\pi Q_o)}{f_\epsilon} & \beta_d \sin(2\pi Q_o) \\ -\gamma_d \sin(2\pi Q_o) - \frac{\cos(2\pi Q_o) - \alpha_d \sin(2\pi Q_o)}{f_\epsilon} & \cos(2\pi Q_o) - \alpha_d \sin(2\pi Q_o) \end{bmatrix} \\ &\Rightarrow \text{Tr}[\mathcal{M}_\epsilon] = 2 \cos(2\pi Q_o) - \frac{\beta_d \sin(2\pi Q_o)}{f_\epsilon} \end{aligned} \quad (73)$$

The phase advance  $\psi_o$  over one-turn is intrinsically connected to the trace of the one-turn map as derived in Expression (49). Hence, the natural tune  $Q_\epsilon$  of the perturbed lattice structure is determined by the trace of the distorted one-turn map  $\text{Tr}[\mathcal{M}_\epsilon]$ . The result for  $Q_\epsilon$  is presented in Equation (74).

$$Q_\epsilon \stackrel{(49)}{=} \frac{1}{2\pi} \cdot \arccos\left(\frac{\text{Tr}[\mathcal{M}_\epsilon]}{2}\right) \stackrel{(73)}{=} \frac{1}{2\pi} \cdot \arccos\left(\cos(2\pi Q_o) - \frac{\beta_d \sin(2\pi Q_o)}{2f_\epsilon}\right) \quad (74)$$

If the focal length  $f_\epsilon$  is very large and the integrated strength of the quadrupole error small then the perturbed tune  $Q_\epsilon$  is close to the unperturbed tune  $Q_o$ . Consequently, a perturbation treatment can be applied to Equation (74) by exploiting the Ansatz:  $Q_\epsilon = Q_o + \Delta Q$ . This Ansatz results in a tune shift  $\Delta Q$  as presented in Equation (75).

$$\begin{aligned} \cos(2\pi Q_o) - \frac{\beta_d \sin(2\pi Q_o)}{2f_\epsilon} &\stackrel{(74)}{=} \cos(2\pi Q_\epsilon) \\ \Leftrightarrow \cos(2\pi Q_o) - \frac{\beta_d \sin(2\pi Q_o)}{2f_\epsilon} &= \cos(2\pi Q_o + 2\pi \Delta Q) \\ \Leftrightarrow \cos(2\pi Q_o) - \frac{\beta_d \sin(2\pi Q_o)}{2f_\epsilon} &= \cos(2\pi Q_o) \cdot \cos(2\pi \Delta Q) - \sin(2\pi Q_o) \cdot \sin(2\pi \Delta Q) \\ \Rightarrow \cos(2\pi Q_o) - \frac{\beta_d \sin(2\pi Q_o)}{2f_\epsilon} &\approx \cos(2\pi Q_o) - 2\pi \Delta Q \sin(2\pi Q_o) \\ &\Leftrightarrow \Delta Q = \frac{\beta_d}{4\pi f_\epsilon} \end{aligned} \quad (75)$$

As a result, the tune shift generated by a quadrupole field error depends on the local beta function at the point of the distortion. As seen for the closed orbit offset, caused by dipole field errors, the beta function serves as a sensitivity coefficient regarding the field error impact on the global quantities of the machine. In general, the perturbed tune increases if the focal length of the quadrupole field error

is larger than zero, which corresponds to a focusing quadrupole. Consequently, the lattice structure focuses more than expected, which results in more betatron oscillations and a positive tune shift. Furthermore, a pure quadrupole field error would shift the tune one way in one plane and the other way in the perpendicular plane as elaborated in Expression (24). If there are several quadrupole errors around the ring, the averaged strength of the perturbation has to be replaced by a field error distribution. For this reason, the focal length  $f_\epsilon$  has been replaced by the integral over the quadrupole error distribution  $\delta k(s)$ , as illustrated in (76).

$$\Delta Q \stackrel{(72)}{=} \frac{1}{4\pi} \cdot \oint \beta(s) \cdot \delta k(s) ds \quad (76)$$

The relationship between tune shift and beta function can be exploited in order to measure the beta function indirectly. Generally, the strength of a single quadrupole in the ring is varied and the resulting tune shift is measured. As a consequence, the beta function at the point of the quadrupole can be calculated by using Equation (75). Due to the fact that the beta function weights the impact of perturbations on beam quantities, large values of the beta function indicate regions of high sensitivity for field errors, as in the arcs.

#### 5.4.3 Beta beating due to quadrupolar errors

In addition to the tune shift  $\Delta Q$  from Equation (75), a quadrupole field error results in a distortion of the beta function  $\Delta\beta$ . This effect is commonly known as beta beating. In order to quantify the distortion of the beta function  $\Delta\beta$ , the perturbed one-turn map (73) will be considered at an arbitrary beamline position  $s$  and Equation (77) will be applied to determine the perturb beta function  $\beta_\epsilon(s)$ .

$$\beta_\epsilon(s) \stackrel{(49)}{=} \frac{\mathcal{M}_\epsilon(s + L|s)[1, 2]}{\sin \psi_\epsilon} \quad (77)$$

The general Ansatz for the derivation of  $\Delta\beta$  is presented in Expression (78) and (79) respectively. The phase advance  $\Delta\psi$  between the impact position of the distortion at  $s_d$  and the point of interest at  $s$  has to be positive since the magnetic field perturbation can only propagate forward in time and beamline position respectively. As a result of causality, the case differentiation between up- and downstream has to be considered as previously presented in Equation (65) and (66). The Ansatz is executed in Equation (80), where the unperturbed phase advance over one turn is labelled as  $\psi_o$ . The distorted one-turn map  $\mathcal{M}_\epsilon(s + L|s)$  for a quadrupole perturbation incorporates the tune shift as well as  $\beta$ -beating.

$$\text{Downstream: } s \geq s_d \left\{ \begin{array}{l} \mathcal{M}_\epsilon(s + L|s) = \mathcal{M}(s|s_d) \cdot \begin{bmatrix} 1 & 0 \\ -f_\epsilon^{-1} & 1 \end{bmatrix} \cdot \mathcal{M}(s_d + L|s) \\ \Leftrightarrow \mathcal{M}_\epsilon(s + L|s) = \mathcal{M}(s|s_d + L) \cdot \mathcal{M}_o(s_d + L|s_d) \cdot \begin{bmatrix} 1 & 0 \\ -f_\epsilon^{-1} & 1 \end{bmatrix} \cdot \mathcal{M}(s_d + L|s) \\ \Leftrightarrow \mathcal{M}_\epsilon(s + L|s) \stackrel{(73)}{=} [\mathcal{M}(s_d + L|s)]^{-1} \cdot \mathcal{M}_\epsilon(s_d + L|s_d) \cdot \mathcal{M}(s_d + L|s) \end{array} \right. \quad (78)$$

$$\text{Upstream: } s < s_d \left\{ \begin{array}{l} \mathcal{M}_\epsilon(s + L|s) = \mathcal{M}(s + L|s_d) \cdot \begin{bmatrix} 1 & 0 \\ -f_\epsilon^{-1} & 1 \end{bmatrix} \cdot \mathcal{M}(s_d|s) \\ \Leftrightarrow \mathcal{M}_\epsilon(s + L|s) = \mathcal{M}(s + L|s_d + L) \cdot \mathcal{M}_o(s_d + L|s_d) \cdot \begin{bmatrix} 1 & 0 \\ -f_\epsilon^{-1} & 1 \end{bmatrix} \cdot \mathcal{M}(s_d|s) \\ \Leftrightarrow \mathcal{M}_\epsilon(s + L|s) \stackrel{(73)}{=} [\mathcal{M}(s_d|s)]^{-1} \cdot \mathcal{M}_\epsilon(s_d + L|s_d) \cdot \mathcal{M}(s_d|s) \end{array} \right. \quad (79)$$

$$\begin{aligned}
\mathcal{M}_\epsilon(s+L|s) &\stackrel{(46)}{=} \begin{bmatrix} \sqrt{\frac{\beta}{\beta_d}} \cdot (\cos \Delta\psi - \alpha_d \sin \Delta\psi) & -\sqrt{\beta_d \beta} \cdot \sin \Delta\psi \\ \frac{\alpha_d - \alpha}{\sqrt{\beta_d \beta}} \cdot \cos \Delta\psi + \frac{1 + \alpha_d \alpha}{\sqrt{\beta_d \beta}} \cdot \sin \Delta\psi & \sqrt{\frac{\beta_d}{\beta}} \cdot (\cos \Delta\psi + \alpha \sin \Delta\psi) \end{bmatrix} \\
&\times \begin{bmatrix} \cos \psi_o + \alpha_d \sin \psi_o - \frac{\beta_d \sin \psi_o}{f_\epsilon} & \beta_d \sin \psi_o \\ -\gamma_d \sin \psi_o - \frac{\cos \psi_o - \alpha_d \sin \psi_o}{f_\epsilon} & \cos \psi_o - \alpha_d \sin \psi_o \end{bmatrix} \\
&\times \begin{bmatrix} \sqrt{\frac{\beta_d}{\beta}} \cdot (\cos \Delta\psi + \alpha \sin \Delta\psi) & \sqrt{\beta \beta_d} \cdot \sin \Delta\psi \\ \frac{\alpha - \alpha_d}{\sqrt{\beta \beta_d}} \cdot \cos \Delta\psi - \frac{1 + \alpha \alpha_d}{\sqrt{\beta \beta_d}} \cdot \sin \Delta\psi & \sqrt{\frac{\beta}{\beta_d}} \cdot (\cos \Delta\psi - \alpha_d \sin \Delta\psi) \end{bmatrix} \\
&\hookrightarrow \text{with: } \begin{cases} \text{Downstream: } s \geq s_d, & \Delta\psi \equiv \psi(s_d + L) - \psi(s) = \psi(s_d) - \psi(s) + 2\pi Q_o \\ \text{Upstream: } s < s_d, & \Delta\psi \equiv \psi(s_d) - \psi(s) \end{cases}
\end{aligned} \tag{80}$$

$$\begin{aligned}
\Rightarrow \mathcal{M}_\epsilon(s+L|s)[1,2] &= \beta \sin \Delta\psi \cdot (\cos \Delta\psi - \alpha_d \sin \Delta\psi) \cdot \left( \cos \psi_o + \alpha_d \sin \psi_o - \frac{\beta_d \sin \psi_o}{f_\epsilon} \right) \\
&+ \beta_d \beta \sin^2 \Delta\psi \cdot \left( \frac{\cos \psi_o - \alpha_d \sin \psi_o}{f_\epsilon} \right) + \beta \sin \psi_o \cdot (\cos \Delta\psi - \alpha_d \sin \Delta\psi)^2 \\
&- \beta \sin \Delta\psi \cdot (\cos \Delta\psi - \alpha_d \sin \Delta\psi) \cdot (\cos \psi_o - \alpha_d \sin \psi_o) + \beta_d \beta \sin \psi_o \cdot \sin^2 \Delta\psi \cdot \left( \frac{1 + \alpha_d^2}{\beta_d} \right) \\
\Rightarrow \mathcal{M}_\epsilon(s+L|s)[1,2] &= \frac{\beta_d \beta}{f_\epsilon} \sin \Delta\psi \cdot (\sin \Delta\psi \cos \psi_o - \cos \Delta\psi \sin \psi_o) + \beta \sin \psi_o \\
\Rightarrow \mathcal{M}_\epsilon(s+L|s)[1,2] &= \frac{\beta_d \beta}{f_\epsilon} \sin \Delta\psi \cdot \sin(\Delta\psi - \psi_o) + \beta \sin \psi_o
\end{aligned}$$

For the explicit calculation of the matrix products in Expression (80), the case differentiation has been shifted into the declaration of the phase advance  $\Delta\psi$ . The matrix element (80) contains the information about the distorted beta function according to Equation (77).

$$\begin{aligned}
&\frac{\beta_d \beta}{f_\epsilon} \sin \Delta\psi \cdot \sin(\Delta\psi - \psi_o) + \beta \sin \psi_o \stackrel{!}{=} \beta_\epsilon \sin \psi_\epsilon \\
\Rightarrow \frac{\beta_d \beta}{f_\epsilon} \sin \Delta\psi \cdot \sin(\Delta\psi - \psi_o) + \beta \sin \psi_o &= (\beta + \Delta\beta) \cdot \sin(\psi_o + 2\pi \Delta Q) \\
\Rightarrow \frac{\beta_d \beta}{f_\epsilon} \sin \Delta\psi \cdot \sin(\Delta\psi - \psi_o) + \beta \sin \psi_o &\approx (\beta + \Delta\beta) \cdot [\sin \psi_o + 2\pi \Delta Q \cos \psi_o] \\
\Rightarrow \frac{\beta_d \beta}{f_\epsilon} \sin \Delta\psi \cdot \sin(\Delta\psi - \psi_o) + \beta \sin \psi_o &= \beta \sin \psi_o + \Delta\beta \sin \psi_o + 2\pi \beta \Delta Q \cos \psi_o + \mathcal{O}(\Delta^2)
\end{aligned} \tag{81}$$

In Expression (81),  $\psi_\epsilon$  denotes the perturbed tune due to the tune shift  $\Delta Q$  (75) and  $\beta_\epsilon$  labels the distorted beta function on account of the beta beat  $\Delta\beta$ . At this stage, only perturbations of first order are considered. The case differentiation can be merged together by introducing absolute values for the phase advance. If several quadrupole distortions with field distribution  $\delta k(s)$  are considered, the beta beating impact accumulates as an integral over the ensemble of quadrupole perturbations.

$$\begin{aligned}
\frac{\Delta\beta}{\beta}(s) &\stackrel{(75)}{\approx} -\frac{\beta_d(s_d)}{2f_\epsilon \sin \psi_o} \cdot [\cos \psi_o - 2 \sin \Delta\psi \cdot \sin(\Delta\psi - \psi_o)] \\
\Leftrightarrow \frac{\Delta\beta}{\beta}(s) &= -\frac{\beta_d(s_d)}{2f_\epsilon \sin \psi_o} \cdot \cos(\psi_o - 2\Delta\psi) = -\frac{\beta_d(s_d)}{2f_\epsilon \sin \psi_o} \cdot \begin{cases} s \geq s_d : & \cos(2[\psi(s) - \psi(s_d) - \pi Q_o]) \\ s < s_d : & \cos(2[\psi(s) - \psi(s_d) + \pi Q_o]) \end{cases} \quad (82) \\
\Leftrightarrow \frac{\Delta\beta}{\beta}(s) &= -\frac{\beta_d(s_d)}{2f_\epsilon \sin(2\pi Q_o)} \cdot \cos\left(2\left|\psi(s) - \psi(s_d)\right| - 2\pi Q_o\right)
\end{aligned}$$

The effect of beta beating, described by Equation (82), quantifies the impact of a single, infinitesimal quadrupole distortion on the particle trajectory. In order to account for several quadrupole distortions, which are distributed around the ring, it is required to integrate over the closed domain starting from the observation point  $s$ . The accumulated beta beating impact is given by Equation (83), where  $\delta k(\xi)$  describes the distribution of the quadrupole field perturbation.

$$\frac{\Delta\beta}{\beta}(s) \stackrel{(72)}{=} -\frac{1}{2 \sin(2\pi Q_o)} \cdot \int_s^{s+L} \beta_d(\xi) \cdot \delta k(\xi) \cdot \cos\left(2\left|\psi(s) - \psi(\xi)\right| - 2\pi Q_o\right) d\xi \quad (83)$$

For demonstrating the coherence of Equation (82) and (83), a quadrupole field distortion of the shape  $\delta k(\xi) = \Delta k \cdot \delta(\xi - s_d)$  is considered, where  $\Delta k$  quantifies the strength of the perturbation and  $\delta(\xi)$  corresponds to the Dirac-Delta-distribution. As required for consistency, the integrated beta beating impact (84) for an infinitesimal quadrupole error results in Expression (82). In order to execute the integration, the case differentiation between up- and downstream has to be reconsidered. The integration limits will induce the requirement of taking the absolute value of the phase advance. Consequently, Equation (84) validates the beta beating formalism.

$$\begin{aligned}
\frac{\Delta\beta}{\beta}(s) &= -\frac{1}{2 \sin(2\pi Q_o)} \cdot \int_s^{s+L} \beta_d(\xi) \cdot \Delta k \cdot \delta(\xi - s_d) \cdot \cos\left(2\left|\psi(s) - \psi(\xi)\right| - 2\pi Q_o\right) d\xi \\
\Leftrightarrow \frac{\Delta\beta}{\beta}(s) &= -\frac{\Delta k}{2 \sin(2\pi Q_o)} \cdot \begin{cases} \text{downstream: } s \geq s_d, & \int_L^{s+L} \beta_d(\xi) \cdot \delta(\xi - s_d) \cdot \cos\left(2\left|\psi(s) - \psi(\xi)\right| - 2\pi Q_o\right) d\xi \\ \text{upstream: } s < s_d, & \int_s^L \beta_d(\xi) \cdot \delta(\xi - s_d) \cdot \cos\left(2\left|\psi(s) - \psi(\xi)\right| - 2\pi Q_o\right) d\xi \end{cases} \\
\Leftrightarrow \frac{\Delta\beta}{\beta}(s) &= -\frac{\Delta k}{2 \sin(2\pi Q_o)} \cdot \begin{cases} \text{downstream: } s \geq s_d, & \int_0^s \beta_d(\xi) \cdot \delta(\xi - s_d) \cdot \cos\left(2\left|\psi(s) - (\psi(\xi) + 2\pi Q_o)\right| - 2\pi Q_o\right) d\xi \\ \text{upstream: } s < s_d, & \int_s^L \beta_d(\xi) \cdot \delta(\xi - s_d) \cdot \cos\left(2\left|\psi(s) - \psi(\xi)\right| - 2\pi Q_o\right) d\xi \end{cases} \\
\Leftrightarrow \frac{\Delta\beta}{\beta}(s) &= -\frac{\beta_d(s_d) \cdot \Delta k}{2 \sin(2\pi Q_o)} \cdot \cos\left(2\left|\psi(s) - \psi(s_d)\right| - 2\pi Q_o\right) \quad \blacksquare \quad (84)
\end{aligned}$$

As a consequence of beta beating, the beta function will diverge if the natural tune in the corresponding plane is equal to a half-integer value. Hence, the second order tune resonances, indicated as blue lines in Figure (12), are caused by quadrupole field errors.

#### 5.4.4 Exact beta beating equation

In the previous derivation of the beta beating Equation (82) a Taylor-expansion up to the first order in the beating quantities was applied. Hence, higher order perturbations  $\mathcal{O}(\Delta^2)$  were implicitly neglected. It turns out that this approximation is not required and the first lines of Expression (81) and (75) can be used to derive an exact equation for the beta beating effect. The exact mathematical description of the beta beating phenomenon is demonstrated in Expression (85). Nevertheless, first order perturbation theory is sufficiently accurate for describing the beam dynamics if the residual error from simulation approximations dominates over the residual error from first order perturbation theory in the beating quantities. For the simulations presented in Chapter 9.4, higher order field contributions of the bending dipoles were not included into the consideration. Consequently, by applying an orbit bump additional feed-downs from the bending dipoles are expected which generate supplementary beating waves. The impact of the bending dipole artefacts on the residual error of the correction method is assumed to be larger than the residual error from first order perturbation theory. As a result, the simulation studies conducted within this thesis are based on first order perturbation theory in the beating quantities.

$$\begin{aligned}
(1) : \quad \frac{\Delta\beta}{\beta}(s) &\equiv \frac{\beta_\epsilon - \beta}{\beta} \stackrel{(81)}{=} \frac{\beta_d}{f_\epsilon} \frac{\sin \Delta\psi \cdot \sin(\Delta\psi - \psi_o)}{\sin \psi_\epsilon} + \frac{\sin \psi_o}{\sin \psi_\epsilon} - 1 \\
(2) : \quad \frac{1}{f_\epsilon} &\stackrel{(75)}{=} 2 \cdot \frac{\cos \psi_o - \cos \psi_\epsilon}{\beta_d \cdot \sin \psi_o} \\
[(1) \circ (2)] : \quad \frac{\Delta\beta}{\beta}(s) &= \frac{2(\cos \psi_o - \cos \psi_\epsilon) \cdot \sin \Delta\psi \cdot \sin(\Delta\psi - \psi_o)}{\sin \psi_o \cdot \sin \psi_\epsilon} + \frac{\sin \psi_o}{\sin \psi_\epsilon} - 1 \\
&\Leftrightarrow \frac{\Delta\beta}{\beta}(s) = \frac{(\cos \psi_o - \cos \psi_\epsilon) \cdot (\cos \psi_o - \cos(\psi_o - 2\Delta\psi))}{\sin \psi_o \cdot \sin \psi_\epsilon} + \frac{\sin \psi_o}{\sin \psi_\epsilon} - 1 \\
&\Leftrightarrow \frac{\Delta\beta}{\beta}(s) = \frac{2(\cos \psi_o - \cos \psi_\epsilon) \cdot (\cos \psi_o - \cos(\psi_o - 2\Delta\psi)) + 2\sin^2(\psi_o)}{\cos(\psi_\epsilon - \psi_o) - \cos(\psi_\epsilon + \psi_o)} - 1 \\
&\Leftrightarrow \frac{\Delta\beta}{\beta}(s) = \frac{2 - 2\cos \psi_o \cdot \cos \psi_\epsilon - 2(\cos \psi_o - \cos \psi_\epsilon) \cdot \cos(\psi_o - 2\Delta\psi)}{\cos(\psi_\epsilon - \psi_o) - \cos(\psi_\epsilon + \psi_o)} - 1 \\
&\Leftrightarrow \frac{\Delta\beta}{\beta}(s) = \frac{2 - 2\cos \psi_o \cdot \cos \psi_\epsilon - 4\sin(\pi Q_-) \cdot \sin(\pi Q_+) \cdot \cos(\psi_o - 2\Delta\psi)}{\cos(\psi_\epsilon - \psi_o) - \cos(\psi_\epsilon + \psi_o)} - 1 \tag{85} \\
&\Leftrightarrow \frac{\Delta\beta}{\beta}(s) = \frac{2\sin^2(\pi Q_+) + 2\sin^2(\pi Q_-) - 4\sin(\pi Q_-) \cdot \sin(\pi Q_+) \cdot \cos(\psi_o - 2\Delta\psi)}{\cos(\psi_\epsilon - \psi_o) - \cos(\psi_\epsilon + \psi_o)} - 1 \\
&\Leftrightarrow \frac{\Delta\beta}{\beta}(s) = \frac{\sin^2(\pi Q_+) + \sin^2(\pi Q_-) - 2\sin(\pi Q_-) \cdot \sin(\pi Q_+) \cdot \cos(\psi_o - 2\Delta\psi)}{\sin^2(\pi Q_+) - \sin^2(\pi Q_-)} - 1 \\
&\Leftrightarrow \frac{\Delta\beta}{\beta}(s) = \frac{1 + \lambda^2 - 2\lambda \cdot \cos(\psi_o - 2\Delta\psi)}{1 - \lambda^2} - 1 \quad , \quad \text{with: } \lambda \equiv \frac{\sin(\pi Q_-)}{\sin(\pi Q_+)} \\
&\Leftrightarrow \frac{\Delta\beta}{\beta}(s) = \frac{2\lambda^2 - 2\lambda \cdot \cos(\psi_o - 2\Delta\psi)}{1 - \lambda^2} \\
&\Leftrightarrow \frac{\Delta\beta}{\beta}(s) = \frac{2\lambda^2 - 2\lambda \cdot \cos\left(2\left|\psi(s) - \psi(s_d)\right| - 2\pi Q_o\right)}{1 - \lambda^2}
\end{aligned}$$

Here, the notation of  $Q_- \equiv Q_\epsilon - Q_o$  and  $Q_+ \equiv Q_\epsilon + Q_o$  was adapted for simplification. Furthermore, a new parameter  $\lambda$  was defined which quantifies the ratio between the modes of  $Q_-$  and  $Q_+$ . Expression (82) and (85) are manifestations of the same phenomenon in different parameterisations. The approximative Equation (82) only depends on unperturbed beam parameters as well as the magnitude of the local quadrupole perturbation  $\Delta k$ . In contrast, the exact Equation (85) does not depend on the local perturbation magnitude  $\Delta k$  but on the global tune shift  $\Delta Q$  instead.

### 5.4.5 Observed beta beating

In general, dipole field errors cause first order resonances and quadrupole field distortions generate second order resonances. This series is continued by sextupole field imperfections, which lead to resonances of third order. In contrast to the tune shift  $\Delta Q$ , the distortion of the beta function  $\Delta\beta$  oscillates around the ring with twice the betatron frequency  $\psi(s)$ . As previously observed, the beta function acts as a sensitivity coefficient regarding the perturbation of the quadrupole field strength  $\delta k(s_d)$ . Figure (13) visualizes the magnitude of beta beating in the LHC. The relative perturbation increases inversely proportional to the beta function  $\beta^*$  at the interaction point.

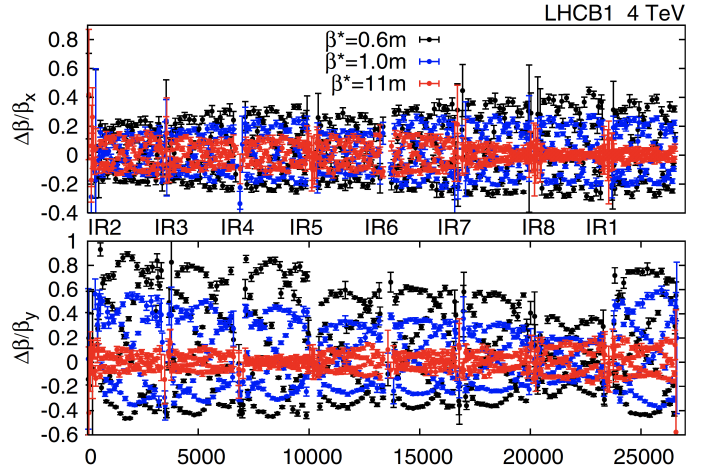


Figure 13: Effect of beta beating at the LHC for different optic configurations. The data sets are characterised by the value of the beta function at the interaction point, labelled as  $\beta^*$  [8].

### 5.4.6 Feed-down from sextupoles

If a particle beam passes a sextupole with an offset from the closed orbit, field components of lower order are triggered. Suppose a closed orbit offset of  $\Delta x_o$  is present at the location of a sextupole. The offset from the closed orbit induces an additional quadrupole component  $\mathcal{O}(x)$  as well as a dipole component  $\mathcal{O}(1)$ . This phenomenon is proven by Equation (86).

$$\begin{aligned} B_y(x) &\stackrel{(14)}{=} \frac{k_2}{2} \cdot (x + \Delta x_o)^2 \\ \Leftrightarrow B_y(x) &= \frac{k_2}{2} \cdot x^2 + k_2 \cdot \Delta x_o \cdot x + \frac{k_2}{2} \cdot \Delta x_o^2 \end{aligned} \quad (86)$$

An additional dipole component will induce a closed orbit offset and change the observed orbit of synchronous particles according to equation (61). In practice, the oscillations around the closed orbit are matter of interest. Hence, the difference between model orbit and closed orbit is not considered in the following studies. The quadrupole feed-down can be used for the elimination of quadrupolar distortions at locations where it is unfeasible to correct with lattice quadrupoles.

### 5.4.7 Alpha beating due to quadrupolar errors

A quadrupolar distortion perturbs the focusing properties of the lattice. Hence, all optical characteristics get distorted. The ellipse equation (42) teaches us that under constant action  $J$  a perturbation in beta will also be noticeable in alpha. Furthermore, an overall tune shift  $\Delta Q$  will result from an accumulation in phase beating. As a result, it has to be considered that all beam parameters will be affected by a quadrupolar perturbation. The magnitude of alpha beating, under thin lens approximation, is determined by Equation (87).

$$\begin{aligned}
\alpha(s) &\stackrel{(42)}{=} -\frac{\beta'(s)}{2} \longrightarrow \Delta\alpha(s) = -\frac{\Delta\beta'(s)}{2} \\
\Delta\alpha(s) &\stackrel{(82)}{=} -\frac{\beta'(s)}{2} \cdot \frac{\Delta\beta}{\beta}(s) - \frac{\beta(s) \cdot \beta_d(s_d)}{4f_\epsilon \sin(2\pi Q_o)} \cdot \sin\left(2\left|\psi(s) - \psi(s_d)\right| - 2\pi Q_o\right) \cdot \theta(\psi(s) - \psi(s_d)) \cdot 2\psi'(s) \\
\Leftrightarrow \Delta\alpha(s) &\stackrel{(39)}{=} \alpha(s) \cdot \frac{\Delta\beta}{\beta}(s) - \theta(s - s_d) \cdot \frac{\beta_d}{2f_\epsilon \sin\psi_o} \cdot \sin\left(2\left|\psi(s) - \psi_d\right| - \psi_o\right)
\end{aligned} \tag{87}$$

#### 5.4.8 Mismatch vector

The phenomena of beta beating (84) and alpha beating (87) can be merged together by constructing a mismatch vector  $\vec{\mu}$ . The definition is stated in Equation (88).

$$\vec{\mu}(s) \equiv \begin{pmatrix} \frac{\Delta\beta}{\beta}(s) \\ \theta(s - s_d) \cdot \left[ \Delta\alpha(s) - \alpha(s) \cdot \frac{\Delta\beta}{\beta}(s) \right] \end{pmatrix} = -\frac{\beta_d}{2f_\epsilon \sin\psi_o} \cdot \begin{pmatrix} \cos\left(2\left|\psi(s) - \psi_d\right| - \psi_o\right) \\ \sin\left(2\left|\psi(s) - \psi_d\right| - \psi_o\right) \end{pmatrix} \tag{88}$$

The mismatch vector  $\mu$  has zero norm if the lattice is unperturbed. If a perturbation comes into play then the vector spans a circle with certain radius. This radius will change if there are several distortions around the ring. Hence, a sudden change in the norm of the mismatch vector can be associated with the location of a distortion. As a consequence, the norm of the mismatch vector can be used as an error detection indicator.

#### 5.4.9 Phase beating due to quadrupolar errors

As pointed out earlier, a quadrupolar distortion will also perturb the phase advance between the lattice elements. In order to quantify this effect it is advisable to choose a similar approach as for beta beating. Hence, the distorted one-turn map will be considered at a different location.

$$\begin{aligned}
\mathcal{M}_\epsilon(s_f + L|s_i) &= \mathcal{M}_\epsilon(s_f + L|s_f) \cdot \mathcal{M}(s_f|s_i) \quad , \quad \forall \{s_i, s_f < s_d \vee s_i, s_f > s_d\} \mid s_i < s_f \\
\Leftrightarrow \mathcal{M}_\epsilon(s_f + L|s_i) &\stackrel{(46)}{=} \mathcal{M}_\epsilon(s_f + L|s_f) \cdot \begin{bmatrix} \sqrt{\frac{\beta_f}{\beta_i}} \cdot (\cos\psi_{fi} + \alpha_i \sin\psi_{fi}) & \sqrt{\beta_i\beta_f} \cdot \sin\psi_{fi} \\ \frac{\alpha_i - \alpha_f}{\sqrt{\beta_i\beta_f}} \cdot \cos\psi_{fi} - \frac{1 + \alpha_i\alpha_f}{\sqrt{\beta_i\beta_f}} \cdot \sin\psi_{fi} & \sqrt{\frac{\beta_i}{\beta_f}} \cdot (\cos\psi_{fi} - \alpha_f \sin\psi_{fi}) \end{bmatrix}
\end{aligned} \tag{89}$$

The perturbed transfer map  $\mathcal{M}_\epsilon(s_f + L|s_i)$  corresponds to the nominal transfer map from Equation (46) in the picture of the distorted twiss parameters. Since it is desired to evaluate the phase beating, only entry  $\mathcal{M}_\epsilon(s_f + L|s_i)[1,2]$  has to be evaluated. Perturbation theory is applied to the matrix element in Equation (90).

$$\begin{aligned}
\mathcal{M}_\epsilon(s_f + L|s_i)[1, 2] &\stackrel{(46)}{=} \sqrt{\beta_i^{(\epsilon)} \beta_f^{(\epsilon)}} \cdot \sin(\psi_{fi}^{(\epsilon)} + \psi_o^{(\epsilon)}) \\
\Leftrightarrow \mathcal{M}_\epsilon(s_f + L|s_i)[1, 2] &= \sqrt{(\beta_i + \Delta\beta_i) \cdot (\beta_f + \Delta\beta_f)} \cdot \sin(\psi_{fi} + \Delta\psi_{fi} + \psi_o + 2\pi\Delta Q) + \mathcal{O}(\Delta^2) \\
\Leftrightarrow \mathcal{M}_\epsilon(s_f + L|s_i)[1, 2] &= \sqrt{\beta_i \beta_f \cdot \left(1 + \frac{\Delta\beta}{\beta}\Big|_i + \frac{\Delta\beta}{\beta}\Big|_f\right)} \\
&\quad \times \left[ (\sin \psi_{fi} \cdot \cos \Delta\psi_{fi} + \cos \psi_{fi} \cdot \sin \Delta\psi_{fi}) \cdot (\cos \psi_o \cdot \cos(2\pi\Delta Q) - \sin \psi_o \cdot \sin(2\pi\Delta Q)) \right. \\
&\quad \left. + (\cos \psi_{fi} \cdot \cos \Delta\psi_{fi} - \sin \psi_{fi} \cdot \sin \Delta\psi_{fi}) \cdot (\sin \psi_o \cdot \cos(2\pi\Delta Q) + \cos \psi_o \cdot \sin(2\pi\Delta Q)) \right] \\
&\quad + \mathcal{O}(\Delta^2) \\
\Leftrightarrow \mathcal{M}_\epsilon(s_f + L|s_i)[1, 2] &= \sqrt{\beta_i \beta_f \cdot \left(1 + \frac{\Delta\beta}{\beta}\Big|_i + \frac{\Delta\beta}{\beta}\Big|_f\right)} \cdot \left[ (\sin \psi_{fi} + \Delta\psi_{fi} \cos \psi_{fi}) \cdot (\cos \psi_o - 2\pi\Delta Q \sin \psi_o) \right. \\
&\quad \left. + (\cos \psi_{fi} - \Delta\psi_{fi} \sin \psi_{fi}) \cdot (\sin \psi_o + 2\pi\Delta Q \cos \psi_o) \right] + \mathcal{O}(\Delta^2) \\
\Leftrightarrow \mathcal{M}_\epsilon(s_f + L|s_i)[1, 2] &= \sqrt{\beta_i \beta_f \cdot \left(1 + \frac{\Delta\beta}{\beta}\Big|_i + \frac{\Delta\beta}{\beta}\Big|_f\right)} \cdot \left[ \sin(\psi_{fi} + \psi_o) + (\Delta\psi_{fi} + 2\pi\Delta Q) \cdot \cos(\psi_{fi} + \psi_o) \right] + \mathcal{O}(\Delta^2) \\
\Leftrightarrow \mathcal{M}_\epsilon(s_f + L|s_i)[1, 2] &= \left( \sqrt{\beta_i \beta_f} + \frac{\sqrt{\beta_i \beta_f}}{2} \left( \frac{\Delta\beta}{\beta}\Big|_i + \frac{\Delta\beta}{\beta}\Big|_f \right) \right) \cdot \left[ \sin(\psi_{fi} + \psi_o) + (\Delta\psi_{fi} + 2\pi\Delta Q) \cdot \cos(\psi_{fi} + \psi_o) \right] + \mathcal{O}(\Delta^2) \\
\Leftrightarrow \mathcal{M}_\epsilon(s_f + L|s_i)[1, 2] &= \sqrt{\beta_i \beta_f} \sin(\psi_{fi} + \psi_o) + \sqrt{\beta_i \beta_f} (\Delta\psi_{fi} + 2\pi\Delta Q) \cdot \cos(\psi_{fi} + \psi_o) \\
&\quad + \frac{\sqrt{\beta_i \beta_f}}{2} \left( \frac{\Delta\beta}{\beta}\Big|_i + \frac{\Delta\beta}{\beta}\Big|_f \right) \cdot \sin(\psi_{fi} + \psi_o) + \mathcal{O}(\Delta^2)
\end{aligned} \tag{90}$$

The result of first order perturbation theory (90) has to be compared to the corresponding element of the transfer map  $\mathcal{M}_\epsilon(s_f + L|s_i)[1, 2]$ . For that reason, the Ansatz in Equation (89) has to be calculated in detail, what is done in Equation (91).

$$\begin{aligned}
\mathcal{M}_\epsilon(s_f + L|s_i)[1, 2] &= [\hat{\mathbf{e}}_1^T \times \mathcal{M}_\epsilon(s_f + L|s_f)] \times [\mathcal{M}(s_f|s_i) \times \hat{\mathbf{e}}_2] \\
&\stackrel{(80)}{\Leftrightarrow} \begin{pmatrix} \cos \psi_o + \alpha_f \sin \psi_o + \frac{\beta_d}{f_\epsilon} \sin(\Delta\psi_f - \psi_o) \cdot [\cos \Delta\psi_f + \alpha_f \sin \Delta\psi_f] \\ \frac{\beta_d \beta_f}{f_\epsilon} \sin \Delta\psi_f \cdot \sin(\Delta\psi_f - \psi_o) + \beta_f \sin \psi_o \end{pmatrix}^T \times \begin{pmatrix} \sqrt{\beta_i \beta_f} \sin \psi_{fi} \\ \sqrt{\frac{\beta_i}{\beta_f}} \cdot [\cos \psi_{fi} - \alpha_f \sin \psi_{fi}] \end{pmatrix} \\
&\hookrightarrow \text{with: } \begin{cases} \text{Downstream: } s_f > s_d, & \Delta\psi_f \equiv \psi(s_d + L) - \psi(s_f) = \psi(s_d) - \psi(s_f) + 2\pi Q_o \\ \text{Upstream: } s_f < s_d, & \Delta\psi_f \equiv \psi(s_d) - \psi(s_f) \end{cases} \\
&\Leftrightarrow \sqrt{\beta_i \beta_f} \sin(\psi_{fi} + \psi_o) + \frac{\sqrt{\beta_i \beta_f} \beta_d}{f_\epsilon} \sin(\Delta\psi_f - \psi_o) \cdot \sin(\psi_{fi} + \Delta\psi_f)
\end{aligned} \tag{91}$$

Equating (90) with (91) results in a conditional equation for the phase beating  $\Delta\psi_{fi}$  between the element at  $s_i$  and the element at  $s_f$ . The conditional equation is given by (92).

$$\begin{aligned}
\frac{\beta_d}{f_\epsilon} \sin(\Delta\psi_f - \psi_o) \cdot \sin(\psi_{fi} + \Delta\psi_f) &\stackrel{!}{\approx} (\Delta\psi_{fi} + 2\pi\Delta Q) \cdot \cos(\psi_{fi} + \psi_o) + \frac{1}{2} \left( \frac{\Delta\beta}{\beta}\Big|_i + \frac{\Delta\beta}{\beta}\Big|_f \right) \cdot \sin(\psi_{fi} + \psi_o) \\
\Leftrightarrow \Delta\psi_{fi} + 2\pi\Delta Q &= \frac{\beta_d \sin(\Delta\psi_f - \psi_o) \cdot \sin(\psi_{fi} + \Delta\psi_f)}{\cos(\psi_{fi} + \psi_o)} - \frac{1}{2} \left( \frac{\Delta\beta}{\beta}\Big|_i + \frac{\Delta\beta}{\beta}\Big|_f \right) \cdot \tan(\psi_{fi} + \psi_o)
\end{aligned} \tag{92}$$

Substituting Equation (82) into Equation (92) results in Relation (93).



$$\Delta\psi_{fi} + 2\pi\Delta Q \stackrel{(82)}{=} \frac{\beta_d}{2f_\epsilon \sin\psi_o} \cdot \left[ \frac{2\sin(\Delta\psi_f - \psi_o) \cdot \sin(\psi_{fi} + \Delta\psi_f) \cdot \sin\psi_o}{\cos(\psi_{fi} + \psi_o)} + \frac{\cos(\psi_o - 2\Delta\psi_i) + \cos(\psi_o - 2\Delta\psi_f)}{2\cotan(\psi_{fi} + \psi_o)} \right]$$

$$\Leftrightarrow \text{with: } \begin{cases} \text{Downstream: } s_f > s_d, & \Delta\psi_f \equiv \psi(s_d + L) - \psi(s_f) = \psi(s_d) - \psi(s_f) + 2\pi Q_o \\ & s_i > s_d, & \Delta\psi_i \equiv \psi(s_d + L) - \psi(s_i) = \psi(s_d) - \psi(s_i) + 2\pi Q_o \\ \text{Upstream: } s_f < s_d, & \Delta\psi_f \equiv \psi(s_d) - \psi(s_f) \\ & s_i < s_d, & \Delta\psi_i \equiv \psi(s_d) - \psi(s_i) \end{cases} \quad (93)$$

$$\Leftrightarrow \Delta\psi_{fi} = \frac{\beta_d}{2f_\epsilon \sin\psi_o} \cdot \sin\psi_{fi} \cdot \begin{cases} \cos(\psi_o - \psi(s_f) - \psi(s_i) + 2\psi(s_d)) & , \quad \forall s_i, s_f > s_d \\ \cos(\psi_o + \psi(s_f) + \psi(s_i) - 2\psi(s_d)) & , \quad \forall s_i, s_f < s_d \end{cases}$$

$$\Leftrightarrow \Delta\psi_{fi} = \frac{\beta_d}{2f_\epsilon \sin(2\pi Q_o)} \cdot \sin\psi_{fi} \cdot \cos(2\pi Q_o - |\psi_f - \psi_d| - |\psi_i - \psi_d|) \quad , \quad \forall \{s_i, s_f < s_d \vee s_i, s_f > s_d\}$$

Equation (93) quantifies phase beating up to first order terms in perturbation theory. Ansatz (89) is valid for the two main scenarios. The downstream case, where  $s_d < s_i < s_f$  holds, and the upstream case, where  $s_i < s_f < s_d$  is fulfilled. Nevertheless, the phase distortion around the perturbation, where  $s_i < s_d < s_f$  is valid, is not covered by this Ansatz. Consequently, this case has to be treated separately, what is done by Ansatz (94).

$$\mathcal{M}_\epsilon(s_f|s_i) = \mathcal{M}(s_f|s_d) \cdot \begin{bmatrix} 1 & 0 \\ -f_\epsilon^{-1} & 1 \end{bmatrix} \cdot \mathcal{M}(s_d|s_i) \quad , \quad \forall s_i < s_d < s_f \quad (94)$$

$$\Rightarrow \mathcal{M}_\epsilon(s_f|s_i)[1, 2] \stackrel{(46)}{=} \sqrt{\beta_i \beta_f} \cdot \left[ \sin\psi_{fi} - \frac{\beta_d}{f_\epsilon} \sin\psi_{di} \cdot \sin\psi_{fd} \right]$$

Analogous to the perturbation treatment in Equation (90), a similar expression can be derived for the matrix element  $\mathcal{M}_\epsilon(s_f|s_i)[1, 2]$ . The resulting expression, considering first order terms only, is presented in Equation (95).

$$\mathcal{M}_\epsilon(s_f + L|s_i)[1, 2] \stackrel{(46)}{=} \sqrt{\beta_i^{(\epsilon)} \beta_f^{(\epsilon)}} \cdot \sin\psi_{fi}^{(\epsilon)}$$

$$\Leftrightarrow \mathcal{M}_\epsilon(s_f + L|s_i)[1, 2] = \sqrt{(\beta_i + \Delta\beta_i) \cdot (\beta_f + \Delta\beta_f)} \cdot \sin(\psi_{fi} + \Delta\psi_{fi}) + \mathcal{O}(\Delta^2)$$

$$\Leftrightarrow \mathcal{M}_\epsilon(s_f + L|s_i)[1, 2] = \sqrt{\beta_i \beta_f} \cdot \left( 1 + \frac{\Delta\beta}{\beta} \Big|_i + \frac{\Delta\beta}{\beta} \Big|_f \right) \cdot (\sin\psi_{fi} + \Delta\psi_{fi} \cos\psi_{fi}) + \mathcal{O}(\Delta^2) \quad (95)$$

$$\Leftrightarrow \mathcal{M}_\epsilon(s_f + L|s_i)[1, 2] = \sqrt{\beta_i \beta_f} \cdot \left[ \sin\psi_{fi} + \Delta\psi_{fi} \cos\psi_{fi} + \frac{\sin\psi_{fi}}{2} \left( \frac{\Delta\beta}{\beta} \Big|_i + \frac{\Delta\beta}{\beta} \Big|_f \right) \right] + \mathcal{O}(\Delta^2)$$

Equating (94) with (95) results in a conditional equation for  $\Delta\psi_{fi}$  in case of  $s_i < s_d < s_f$ , which is indicated by Equation (96).

$$-\frac{\beta_d}{f_\epsilon} \sin\psi_{di} \cdot \sin\psi_{fd} \stackrel{!}{\approx} \Delta\psi_{fi} \cos\psi_{fi} + \frac{\sin\psi_{fi}}{2} \left( \frac{\Delta\beta}{\beta} \Big|_i + \frac{\Delta\beta}{\beta} \Big|_f \right)$$

$$\stackrel{(82)}{\Leftrightarrow} \Delta\psi_{fi} = \frac{\beta_d}{2f_\epsilon \sin(2\pi Q_o)} \cdot \left[ \frac{\cos(\psi_o - 2\Delta\psi_i) + \cos(\psi_o - 2\Delta\psi_f)}{2\cotan\psi_{fi}} - \frac{2\sin\psi_{di} \cdot \sin\psi_{fd} \cdot \sin\psi_o}{\cos\psi_{fi}} \right] \quad (96)$$

Since the initial position  $s_i$  is upstream in relation to the location of the distortion at  $s_d$ , the phase advance  $\Delta\psi_i$  is given by  $\Delta\psi_i \equiv \psi(s_d) - \psi(s_i)$ . On the other hand, the final position  $s_f$  is downstream in relation to the location of the distortion at  $s_d$ . As a consequence, the phase advance  $\Delta\psi_f$  is

characterised by  $\Delta\psi_f \equiv \psi(s_d) - \psi(s_f) + 2\pi Q_\circ$ . The resulting equation for  $\Delta\psi_{fi}$  is stated in (97).

$$\Delta\psi_{fi} = \frac{\beta_d}{2f_\epsilon \sin(2\pi Q_\circ)} \cdot \left[ \frac{\cos(\psi_\circ - 2\psi_{di}) + \cos(\psi_\circ - 2\psi_{fd})}{2\cotan\psi_{fi}} - \frac{2\sin\psi_{di} \cdot \sin\psi_{fd} \cdot \sin\psi_\circ}{\cos\psi_{fi}} \right] \quad (97)$$

$$\stackrel{(75)}{\Leftrightarrow} \Delta\psi_{fi} = 2\pi\Delta Q - \frac{\beta_d}{2f_\epsilon \sin(2\pi Q_\circ)} \cdot \cos(\psi_f + \psi_i - 2\psi_d) \cdot \sin(\psi_\circ - \psi_{fi}) \quad , \quad \forall s_i < s_d < s_f$$

It is worth noticing that the case differentiation for  $s_i < s_d < s_f$  is technically necessary but does not represent an independent conditional equation for  $\Delta\psi_{fi}$ . In fact, Equation (97) is composed of a linear combination of the downstream and upstream case of Equation (93). In order to illustrate this in more detail, Equation (98) demonstrates an alternative Ansatz for the derivation of phase beating in case of  $s_i < s_d < s_f$ .

$$\begin{aligned} \Delta\psi_{fi} &= \Delta\psi_{fd+} + \Delta\psi_{d-i} \quad \text{with: } s_{d-} + \epsilon \equiv s_d \equiv s_{d+} - \epsilon \quad \Big| \quad \epsilon \gtrsim 0 \quad , \quad \forall s_i < s_d < s_f \\ \Leftrightarrow \Delta\psi_{fi} &\stackrel{(93)}{=} \frac{\beta_d}{2f_\epsilon \sin(2\pi Q_\circ)} \cdot [\sin\psi_{fd+} \cdot \cos(2\pi Q_\circ - \psi_{fd} - \psi_{d+}) + \sin\psi_{d-i} \cdot \cos(2\pi Q_\circ - \psi_{dd-} - \psi_{di})] \\ \Rightarrow \Delta\psi_{fi} &\approx \frac{\beta_d}{2f_\epsilon \sin(2\pi Q_\circ)} \cdot [\sin\psi_{fd} \cdot \cos(2\pi Q_\circ - \psi_{fd}) + \sin\psi_{di} \cdot \cos(2\pi Q_\circ - \psi_{di})] \\ \Leftrightarrow \Delta\psi_{fi} &= 2\pi\Delta Q_\circ - \frac{\beta_d}{4f_\epsilon \sin(2\pi Q_\circ)} \cdot [\sin(2\pi Q_\circ - 2\psi_{fd}) + \sin(2\pi Q_\circ - 2\psi_{di})] \\ \Leftrightarrow \Delta\psi_{fi} &= 2\pi\Delta Q_\circ - \frac{\beta_d}{2f_\epsilon \sin(2\pi Q_\circ)} \cdot \cos(\psi_f + \psi_i - 2\psi_d) \cdot \sin(\psi_\circ - \psi_{fi}) \quad , \quad \forall s_i < s_d < s_f \quad \blacksquare \end{aligned} \quad (98)$$

In conclusion, Equation (93) fully incorporates the effect of phase beating mathematically.

## 5.5 Off-momentum particles

In the previous sections, the dynamics of particles with design momentum has been studied. Naturally, the particle momentum can fluctuate around the expectation value which results in off-momentum particles. If the momentum of a particle shows a deviation  $\Delta p$  from the reference value  $\langle p \rangle$ , the equation of motion has to be modified accordingly. The equation of motion is characterised by the homogeneous Hill's differential Equation (20) and can be adapted to off-momentum particles by expanding the momentum around the expectation value  $\langle p \rangle$ . By applying a Taylor expansion to the rational dependency on  $p$ , as presented in Equation (21) for the horizontal displacement  $x$ , the equation of motion for off-momentum particles results. This differential equation has a constant inhomogeneity and is known as the inhomogeneous Hill's differential Equation (99).

$$\begin{aligned} 0 &\stackrel{(20)}{=} \left( \frac{\rho_0}{\rho_0 + x} \right)^2 \cdot x'' - \frac{1}{\rho_0 + x} + \frac{qB_y}{\gamma m v_s} \\ \Leftrightarrow 0 &= x'' - \frac{\rho_0 + x}{\rho_0^2} + \frac{qB_y}{\langle p \rangle + \Delta p} \cdot \left( \frac{\rho_0 + x}{\rho_0} \right)^2 \quad , \quad \text{with: } \gamma m v_s = \langle p \rangle \cdot \left( 1 + \frac{\Delta p}{\langle p \rangle} \right) \equiv \langle p \rangle \cdot (1 + \delta) \\ \Leftrightarrow 0 &= x'' - \frac{\rho_0 + x}{\rho_0^2} + \left( \frac{1}{\rho_0} + gx \right) \cdot \frac{1}{1 + \delta} \cdot \left( 1 + \frac{x}{\rho_0} \right)^2 \quad , \quad \text{with: } \frac{q}{\langle p \rangle} B_y(x) \stackrel{!}{=} \frac{1}{\rho_0} + gx \\ \Leftrightarrow 0 &\stackrel{(21)}{=} x'' - \frac{\rho_0 + x}{\rho_0^2} + \left( \frac{1}{\rho_0} + gx \right) \cdot (1 - \delta) \cdot \left( 1 + \frac{2x}{\rho_0} \right) + \mathcal{O}(\Delta^2) \\ \Leftrightarrow 0 &= x'' - \frac{\rho_0 + x}{\rho_0^2} + \frac{1}{\rho_0} + gx + \frac{2x}{\rho_0^2} - \frac{\delta}{\rho_0} + \mathcal{O}(\Delta^2) = x'' + \left( \frac{1}{\rho_0^2} + g \right) \cdot x - \frac{\delta}{\rho_0} + \mathcal{O}(\Delta^2) \\ \Rightarrow \frac{\delta}{\rho_0} &\stackrel{(23)}{\approx} x'' + k(s) \cdot x \end{aligned} \quad (99)$$

In sum, the homogeneous Hill's differential equation results from a linear expansion in the position displacement and the inhomogeneous version accounts for linear orders of the momentum dispersion  $\delta$  additionally. If higher order terms of the Taylor expansion are considered more sophisticated effects need to be studied, as the chromaticity for example. The driving term of the inhomogeneous Hill's differential equation leads to an overall trajectory which is composed by the sum of the homogeneous  $x_h(s)$  and the inhomogeneous solution  $x_i(s)$ . The inhomogeneous contribution of off-momentum particles can be interpreted as a closed orbit offset around which the homogeneous solution oscillates (betatron motion). Generally, an isolated dispersion term only appears in the horizontal plane since the momentum perturbation causes exclusively higher order distortions  $\mathcal{O}(\Delta^2)$  in the vertical plane. Alternatively, one could argue that the bending radius  $\rho_0$  is infinite in the vertical plane wherefore the inhomogeneity vanishes. Hence, there is no first order dispersion term in the vertical plane.

### 5.5.1 Dispersion function

In general, there is a distinct orbit  $D(s)$  which is followed by a particle with a dispersion  $\delta = 1$ . Since the inhomogeneity in Equation (99) is linear in  $\delta$  the off-momentum dispersion offset  $x_i(s)$  can be expressed as an amplitude modulation of the orbit  $D(s)$ . Consequently, the full trajectory of an off-momentum particle is given by Equation (100), where  $x_\beta(s)$  indicates the betatron oscillations resulting from the homogeneous Hill's differential equation.

$$x(s) = x_h(s) + x_i(s) = x_\beta(s) + \delta \cdot D(s) \quad , \quad \text{with : } D(s) \equiv \frac{\partial x_i(s)}{\partial \delta} \quad (100)$$

The closed orbit shift  $x_i(s) = \delta \cdot D(s)$ , due to the momentum deviation of the particle, has to be understood as an orbit offset in addition to the closed orbit distortion of the lattice dipoles. It is of particular interest to determine the dispersion function  $D(s)$  in order to describe the particle trajectory completely. For simplicity, a lattice which is restricted to zeroth order fields is considered. Hence, the inhomogeneous Hill's differential Equation (99) simplifies to (101).

$$\frac{d^2 D(s)}{ds^2} + \frac{D(s)}{\rho_0^2} = \frac{1}{\rho_0} \quad , \quad \text{with : } x_i(s) \stackrel{(100)}{=} \delta \cdot D(s) \quad (101)$$

The full solution of the inhomogeneous differential Equation (101) consists of the sum of homogeneous and particular solution. In order to find the particular solution a similarity Ansatz according to the shape of the inhomogeneity is made. Hence, the Ansatz for the particular solution is given by  $D_p(s) = \kappa$ , where  $\kappa$  is a constant. Inserting the Ansatz of  $D_p(s)$  into the inhomogeneous Hill's differential Equation (101) leads to the particular solution of  $D_p(s) = \rho_0$ . The homogeneous version of Equation (101) is equivalent to the equation of motion of a harmonic oscillator. Consequently, the homogeneous solution is given by a trigonometric function with two degrees of freedom. As a result, the full solution of the inhomogeneous Hill's differential Equation (101) for dipoles is specified by Expression (102).

$$\begin{aligned} D(s) &= c_1 \cos\left(\frac{s}{\rho_0}\right) + c_2 \sin\left(\frac{s}{\rho_0}\right) + \rho_0 \\ D'(s) &\equiv \frac{dD(s)}{ds} = -\frac{c_1}{\rho_0} \sin\left(\frac{s}{\rho_0}\right) + \frac{c_2}{\rho_0} \cos\left(\frac{s}{\rho_0}\right) \end{aligned} \quad (102)$$

The two degrees of freedom of Equation (102) are fixed by initial conditions for  $D(s_0)$  and  $D'(s_0)$ .

Without loss of generality, the initial values are defined as  $D(s=0) \equiv D_0$  and  $D'(s=0) \equiv D'_0$ . Consequently, the initial values determine the coefficients  $c_1$  and  $c_2$  to be equal to  $c_1 = D_0 - \rho_0$  and  $c_2 = \rho_0 D'_0$ . Replacing the coefficients  $c_1$  and  $c_2$  of the dispersion function by the initial values results in Expression (103).

$$\begin{aligned} D(s) &= D_0 \cos\left(\frac{s}{\rho_0}\right) + \rho_0 D'_0 \sin\left(\frac{s}{\rho_0}\right) + \rho_0 \cdot \left[1 - \cos\left(\frac{s}{\rho_0}\right)\right] \\ D'(s) &= -\frac{D_0}{\rho_0} \sin\left(\frac{s}{\rho_0}\right) + D'_0 \cos\left(\frac{s}{\rho_0}\right) + \sin\left(\frac{s}{\rho_0}\right) \end{aligned} \quad (103)$$

The system of linear equations in Expression (103) can be cast into a matrix formalism as indicated in Equation (104).

$$\begin{pmatrix} D(s) \\ D'(s) \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{s}{\rho_0}\right) & \rho_0 \sin\left(\frac{s}{\rho_0}\right) & \rho_0 \cdot \left[1 - \cos\left(\frac{s}{\rho_0}\right)\right] \\ -\frac{1}{\rho_0} \sin\left(\frac{s}{\rho_0}\right) & \cos\left(\frac{s}{\rho_0}\right) & \sin\left(\frac{s}{\rho_0}\right) \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} D_0 \\ D'_0 \\ 1 \end{pmatrix} \quad (104)$$

The transfer matrix in Equation (104) maps the dispersion vector through a dipole segment in the lattice. The transfer matrix shows a weak focusing effect of the dispersion in the dipole element due to the trigonometric dependency. This is plausible since the dispersion function is a closed orbit offset which is affected by the focusing properties of the lattice. Furthermore, the dipole itself generates additional dispersion because of the non-zero third component of the dispersion vector. As a consequence, each particle will have a different closed orbit offset due to the generated dispersion in the dipoles of the lattice. This gives rise to favour a statistical beam description of the dynamics instead of a particle by particle description. Generally, the upper-left two by two matrix in the dispersion transfer matrix (104) is equivalent to the transfer matrix of a dipole in the horizontal plane, as derived in Expression (35). This is a more general structure which can be exploited to densify the formalism. The particle trajectory through an arbitrary beamline element with transfer matrix  $\mathcal{M}(s_f|s_i)$  is given by Equation (105).

$$\begin{pmatrix} x(s) \\ x'(s) \\ \delta \end{pmatrix} \stackrel{(100)}{=} \begin{pmatrix} \mathcal{M}[1,1] & \mathcal{M}[1,2] & D(s) \\ \mathcal{M}[2,1] & \mathcal{M}[2,2] & D'(s) \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ x'_0 \\ \delta \end{pmatrix} \quad (105)$$

It is important to emphasize, that no dispersion is generated inside a quadrupole magnet since quadrupole fields do not contain zeroth order field components. As a consequence, the inhomogeneity of Hill's differential equation vanishes for quadrupoles. Furthermore, quadrupole magnets inherently focus dispersion due to the linear increase of the field strength with the elongation  $x$ . If a thin dipole is considered under the condition  $\theta \equiv \rho_0^{-1}L \ll 1$  the thin lens will effectively act as a kicker to off-momentum particles. The thin lens approximation is applied to the dipole transfer matrix (104) in Equation (106).

$$\begin{pmatrix} D(s) \\ D'(s) \\ 1 \end{pmatrix} \approx \begin{pmatrix} 1 & L & \frac{\theta L}{2} \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} D_0 \\ D'_0 \\ 1 \end{pmatrix} \quad (106)$$

### 5.5.2 Dispersion in a FODO-cell

Dispersion is driven by dipoles and the impact of dispersion on a FODO-cell can be studied by replacing the drift elements by thin bending dipoles. The symmetrical one-cell map for the bending FODO-cell is given by Equation (107).

$$\mathcal{M}_{\text{FODO}}(s + L|s) = \mathcal{M}_{\text{FQ}}(2f) \cdot \mathcal{M}_{\text{dipole}} \cdot \mathcal{M}_{\text{DQ}}(-f) \cdot \mathcal{M}_{\text{dipole}} \cdot \mathcal{M}_{\text{FQ}}(2f) \quad (107)$$

Calculating the matrix product in (107), under the thin lens approximation for the dipoles (106) as well as for the quadrupoles (33), results in the one-cell map presented in Equation (108).

$$\mathcal{M}_{\text{FODO}}(s + L|s) = \begin{bmatrix} \left(1 - \frac{L^2}{2f^2}\right) & \left(2L + \frac{L^2}{f}\right) & \left(2\theta L + \frac{\theta L^2}{2f}\right) \\ \left(\frac{L^2}{4f^3} - \frac{L}{2f^2}\right) & \left(1 - \frac{L^2}{2f^2}\right) & \left(2\theta - \frac{\theta L}{2f} - \frac{\theta L^2}{4f^2}\right) \\ 0 & 0 & 1 \end{bmatrix} \quad (108)$$

Here,  $L$  denotes the length of each dipole,  $\theta$  corresponds to the bending angle of the thin dipoles and  $f$  is the focal length of the thin quadrupoles. Note that the upper two-by-two matrix in Expression (108) is equivalent to the one-cell map (58) of a drift FODO-cell. In the LHC the bending FODO-cells are roughly 110 m long [6].

### 5.5.3 Chromaticity

In ray optics, the refraction index, as well as the focal length, of a lens depends on the wavelength. Similarly in accelerator physics, off-momentum particles are affected by a different focusing strength of the quadrupoles, wherefore they oscillate with a different betatron tune around the ring. The chromaticity  $\xi$  quantifies the variation of the betatron tune  $Q$  with the relative momentum deviation  $\delta$ , as presented in Equation (109).

$$\begin{aligned} \xi &\equiv \frac{dQ}{d\delta} \\ \Leftrightarrow \xi &\stackrel{(76)}{=} -\frac{1}{4\pi} \cdot \oint \beta(s) \cdot k(s) \, ds \end{aligned} \quad (109)$$

For the second line of Equation (109), the formula for the integrated tune shift (76) based on a quadrupolar error distribution has been adapted. This is valid if the perturbation in the quadrupole strength  $\delta k$  is replaced by the equivalent contribution from the effect of dispersion. In Equation (14), the magnetic field gradient  $k$  has been normalized by the beam rigidity (11) in order to remove the dependence on the particle momentum  $p$ . Physically, the required magnetic flux density  $B_0$  for guiding a particle is determined by the product of  $k$  times the beam rigidity. As a consequence, a more rigid beam will need stronger fields in order to follow the design orbit. But if the magnetic field strengths are fixed, a more rigid beam will experience a lack of guidance. Hence, the effect of dispersion can be interpreted as a deviation in the magnetic field strength from the desired field, which would satisfy the design orbit. Since the beam rigidity is linear in  $p$ , a momentum perturbation  $\delta$  will trigger a quadrupolar distortion of  $f_\epsilon^{-1} \mapsto -\delta \cdot k(s)$ . As a result, Equation (109) is obtained which characterises the effect of chromaticity. The chromaticity  $\xi$  is a global quantity of an accelerator, like the natural tune  $Q$ . For strong focusing storage rings, the chromaticity is always negative and cannot be made zero. This results from the fact, that the beta function is large in focusing quadrupoles ( $k_1 > 0$ ) and small in

defocusing quadrupoles ( $k_1 < 0$ ). Consequently, the integral will be positive for focusing storage rings and the chromaticity will become negative. Typically, chromaticity is corrected by attaching sextupole magnets to the quadrupoles [32], which will generate additional quadrupolar feed-downs.

#### 5.5.4 Dispersion beating due to quadrupolar errors

Off-momentum particles follow a different orbit than the design orbit due to dispersion. As a consequence, off-momentum particles will pass the quadrupoles of the lattice with a non-zero expectation value in the displacement coordinate  $x$ . As a result, a dipolar feed-down is triggered, which generates ancillary dispersion and co-determines the off-momentum closed orbit. In case of a quadrupole perturbation, the off-momentum closed orbit will change due to the distortion in the generated dipolar feed-down. The impact on the off-momentum closed orbit is characterised by the relative field strength variation  $\epsilon$  of the dipolar feed-down. Hence, off-momentum particles will experience a dipolar kick  $d\theta$  at the location  $s_d$  of the quadrupole imperfection, as given by Equation (110).

$$\begin{aligned}
 d\theta &\stackrel{(64)}{=} -\Delta k_0(s_d) \cdot ds \\
 \Leftrightarrow d\theta &\stackrel{(14)}{=} -\Delta k_1(s_d) \cdot x_i(\delta, s_d) \cdot ds \\
 \Leftrightarrow d\theta &\stackrel{(100)}{=} -\epsilon \cdot k_1(s_d) \cdot \delta \cdot D(s_d) \cdot ds \\
 \Leftrightarrow d\theta &\stackrel{(72)}{=} -\frac{\delta \cdot D(s_d)}{f_\epsilon}
 \end{aligned} \tag{110}$$

The dipole kick  $d\theta$  on the off-momentum closed orbit will propagate around the ring according to Equation (68). Since the relative momentum deviation  $\delta$  is a characteristic quantity of a particle and constant around the ring (neglecting acceleration due to RF-cavities and deceleration due to synchrotron radiation), the effect of a quadrupole distortion directly transforms into a perturbation of the dispersion function  $D(s)$ , as illustrated by Equation (111).

$$\begin{aligned}
 \Delta x_i(s) &\stackrel{(68)}{=} \frac{\sqrt{\beta(s_d) \cdot \beta(s)} \cdot d\theta}{2 \sin(\pi Q_o)} \cdot \cos\left(\left|\psi(s) - \psi(s_d)\right| - \pi Q_o\right) \\
 \Leftrightarrow \Delta x_i(s) &\stackrel{(110)}{=} -\frac{\sqrt{\beta(s_d) \cdot \beta(s)} \cdot \delta \cdot D(s_d)}{2 f_\epsilon \sin(\pi Q_o)} \cdot \cos\left(\left|\psi(s) - \psi(s_d)\right| - \pi Q_o\right) \\
 \Leftrightarrow \Delta D(s) &\stackrel{(100)}{=} -\frac{\sqrt{\beta(s_d) \cdot \beta(s)} \cdot D(s_d)}{2 f_\epsilon \sin(\pi Q_o)} \cdot \cos\left(\left|\psi(s) - \psi(s_d)\right| - \pi Q_o\right)
 \end{aligned} \tag{111}$$

In other words, the quadrupolar perturbation induces a dispersion kick [33], which will propagate around the ring and cause dispersion beating.

## 6 AC-dipole formalism

### 6.1 Normalized coordinates in the Floquet picture

The one-turn map from Equation (47) can be projected onto a circle by applying a matrix decomposition according to Equation (112).

$$\begin{aligned}
\vec{\eta}_{n+1}(s) &\stackrel{(47)}{=} \begin{bmatrix} \cos(2\pi Q_\circ) + \alpha(s) \cdot \sin(2\pi Q_\circ) & \beta(s) \cdot \sin(2\pi Q_\circ) \\ -\gamma(s) \cdot \sin(2\pi Q_\circ) & \cos(2\pi Q_\circ) - \alpha(s) \cdot \sin(2\pi Q_\circ) \end{bmatrix} \cdot \vec{\eta}_n(s) \\
\Leftrightarrow \vec{\eta}_{n+1}(s) &= \begin{bmatrix} \frac{1}{\sqrt{\beta(s)}} & 0 \\ \frac{\alpha(s)}{\sqrt{\beta(s)}} & \sqrt{\beta(s)} \end{bmatrix}^{-1} \times \begin{bmatrix} \cos(2\pi Q_\circ) & \sin(2\pi Q_\circ) \\ -\sin(2\pi Q_\circ) & \cos(2\pi Q_\circ) \end{bmatrix} \times \begin{bmatrix} \frac{1}{\sqrt{\beta(s)}} & 0 \\ \frac{\alpha(s)}{\sqrt{\beta(s)}} & \sqrt{\beta(s)} \end{bmatrix} \cdot \vec{\eta}_n(s) \\
\Rightarrow \vec{\eta}_{n+1}(s) &= \begin{bmatrix} \sqrt{\beta(s)} & 0 \\ -\frac{\alpha(s)}{\sqrt{\beta(s)}} & \frac{1}{\sqrt{\beta(s)}} \end{bmatrix} \times \begin{bmatrix} \cos(2\pi Q_\circ) & \sin(2\pi Q_\circ) \\ -\sin(2\pi Q_\circ) & \cos(2\pi Q_\circ) \end{bmatrix} \times \begin{bmatrix} \frac{1}{\sqrt{\beta(s)}} & 0 \\ \frac{\alpha(s)}{\sqrt{\beta(s)}} & \sqrt{\beta(s)} \end{bmatrix} \cdot \vec{\eta}_n(s) \\
\Rightarrow \vec{\xi}_{n+1}(s) &\equiv \begin{bmatrix} \cos(2\pi Q_\circ) & \sin(2\pi Q_\circ) \\ -\sin(2\pi Q_\circ) & \cos(2\pi Q_\circ) \end{bmatrix} \cdot \vec{\xi}_n(s)
\end{aligned} \tag{112}$$

In contrast to the elements of the phase space vector  $\vec{\eta}$ , both elements of the normalized phase space vector  $\vec{\xi}$  are dimensioned in  $\sqrt{\text{m} \cdot \text{rad}}$  and are commonly termed as the Courant-Snyder coordinates. The normalized one-turn map represents a rotation by an angle of  $\angle(\vec{\xi}_n, \vec{\xi}_{n+1}) = 2\pi Q_\circ$ . In other words, the normalised phase space vector  $\vec{\xi}$  spans a circle and  $\vec{\eta}$  an ellipse.

### 6.2 Phasor notation

In the Floquet picture of normalized coordinates the one-turn map is simply a rotation of the state vector  $\vec{\xi}$  in phase space by  $2\pi Q_\circ$ . Consequently, the vector representation can be replaced by a phasor notation (113). The phasor  $\Xi$  is constructed by the normalised horizontal coordinate  $\tilde{x} \equiv \xi_1$  and the corresponding derivative  $\tilde{x}' \equiv \xi_2$  with respect to the independent variable  $s$ .

$$\Xi \equiv \tilde{x}' + i\tilde{x} \tag{113}$$

Adapting the phasor notation for the one-turn map results in Expression (114).

$$\Xi_{n+1} = e^{i2\pi Q_\circ} \cdot \Xi_n \equiv R_\circ \cdot \Xi_n \tag{114}$$

The impact of a single AC-dipole on the beam trajectory is given by the ratio of the integrated strength and the rigidity of the bending dipoles. The integrated strength of the AC-dipole depends on time, since the magnetic field of the AC-dipole (115) varies as a sinusoidal oscillation in time. Here,  $f_d$  corresponds to the driven frequency of the AC-dipole.

$$B_d(t) = \hat{B} \cdot \cos(2\pi f_d t) \tag{115}$$

Since the AC-dipole is seen by the particle beam only once per turn, the time of interest is quantized by full steps of the revolution time  $t_\circ = f_\circ^{-1}$ . A time offset  $t_i$  results, if the reference point of the lattice does not coincide with the position of the AC-dipole or if the AC-dipole is not triggered in coherence

with the particle beam. The time offset  $t_i$  can be absorbed into an additional phase  $\phi$  in the argument of the cosine. Consequently, the integrated strength of an AC-dipole is given by Equation (116). Here,  $L$  denotes the impact length of the AC-dipole on the reference particle and  $Q_d$  is defined as the driven tune of the AC-dipole [16].

$$L \cdot B_d(t) = L \cdot \hat{B} \cdot \cos(2\pi f_d \cdot (t_i + nt_o)) = L \cdot \hat{B} \cdot \cos\left(2\pi \cdot \frac{f_d}{f_o} \cdot n + \phi\right) \quad , \quad \text{with: } n \in \mathbb{N} \quad (116)$$

$$\Leftrightarrow L \cdot B_d(n) \equiv L \cdot \hat{B} \cdot \cos(2\pi Q_d \cdot n + \phi)$$

The deflection angle due to the AC-dipole kick is given by the ratio of the integrated field strength and the beam rigidity as presented in Expression (117).

$$d\theta_n = \frac{L \cdot B_d(n)}{\rho \cdot B_y^{(0)}} \stackrel{(116)}{=} \frac{L \cdot \hat{B}}{\rho \cdot B_y^{(0)}} \cdot \cos(2\pi Q_d \cdot n + \phi) \quad (117)$$

In the normalized phase space the AC-dipole deflection angle transforms into Expression (118) by exploiting the relationship from Equation (112).

$$\vec{\xi}_d = \begin{bmatrix} \frac{1}{\sqrt{\beta(s_d)}} & 0 \\ -\frac{\alpha(s_d)}{\sqrt{\beta(s_d)}} & \sqrt{\beta(s_d)} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ d\theta_n \end{bmatrix} \Bigg\} \Rightarrow d\tilde{x}'_n = \frac{L \cdot \hat{B}}{\rho \cdot B_y^{(0)}} \cdot \sqrt{\beta(s_d)} \cdot \cos(2\pi Q_d \cdot n + \phi) \triangleq d\Xi_n$$

$$\Leftrightarrow d\Xi_n \equiv \delta \cdot \cos(2\pi Q_d \cdot n + \phi) \quad (118)$$

$$\Leftrightarrow d\Xi_n = \frac{\delta}{2} \cdot \left[ e^{i(2\pi Q_d \cdot n + \phi)} + e^{-i(2\pi Q_d \cdot n + \phi)} \right]$$

$$\Leftrightarrow d\Xi_n \equiv \frac{\delta}{2} \cdot \left[ R_d^n \cdot e^{i\phi} + R_d^{-n} \cdot e^{-i\phi} \right]$$

### 6.3 Exact general solution for the linear driven motion

The net displacement phasor after  $N$  turns is given by the accumulation of the turn-by-turn impact of the AC-dipole [15]. If the input phasor  $\Xi_0$  at the downstream front of the AC-dipole is known, the phasor after  $N$  turns is determined by Equation (119).

$$\Xi_N = R_o^N \cdot \Xi_0 + (R_o^N \cdot d\Xi_0 + R_o^{N-1} \cdot d\Xi_1 + \dots + R_o^1 \cdot d\Xi_{N-1})$$

$$\Leftrightarrow \Xi_N \stackrel{(118)}{=} R_o^N \cdot \Xi_0 + \frac{\delta}{2} \cdot R_o^N \cdot e^{i\phi} \cdot \left[ 1 + \left(\frac{R_d}{R_o}\right) + \left(\frac{R_d}{R_o}\right)^2 + \dots + \left(\frac{R_d}{R_o}\right)^{N-1} \right] \quad (119)$$

$$+ \frac{\delta}{2} \cdot R_o^N \cdot e^{-i\phi} \cdot \left[ 1 + (R_d \cdot R_o)^{-1} + (R_d \cdot R_o)^{-2} + \dots + (R_d \cdot R_o)^{-(N-1)} \right]$$

The partial sums in the squared brackets of Equation (119) can be substituted by using the identity presented in Expression (120).

$$\sum_{k=0}^{N-1} q^k = \frac{q^N - 1}{q - 1} \Rightarrow \begin{cases} \sum_{k=0}^{N-1} \left(\frac{R_d}{R_o}\right)^k = \sum_{k=0}^{N-1} \left(e^{i2\pi(Q_d - Q_o)}\right)^k = -i \frac{(e^{i2\pi(Q_d - Q_o)N} - 1) \cdot e^{-i\pi(Q_d - Q_o)}}{2 \sin(\pi(Q_d - Q_o))} \\ \sum_{k=0}^{N-1} \left(\frac{1}{R_d \cdot R_o}\right)^k = \sum_{k=0}^{N-1} \left(e^{-i2\pi(Q_d + Q_o)}\right)^k = i \frac{(e^{-i2\pi(Q_d + Q_o)N} - 1) \cdot e^{i\pi(Q_d + Q_o)}}{2 \sin(\pi(Q_d + Q_o))} \end{cases} \quad (120)$$



Hence, the exact solution of the equation of motion is given by Equation (121). Please note, that Equation (120) differs from the original derivation by Peggs [15] concerning a missing phase shift of  $90^\circ$ . The phase shift originates from the fact that the sinus is inherently normalised by  $2i$ .

$$\begin{aligned}
\Xi_N &\stackrel{(119)}{=} R_o^N \cdot \Xi_0 - i \frac{\delta \cdot R_o^N}{4} \cdot \left[ \frac{(e^{i2\pi(Q_d - Q_o)N} - 1) \cdot e^{-i\pi(Q_d - Q_o) + i\phi}}{\sin(\pi(Q_d - Q_o))} - \frac{(e^{-i2\pi(Q_d + Q_o)N} - 1) \cdot e^{i\pi(Q_d + Q_o) - i\phi}}{\sin(\pi(Q_d + Q_o))} \right] \\
&\Leftrightarrow \Xi_N \equiv R_o^N \cdot \Xi_0 + R_o^N \cdot \delta_- \cdot (1 - e^{i2\pi N Q_-}) + R_o^N \cdot \delta_+ \cdot (e^{-i2\pi N Q_+} - 1) \\
&\Leftrightarrow \Xi_N = R_o^N \cdot (\Xi_0 + \delta_- - \delta_+) - \delta_- \cdot e^{i2\pi N Q_d} + \delta_+ \cdot e^{-i2\pi N Q_d} \\
&\Leftrightarrow \Xi_N \equiv \hat{\Xi}_0 \cdot e^{i2\pi N Q_o} - \delta_- \cdot e^{i2\pi N Q_d} + \delta_+ \cdot e^{-i2\pi N Q_d}
\end{aligned} \tag{121}$$

The phasor  $\Xi_n$  contains two beating terms which scale inversely with  $Q_- \equiv Q_d - Q_o$  and  $Q_+ \equiv Q_d + Q_o$  respectively. All factors independent of the total revolution number  $N$  are absorbed into two complex AC-dipole strengths  $\delta_-$  and  $\delta_+$ . For completeness,  $\delta_-$  and  $\delta_+$  are explicitly presented in Equation (122). Furthermore, the constant terms were unified into a single phasor  $\hat{\Xi}_0$ , which is given by the initial conditions.

$$\delta_- = \frac{\delta}{4} \cdot \frac{\exp\left[-i\left(\pi Q_- - \phi - \frac{\pi}{2}\right)\right]}{\sin(\pi Q_-)} \quad , \quad \delta_+ = \frac{\delta}{4} \cdot \frac{\exp\left[i\left(\pi Q_+ - \phi + \frac{\pi}{2}\right)\right]}{\sin(\pi Q_+)} \tag{122}$$

The phasor  $\Xi_N$  from Equation (121) refers to the position  $s_{d-}$  in front of the AC-dipole. In order to propagate the phasor to an arbitrary position  $s$ , a rotation by the phase advance  $\psi(s|s_{d-}) \equiv \Delta\psi$  has to be applied. This is illustrated by Equation (123).

$$\begin{aligned}
\Xi_N(s) &= e^{i\psi(s|s_{d-})} \cdot \Xi_N(s_{d-}) \\
&\Leftrightarrow \Xi_N(s) \stackrel{(121)}{=} \hat{\Xi}_0 \cdot e^{i(N\psi_o + \Delta\psi)} - \delta_- \cdot e^{i(N\psi_d + \Delta\psi)} + \delta_+ \cdot e^{-i(N\psi_d - \Delta\psi)} \\
&\Rightarrow \vec{\xi}_N(s) \stackrel{(113)}{=} \begin{bmatrix} \text{Im}(\Xi_N(s)) \\ \text{Re}(\Xi_N(s)) \end{bmatrix} \\
&\Rightarrow \vec{\eta}_N(s) \stackrel{(112)}{=} \begin{bmatrix} \sqrt{\beta(s)} \cdot \text{Im}(\Xi_N(s)) \\ -\frac{\alpha(s)}{\sqrt{\beta(s)}} \cdot \text{Im}(\Xi_N(s)) + \frac{1}{\sqrt{\beta(s)}} \cdot \text{Re}(\Xi_N(s)) \end{bmatrix} \\
&\Rightarrow x(s + N \times \mathcal{C}) \stackrel{(113)}{=} \sqrt{\beta(s)} \cdot \text{Im}\left(\hat{\Xi}_0 \cdot e^{i(N\psi_o + \Delta\psi)}\right) \\
&\quad - \sqrt{\beta(s)} \cdot \text{Im}\left(\delta_- \cdot e^{i(N\psi_d + \Delta\psi)}\right) + \sqrt{\beta(s)} \cdot \text{Im}\left(\delta_+ \cdot e^{-i(N\psi_d - \Delta\psi)}\right) \\
&\Leftrightarrow x(s + N \times \mathcal{C}) \stackrel{(113)}{=} \sqrt{\beta(s)} \cdot \text{Im}\left(\hat{\Xi}_0 \cdot e^{i(N\psi_o + \Delta\psi)}\right) \\
&\quad - \frac{\sqrt{\beta(s)} \cdot \delta}{4 \sin \pi Q_-} \cdot \sin\left(N\psi_d + \Delta\psi - \pi Q_- + \phi + \frac{\pi}{2}\right) \\
&\quad - \frac{\sqrt{\beta(s)} \cdot \delta}{4 \sin \pi Q_+} \cdot \sin\left(N\psi_d - \Delta\psi - \pi Q_+ + \phi - \frac{\pi}{2}\right) \\
&\Leftrightarrow x(s + N \times \mathcal{C}) \stackrel{(118)}{=} \sqrt{\beta(s)} \cdot \text{Im}\left(\hat{\Xi}_0 \cdot e^{i(N\psi_o + \Delta\psi)}\right) \\
&\quad + \frac{\theta_d \sqrt{\beta(s)} \cdot \beta(s_d)}{4} \cdot \left[ \frac{\cos(N\psi_d - \Delta\psi - \pi Q_+ + \phi)}{\sin(\pi Q_+)} - \frac{\cos(N\psi_d + \Delta\psi - \pi Q_- + \phi)}{\sin(\pi Q_-)} \right]
\end{aligned} \tag{123}$$

In the last line of Expression (123) a new quantity  $\theta_d$  was implicitly defined, which characterises the amplitude of the AC-dipole kick. In the following, the remaining two terms will be treated separately.

The phase advance  $\Delta\psi$  requires a case differentiation between upstream and downstream. In the downstream scenario, where  $s \geq s_d$  applies, an additional AC-dipole kick has to be considered since the phasor  $\Xi_N$  from Equation (121) refers to the position  $s_{d-}$  in front of the AC-dipole. Hence, by propagating the phase space vector in downstream direction it is unavoidable to pass the AC-dipole again. As a consequence,  $\Xi_N$  has to be transformed to  $\Xi_{N+1}$  in the downstream case. This transformation will implicitly enlarge the phase advance  $\Delta\psi$  by one turn. As a result, the phase advance  $\Delta\psi$  has to be mapped to  $\Delta\psi \mapsto (\Delta\psi - \psi_o)$  simultaneously. In other words, the phasor needs to be propagated backwards from  $s_{d-}$  to the desired position  $s$  after applying the transformation  $\Xi_N \mapsto \Xi_{N+1}$ . In the upstream scenario, where  $s < s_d$  applies, Equation (123) is applicable without any transformations. As a side note, the upstream scenario implies  $\Delta\psi \leq 0$ . Hence, the phasor at position  $s_{d-}$  is propagated backwards to the desired position  $s$ . Finally, the case differentiation for the phase advance  $\Delta\psi$  is summarized in Expression (124).

$$x(s + N \times C) \stackrel{(123)}{=} \begin{cases} \text{Downstream: } s \geq s_d, & x(s, N \mapsto (N+1), \psi(s|s_{d-}) \mapsto (\psi(s|s_{d-}) - \psi_o)) \\ \text{Upstream: } s < s_d, & x(s, N, \psi(s|s_{d-})) \end{cases} \quad (124)$$

For the moment the first term of Equation (123) is left out and the second term is considered exclusively. Expression (125) results after applying the case differentiation to the second term.

$$\begin{aligned} \frac{4 \cdot x_2(s + N \times C)}{\theta_d \sqrt{\beta(s) \cdot \beta(s_d)}} &\stackrel{(124)}{=} \begin{cases} s \geq s_d, & \frac{\cos(N\psi_d - \psi(s|s_{d-}) + \pi Q_+ + \phi)}{\sin(\pi Q_+)} - \frac{\cos(N\psi_d + \psi(s|s_{d-}) + \pi Q_- + \phi)}{\sin(\pi Q_-)} \\ s < s_d, & \frac{\cos(N\psi_d - \psi(s|s_{d-}) - \pi Q_+ + \phi)}{\sin(\pi Q_+)} - \frac{\cos(N\psi_d + \psi(s|s_{d-}) - \pi Q_- + \phi)}{\sin(\pi Q_-)} \end{cases} \\ \Leftrightarrow \frac{4 \cdot x_2(s + N \times C)}{\theta_d \sqrt{\beta(s) \cdot \beta(s_d)}} &= \frac{\cos(N\psi_d + \theta(s - s_d) \cdot \pi Q_+ - \psi(s|s_{d-}) + \phi)}{\sin(\pi Q_+)} - \frac{\cos(N\psi_d + \theta(s - s_d) \cdot \pi Q_- + \psi(s|s_{d-}) + \phi)}{\sin(\pi Q_-)} \end{aligned} \quad (125)$$

In Expression (125) a particular representation of the Heaviside function  $\theta(x)$  was used, as defined by Equation (126). At this point it has to be emphasised, that the particular choice of the Heaviside function differs from the standard literature [30] in the case of  $s = s_d$ .

$$\theta(x) \equiv \begin{cases} x \geq 0 : & 1 \\ x < 0 : & -1 \end{cases} \quad (126)$$

The influence of the secondary driving term in Equation (125) is quantified by the ratio of the amplitudes, as defined by Equation (127).

$$\lambda \equiv \frac{\sin(\pi Q_-)}{\sin(\pi Q_+)} \quad (127)$$

Substituting  $\lambda$  into Expression (125) results in Equation (128).

$$\begin{aligned} \frac{4 \cdot x_2(s + N \times C)}{\theta_d \sqrt{\beta(s) \cdot \beta(s_d)}} \cdot \sin(\pi Q_-) &= \lambda \cdot \cos(N\psi_d + \theta(s - s_d) \cdot \pi Q_+ - \psi(s|s_{d-}) + \phi) \\ &\quad - \cos(N\psi_d + \theta(s - s_d) \cdot \pi Q_- + \psi(s|s_{d-}) + \phi) \end{aligned} \quad (128)$$

At this point, the addition theorem for harmonic functions (129) should be recalled [31].

$$\sum_i a_i \cdot \cos(x + \delta_i) = a \cdot \cos(x + \delta), \text{ with: } \begin{cases} a^2 \equiv \sum_{i,j} a_i \cdot a_j \cdot \cos(\delta_i - \delta_j) \\ \tan(\delta) \equiv \frac{\sum_i a_i \cdot \sin(\delta_i)}{\sum_i a_i \cdot \cos(\delta_i)} \end{cases} \quad (129)$$

The linear combination of decoherent harmonics with equivalent driving frequencies results in a single wave, as presented in Equation (129). The overall amplitude  $a$  is characterized by Expression (130).

$$a^2 \stackrel{(128)}{=} \lambda^2 + 1 - 2\lambda \cdot \cos(\theta(s - s_d) \cdot \psi_o - 2\psi(s|s_{d-}))$$

$$\Rightarrow a = \sqrt{\lambda^2 + 1 - 2\lambda \cdot \cos\left(2 \left| \psi(s) - \psi(s_{d-}) \right| - 2\pi Q_o\right)}$$
(130)

Furthermore, the resulting initial phase  $\delta$  of the linear combination is determined by Expression (131).

$$\tan \delta \stackrel{(128)}{=} \frac{\lambda \cdot \sin(\theta(s - s_d) \cdot \pi Q_+ - \psi(s|s_{d-})) - \sin(\theta(s - s_d) \cdot \pi Q_- + \psi(s|s_{d-}))}{\lambda \cdot \cos(\theta(s - s_d) \cdot \pi Q_+ - \psi(s|s_{d-})) - \cos(\theta(s - s_d) \cdot \pi Q_- + \psi(s|s_{d-}))}$$

$$\Leftrightarrow \tan \delta \stackrel{(129)}{=} \frac{\sin\left[\pi Q_d \cdot \theta(s - s_d) + \arctan\left(\frac{(\lambda + 1) \cdot \sin(\pi Q_o \cdot \theta(s - s_d) - \psi(s|s_{d-}))}{(\lambda - 1) \cdot \cos(\pi Q_o \cdot \theta(s - s_d) - \psi(s|s_{d-}))}\right)\right]}{\cos\left[\pi Q_d \cdot \theta(s - s_d) + \arctan\left(\frac{(\lambda + 1) \cdot \sin(\pi Q_o \cdot \theta(s - s_d) - \psi(s|s_{d-}))}{(\lambda - 1) \cdot \cos(\pi Q_o \cdot \theta(s - s_d) - \psi(s|s_{d-}))}\right)\right]}$$

$$\Leftrightarrow \delta - \pi Q_d \cdot \theta(s - s_d) = \arctan\left[\frac{\lambda + 1}{\lambda - 1} \cdot \tan(\pi Q_o \cdot \theta(s - s_d) - \psi(s|s_{d-}))\right]$$

$$\Leftrightarrow \tan(\delta - \pi Q_d \cdot \theta(s - s_d)) \stackrel{(127)}{=} \frac{\sin(\pi Q_-) + \sin(\pi Q_+)}{\sin(\pi Q_-) - \sin(\pi Q_+)} \cdot \tan(\pi Q_o \cdot \theta(s - s_d) - \psi(s|s_{d-}))$$
(131)

$$\Leftrightarrow \tan(\delta - \pi Q_d \cdot \theta(s - s_d)) = -\frac{\sin(\pi Q_d) \cdot \cos(\pi Q_o)}{\sin(\pi Q_o) \cdot \cos(\pi Q_d)} \cdot \tan(\pi Q_o \cdot \theta(s - s_d) - \psi(s|s_{d-}))$$

$$\Leftrightarrow \tan(\delta - \pi Q_d \cdot \theta(s - s_d)) = \frac{\tan(\pi Q_d)}{\tan(\pi Q_o)} \cdot \tan(\psi(s|s_{d-}) - \pi Q_o \cdot \theta(s - s_d))$$

$$\Leftrightarrow \tan(\delta \cdot \theta(s - s_d) - \pi Q_d) = \frac{\tan(\pi Q_d)}{\tan(\pi Q_o)} \cdot \tan\left(\left|\psi(s) - \psi(s_{d-})\right| - \pi Q_o\right)$$

$$\Leftrightarrow \delta = \theta(s - s_d) \cdot \left[\arctan\left[\frac{\tan(\pi Q_d)}{\tan(\pi Q_o)} \cdot \tan\left(\left|\psi(s) - \psi(s_{d-})\right| - \pi Q_o\right)\right] + \pi Q_d\right]$$

As a result, the second term of the AC-dipole orbit  $x_2(s + N \times \mathcal{C})$  from Equation (123) can be rewritten as Expression (132).

$$x_2(s + N \times \mathcal{C}) = \frac{\theta_d}{4 \sin(\pi Q_-)} \cdot \sqrt{\beta(s) \cdot \beta(s_d) \cdot \left[\lambda^2 + 1 - 2\lambda \cdot \cos\left(2 \left|\psi(s) - \psi(s_{d-})\right| - 2\pi Q_o\right)\right]}$$

$$\times \cos\left(2\pi Q_d N + \theta(s - s_d) \cdot \left[\arctan\left[\frac{\tan(\pi Q_d)}{\tan(\pi Q_o)} \cdot \tan\left(\left|\psi(s) - \psi(s_{d-})\right| - \pi Q_o\right)\right] + \pi Q_d\right]\right)$$
(132)

$$\Leftrightarrow x_2(s + N \times \mathcal{C}) \stackrel{(85)}{=} \frac{\theta_d \sqrt{\beta(s_d) \cdot (1 - \lambda^2)}}{4 \sin(\pi Q_-)} \cdot \sqrt{\beta(s) \cdot \left[\frac{\Delta\beta}{\beta}\Big|_s + 1\right]}$$

$$\times \cos\left(2\pi Q_d N + \theta(s - s_d) \cdot \left[\arctan\left[\frac{\tan(\pi Q_d)}{\tan(\pi Q_o)} \cdot \tan\left(\left|\psi(s) - \psi(s_{d-})\right| - \pi Q_o\right)\right] + \pi Q_d\right]\right)$$

### 6.3.1 AC-dipole beating

Equation (132) identifies an additional beta beating contribution generated by the AC-dipole. This effect can be understood as follows. The AC-dipole forces the beam to oscillate with the driven tune  $Q_d$  around the reference orbit. Hence, a tune shift is induced by the AC-dipole ( $\Delta Q \equiv Q_d - Q_o$ ), which can be associated with a thin quadrupolar error at the location  $s_d$ . The magnitude of the distortion imposed by the AC-dipole is characterized by Equation (133).

$$\Delta k|_{\text{ACD}} \times \delta(s - s_d) ds \stackrel{(75)}{=} 2 \cdot \frac{\cos \psi_o - \cos \psi_d}{\beta(s_d) \cdot \sin \psi_o} \times \delta(s - s_d) ds = \frac{4 \sin(\pi \Delta Q) \cdot \sin(\pi(2Q_o + \Delta Q))}{\beta(s_d) \cdot \sin \psi_o} \times \delta(s - s_d) ds \quad (133)$$

The resulting effect of AC-dipole beating can also be understood as the result of the interference of two competing driving frequencies [30]. This can be explained by the fact that the beam samples the driving oscillation turn by turn. Hence, the driven frequency will be seen by the beam as  $Q_d$  if  $Q_d < 0.5$  or as  $(1 - Q_d)$  if  $Q_d > 0.5$ . Of course, the physical impact on the beam can also be entirely described by the alias  $((1 - Q_d)$  if  $Q_d < 0.5$  and  $Q_d$  if  $Q_d > 0.5$ ). Since there are two possibilities which cause the same physical effect, the system is degenerated. In the theoretical description (118), two terms are obtained because the real sinusoidal signal already incorporates the degeneracy ( $+Q_d$  and  $-Q_d$ ). The negative frequency component already includes the effect of aliasing, since:  $-Q_d \hat{=} 1 - Q_d$ . The amplitudes of both contributions are then determined by the distance to the resonance condition ( $Q_d = Q_o$ ). In a nutshell, the beam sees the AC-dipole only once per turn and is therefore driven at the frequency  $\min(Q_d, 1 - Q_d)$ . Due to the decomposition of the real sinusoidal oscillation in the derivation, a distinction is implicitly made between the signal and its alias.

### 6.3.2 Coherent betatron motion

Due to the incoherence of the free particle motion the shape of the betatron oscillation is not measurable with a BPM (beam position monitor). As a consequence, AC-dipole excitations are commonly used in accelerators in order to make the lattice functions accessible with BPMs. The synchronising effect of an AC-dipole is visualized in Figure (14). So far, only the second term of the driven particle trajectory (123) has been considered. The first term scales with  $\hat{\Xi}_0 \equiv (\Xi_0 + \delta_- - \delta_+)$ . Here,  $\Xi_0$  incorporates the initial conditions of the single particle trajectory in phasor notation.

Due to the fact that the BPMs measure the induced charge of the entire bunch, an ensemble average is inherently connected with this measurement technique. Hence, it is not possible to obtain information about the single particle trajectories by BPM measurements. As a consequence, the ensemble average of Equation (123) has to be considered if a comparison with the measurement data is desired. In Equation (123) only  $\Xi_0$  depends on initial conditions of the single particle. Consequently, the expectation value of the ensemble average for the second term is already given by Equation (132). For the first term, the expectation value of the phasor  $\hat{\Xi}_0$  is given by  $\langle \hat{\Xi}_0 \rangle = (\delta_- - \delta_+)$  because in the ensemble average  $\langle \Xi_0 \rangle$  vanishes. As a result, the measurable coherent betatron motion will obey Equation (123) with the constraint  $\langle \Xi_0 \rangle = 0$ .

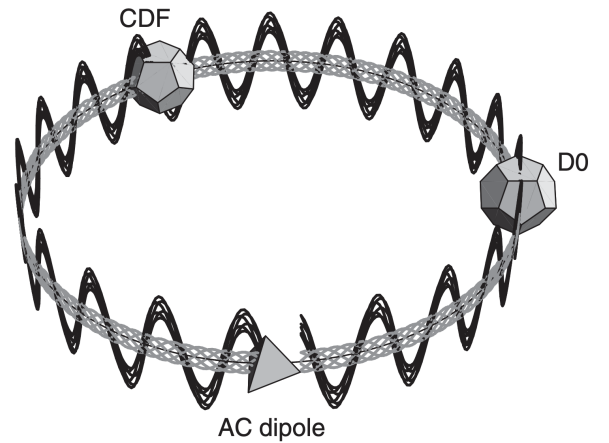


Figure 14: Coherent betatron motion (black) induced by the driving beat of an AC-dipole. The single particle motion (grey) without excitation is shown in comparison [16].

### 6.3.3 Adiabatic ramping and damping

If the ensemble average of the first term in Equation (123) is rearranged as done for the second term, Equation (134) will be obtained. Equation (134) describes a betatron oscillation with the natural tune  $Q_o$ . This term results from the initial Ansatz in Expression (118) of an AC-dipole excitation with constant amplitude  $\delta$ . Hence, a ramping following a step function was implicitly assumed.

$$x_1(s + N \times C) \stackrel{(123)}{=} \frac{\theta_d \sqrt{\beta(s) \cdot \beta(s_d)}}{4} \cdot \left[ \frac{\cos(2\pi N Q_o - \pi Q_- + \psi(s|s_d-) + \phi)}{\sin \pi Q_-} - \frac{\cos(2\pi N Q_o + \pi Q_+ + \psi(s|s_d-) - \phi)}{\sin \pi Q_+} \right] \quad (134)$$

If the ramping progression is characterised by a smoother functional regulation the question arises how this will impact the terms of Equation (121). For that reason, a ramping function following Equation (135) is assumed.

$$\delta(n) = \frac{n}{N} \cdot \hat{\delta} \quad \text{with: } n \in [1, N] \subset \mathbb{N} \quad (135)$$

Mathematically, the turn-by-turn AC-dipole kicks are incorporated into the formalism by the partial sums of Equation (119). For simplification, the partial sums were substituted by the identity in Expression (120). The impact of the ramping function comes into play at this point. Hence, the effect on the partial sum has to be studied in order to investigate the influence of the ramping function on the driven particle trajectory in Equation (123). First of all, it has to be noticed that the second term of Equation (123) arises from the  $q^N$  part of the partial sum in (120) and the non-zero contribution of the ensemble average of the first term of Equation (123) is generated by the  $(-1)$  term in the numerator of (120). With this knowledge in mind, the influence of the ramping function on the bunch trajectory can be studied in detail, as presented by Equation (136).

$$\begin{aligned} \Xi_N &\stackrel{(119)}{\propto} \sum_{k=0}^{N-1} \delta(k) \cdot q^k \stackrel{(135)}{=} \sum_{k=0}^{N-1} \frac{k+1}{N} \cdot \hat{\delta} \cdot q^k = \frac{\hat{\delta}}{N} \cdot \sum_{j=1}^N j \cdot q^{(j-1)} = \frac{\hat{\delta}}{N} \cdot \sum_{j=1}^N \partial_q q^j \\ &\Leftrightarrow \sum_{k=0}^{N-1} \delta(k) \cdot q^k = \frac{\hat{\delta}}{N} \cdot \partial_q \left[ q^N - 1 + \sum_{j=0}^{N-1} q^j \right] \stackrel{(120)}{=} \frac{\hat{\delta}}{N} \cdot \partial_q \left[ q^N - 1 + \frac{q^N - 1}{q - 1} \right] \\ &\Leftrightarrow \sum_{k=0}^{N-1} \delta(k) \cdot q^k = \frac{\hat{\delta}}{N} \cdot \left[ N q^{(N-1)} + \frac{N q^{(N-1)}}{q - 1} - \frac{q^N - 1}{(q - 1)^2} \right] \\ &\Leftrightarrow \sum_{k=0}^{N-1} \delta(k) \cdot q^k = \hat{\delta} \cdot \frac{q^N}{q - 1} - \frac{\hat{\delta}}{N} \cdot \frac{q^N - 1}{(q - 1)^2} \end{aligned} \quad (136)$$

If the AC-dipole excitation is ramped and damped very slowly ( $N \gg 1$ ) the second term in Equation (136) becomes negligible. This implies a small turn-by-turn modification of the AC-dipole amplitude and can be understood as an adiabatic process. As a consequence, if a smooth linear ramping is performed slowly enough, only the term  $q^N$  in the partial sum will be of importance. As pointed out earlier, the  $q^N$  part generates the second term in Equation (123). Hence, under adiabatic ramping and damping the measurable ensemble average of the excited beam motion is entirely determined by Equation (132). In general, adiabaticity is assumed if condition (137) is fulfilled [19]. Here,  $\mathcal{H}(x)$  corresponds to the classical Heaviside step function.

$$N Q_- \times \mathcal{H}(0.5 - Q_-) + N Q_+ \times \mathcal{H}(Q_- - 0.5) \stackrel{!}{=} m \in \mathbb{N} \quad (137)$$

## 6.4 Storage ring revisited

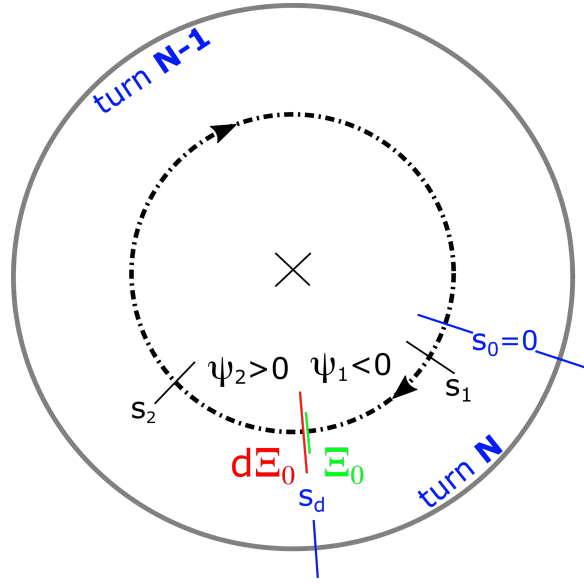


Figure 15: Schematic sketch of the model storage ring example defined by Table (1). The red marker indicates the longitudinal position  $s_d$  of the horizontal AC-dipole with strength  $d\Xi(N)$ . The input phasor  $\Xi(N)$  at  $s_d$  is visualized by a green marker. The impact of the AC-dipole on the particle trajectory has been simulated by applying Equation (121). This calculation is based on a turn by turn analysis at  $s_d$ . At this point it is important to be very precise, since  $\Xi(N)$  is defined as the input phasor at the AC-dipole location the turn number assignment has to be adjusted accordingly. Hence, for the segment by segment simulation the phasor  $\Xi(N)$  is propagated backwards to the beamline position of interest. For example, backpropagating  $\Xi(N)$  to the location  $s_1$  returns the phase space phasor for turn  $N$ . On the other hand, backpropagating  $\Xi(N)$  to the location  $s_2$  results in the phase space phasor for turn  $(N - 1)$ .

The referenced turn number  $N$  is defined as the ratio of the accumulated longitudinal position  $s$  over the circumference  $2\pi\rho$  of the storage ring. The amplitude of the AC-dipole was initially defined by Equation (117). In contrast, MAD-X assigns the kicking amplitude of the AC-dipole with "VOLT" [20]. Hence, the conversion between the theoretical definition and the MAD-X definition of the AC-dipole amplitude has to be considered. An AC-dipole is characterised in MAD-X [20] by Equation (138). Here, "p0c" labels the longitudinal momentum of the reference particle in GeV. Furthermore, it has to be noticed that the parameter "FREQ" corresponds to the driven tune  $Q_d$  of the AC-dipole and that the AC-dipole is by definition in phase with the particle beam. The choice of coherence determines the initial condition  $\phi \stackrel{!}{=} -\pi/2$  for the local phase at the location of the AC-dipole.

$$d\theta = 0.3 \cdot \frac{\text{VOLT}}{\text{p0c}} \cdot \sin(2\pi \cdot \text{FREQ} \cdot \text{turn}) \quad (138)$$

Comparing Equation (117) with the definition from MAD-X (138) results in the conversion relation presented in Equation (139). Here, the ominous factor of 0.3 corresponds to the speed of light in Gm/s. Naturally, this value would not be really exact but the difference in decimal places is absorbed into the definition of the amplitude factor "VOLT". Parameterization of the simulation: VOLT = 0.1 MV, p0c = 0.3458981085 GeV, ramping = [100, 600] and damping = [1400, 1900].

$$d\hat{\theta}_n = \frac{L \cdot \hat{B}}{\rho \cdot B_y^{(0)}} \stackrel{!}{=} 0.3 \cdot \frac{\text{VOLT}}{\text{p0c}} \quad (139)$$

#### 6.4.1 Turn by turn effect of a horizontal AC-dipole at the location $s_d$ of the kicker, without ramping

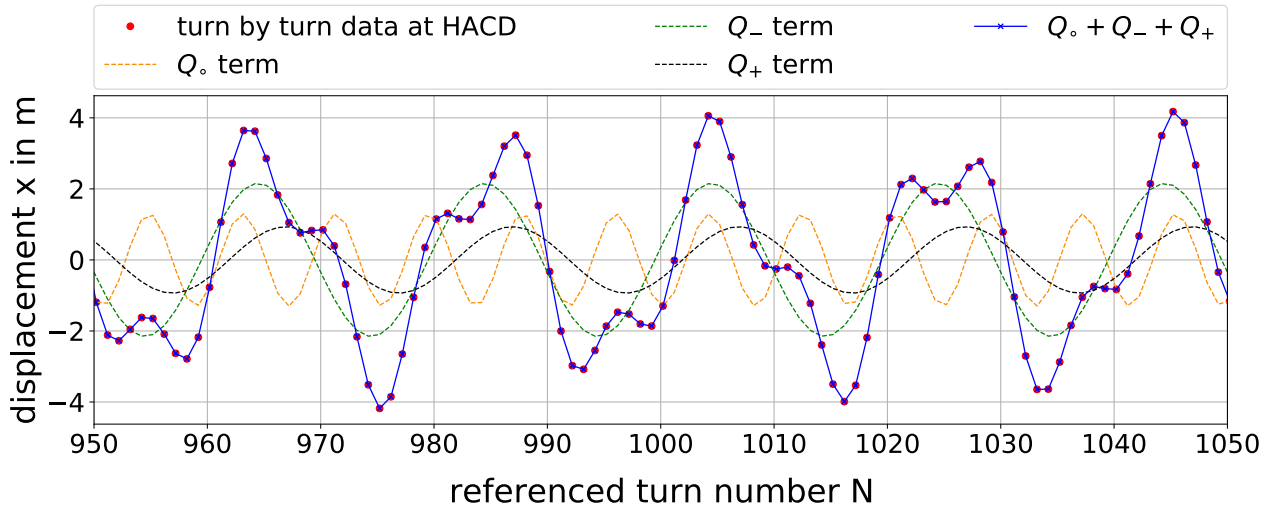


Figure 16: Turn by turn effect of a horizontal AC-dipole ( $U = 0.1$  MV) at the location  $s_d$ . The red dots mark the tracking data obtained by MAD-X. The yellow dashed line corresponds to the term of Equation (121) which propagates with the betatron tune  $Q_o$ . This term is characterised by the parameterisation of the lattice and contains all initial conditions. The green dashed line labels the term of Equation (121) which propagates with the tune  $Q_-$ . Furthermore, the black dashed line shows the propagation of the term which proceeds by the tune  $Q_+$ . The aggregate of all three contributions represents Equation (121) entirely and is plotted as a solid blue line. It is clearly noticeable that the simulation fits the tracking data very accurately.

#### 6.4.2 Segment by segment effect of a horizontal AC-dipole over several turns, without ramping

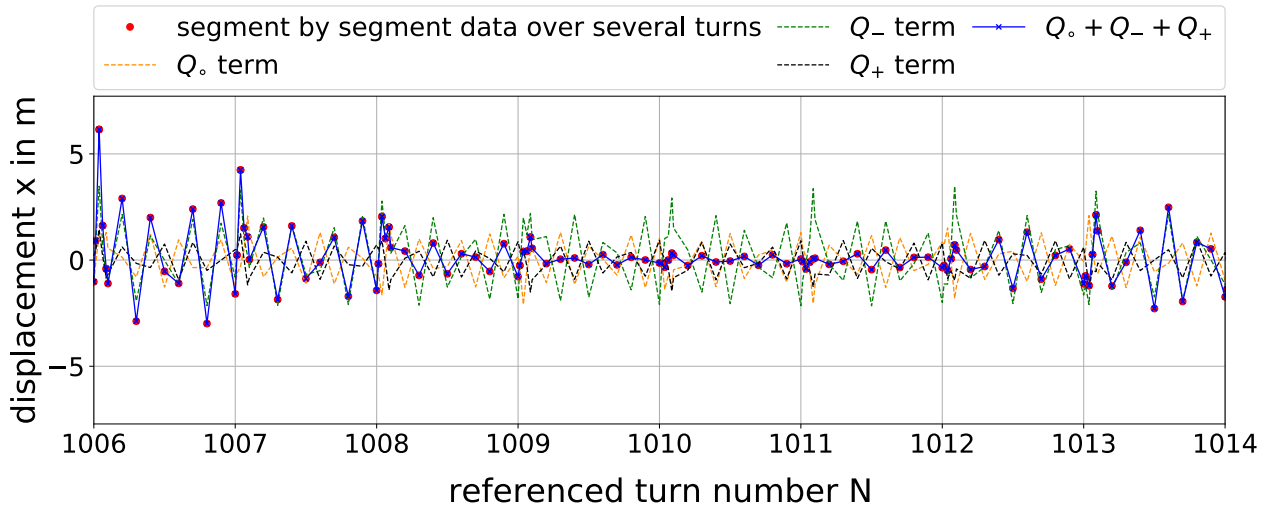


Figure 17: Segment by segment effect of a horizontal AC-dipole ( $U = 0.1$  MV) over several turns. In order to calculate the phasor at an arbitrary location within the lattice, the turn by turn data from graph (16) was propagated by the corresponding transfer matrices. The locations of interest correspond to the labeled positions of Table (1). The visualisation of the data is equivalent to the layout of plot (16). As a result, the segment by segment simulation is also in agreement with the tracking data.



#### 6.4.3 Turn by turn effect of a horizontal AC-dipole at the location $s_d$ of the kicker, with active ramping

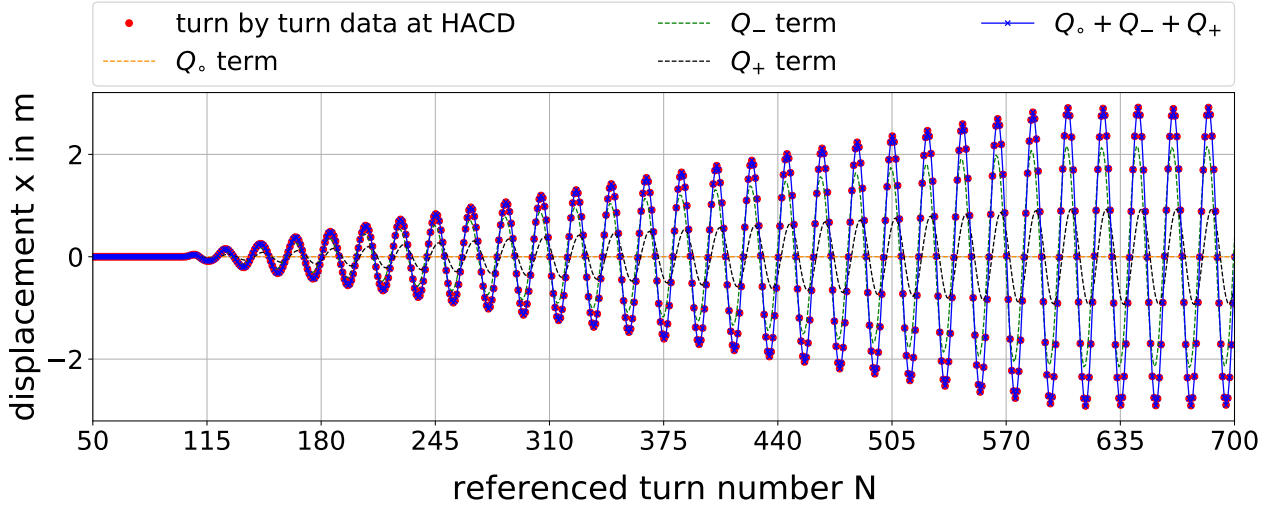


Figure 18: Turn by turn effect of a horizontal AC-dipole ( $U = 0.1$  MV) at the location  $s_d$  with active ramping and damping. The simulation is based on the first line of Equation (119). In contrast to the previous simulation, the AC-dipole amplitude  $\delta_n$  is now a function of the turn number. Hence, the partial sums can not be substituted by the identity presented in Equation (120). This leads to a significant increase of computational costs, wherefore the algorithm was parallelized by exploiting the multiprocessing library of python. The legend of the graph was defined equivalently to Figure (16). In contrast, the betatron term ( $Q_0$ ) only includes the initial phasor  $\Xi_0$  now. As a result of the initial condition  $\Xi_0 = 0$ , the betatron term vanishes completely. In conclusion, the effects of ramping and damping have been implemented successfully into the algorithm and the simulation reconstructs the tracking data accurately.

#### 6.4.4 Segment by segment effect of a horizontal AC-dipole over several turns, with active ramping

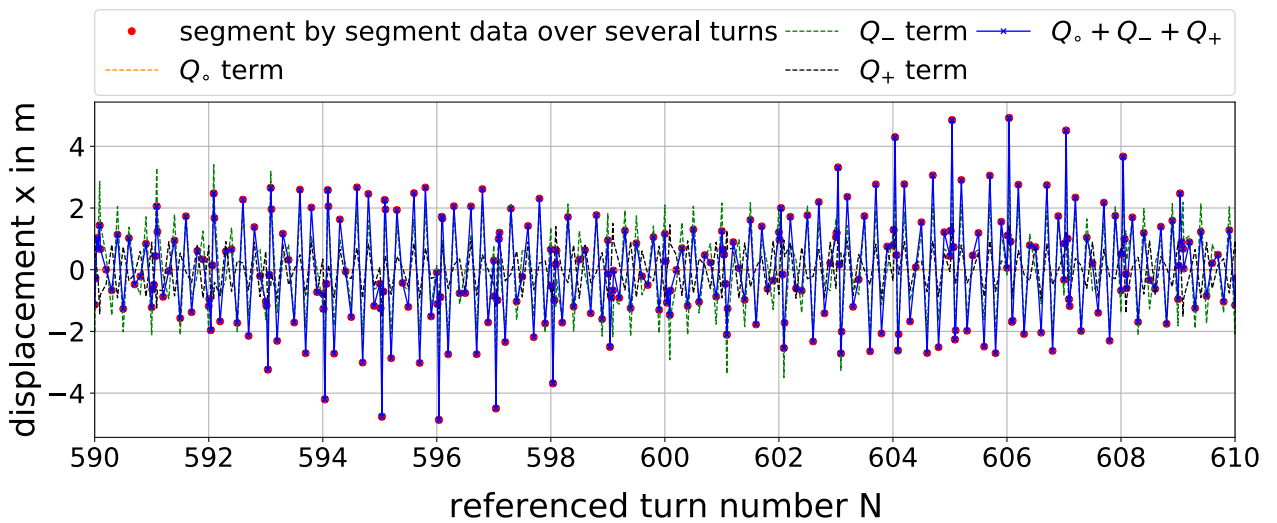


Figure 19: Segment by segment effect of a horizontal AC-dipole ( $U = 0.1$  MV) over several turns with variable amplitude  $\delta_n$ . The elaborations from graph (18) are applicable. The simulation represents the tracking data within computational uncertainties even if ramping is considered.



## 7 Non-linear beam dynamics

So far, only contributions of the multipole expansion (14) up to first order were considered. The algebraic structure of matrices was adapted to cast the equation of motion into transfer maps. If it is required to include higher order magnets into the lattice it is not sufficient anymore to use matrices. Hence, a more sophisticated algebraic structure is needed to account for non-linear elements, like sextupoles, octupoles, decapoles etc.

### 7.1 Symplecticity

In general, symplecticity implies two aspects. First of all, if a transformation is symplectic then it conserves the phase space area. The requirement for this condition is met if the determinant of the transfer map is equal to one in absolute terms. In other words, symplecticity implies Liouville's theorem of phase space volume conservation. Secondly, a symplectic transformation is a canonical transformation and conserves the structure of Hamilton's equations of motion (9). Hence, the Poisson bracket of any two dynamical variables is invariant under a symplectic transformation. In a nutshell, a symplectic mapping conserves Hamilton's equations, which guarantees for energy conservation.

#### 7.1.1 Hamilton equations of motion

In the previous sections we made use of Newtonian mechanics in order to describe the beam dynamics. An alternative approach is to characterise the system dynamics by the Hamiltonian instead of the forces. Hamilton's equations of motion (9) can be cast into a matrix Equation (140) by adopting the Poisson matrix  $\mathbb{S}$  [24].

$$\dot{\vec{\eta}} = \mathbb{S} \cdot \nabla_{\vec{\eta}} H \quad , \quad \text{with: } \mathbb{S} \equiv \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \quad (140)$$

In beam dynamics it is desired to transform the initial phase space vector  $\vec{\eta}_i$  to a final phase space vector  $\vec{\eta}_f$  by a symplectic mapping. The time derivative of the final phase space vector  $\vec{\eta}_f$  can be swapped to the initial phase space vector  $\vec{\eta}_i$  by the transfer map  $\mathcal{M}$ , as presented in Equation (141).

$$\begin{aligned} \dot{\vec{\eta}}_f &= \frac{\partial \vec{\eta}_f}{\partial \vec{\eta}_i} \cdot \dot{\vec{\eta}}_i \equiv \mathcal{M} \cdot \dot{\vec{\eta}}_i \\ \Leftrightarrow \dot{\vec{\eta}}_f &\stackrel{(140)}{=} \mathcal{M} \cdot \mathbb{S} \cdot \nabla_{\vec{\eta}_i} H \\ \Leftrightarrow \dot{\vec{\eta}}_f &= \mathcal{M} \cdot \mathbb{S} \cdot \mathcal{M}^T \cdot \nabla_{\vec{\eta}_f} H \end{aligned} \quad (141)$$

Since the phase space mapping was demanded to be symplectic, Hamilton's equation of motion (140) needs to be invariant. After comparison of coefficients between Equation (141) and Equation (140), the symplecticity condition directly results as formula (142) [21].

$$\mathcal{M} \cdot \mathbb{S} \cdot \mathcal{M}^T \stackrel{!}{=} \mathbb{S} \quad (142)$$

In algebraic terms, the metric of symplectic maps is characterised by the Poisson matrix  $\mathbb{S}$ .

### 7.1.2 Liouville's theorem

As stated above, symplecticity implies Liouville's theorem. Consequently, it is required to check whether the symplecticity condition (142) implies that the determinant of the transfer map  $\mathcal{M}$  is equal to one in absolute terms. Equation (143) underpins the consistency of the symplecticity condition for the transfer map  $\mathcal{M}$ .

$$\begin{aligned} |\mathcal{M}| \cdot |\mathbb{S}| \cdot |\mathcal{M}^T| &\stackrel{!}{=} |\mathbb{S}| \\ \Leftrightarrow |\mathcal{M}|^2 &\stackrel{!}{=} 1 \\ \Leftrightarrow |\mathcal{M}| &\stackrel{!}{=} \pm 1 \end{aligned} \tag{143}$$

## 7.2 Betatron action versus beam emittance

Equation (42) provided some motivation for the particular amplitude definition in the Courant-Snyder Ansatz (38) including the action  $J$ . In this picture, the action  $J$  was identified as the area of the phase space ellipse for a single particle. In the previous chapters the discussion of the theory has been motivated by single particle trajectories. Generally, the goal is to describe the beam motion as a single macroscopic particle, which requires an ensemble study of the single particle trajectories. In the picture of a particle ensemble, the betatron action  $J$  becomes a distribution over all single particle actions. Since single particle trajectories are not observable by BPMs, an expectation value for the betatron action  $J$  is commonly used to characterise the bunch action. The expectation value of the bunch action is then referred to as the bunch emittance  $\epsilon$ . The ensemble average of the action  $J$  quantifies the area of the phase space distribution within the elliptic contour line of 68% confidence level. If a Gaussian distribution is assumed for  $J$  with covariance matrix  $\Sigma$ , then the bunch emittance is intertwined with the single particle action  $J$  according to Equation (144).

$$\begin{aligned} \epsilon &\equiv \langle J_i \rangle \quad , \quad \text{with:} \quad J_i \equiv \frac{1}{2\pi} \oint p_i \, dq_i \\ \Rightarrow \epsilon_x &= \sqrt{|\Sigma|} = \sqrt{\begin{vmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle xx' \rangle & \langle [x']^2 \rangle \end{vmatrix}} = \sqrt{\langle x^2 \rangle \cdot \langle [x']^2 \rangle - [\langle xx' \rangle]^2} \end{aligned} \tag{144}$$

As seen in Figure (7), single particle ellipses are empty since the particle is moving on a stationary orbit in phase space. In comparison, if a particle bunch is considered, the ellipse is full of particles because  $\epsilon$  specifies an expectation value for the betatron action on a given confidence level. Figure (21) visualizes the parameterization of the phase space ellipse by the betatron action  $J$  in real phase space and in normalised phase space. The Hamilton principle of stationary action implies that  $J$  is an invariant of the betatron motion because nature chooses the particle trajectory in such a way that the action is minimised and stationary. The Courant-Snyder Ansatz (38) implicitly assigns the canonical variables of the Hamiltonian  $\mathcal{H}$  as  $\{J, \psi\}$ . If a lattice is only composed of linear elements then the phase space ellipse of the betatron motion can be transformed into a circle with radius  $J$  and angle  $\psi$ , as presented in Equation (112) and visualized in Figure (21). As a consequence, the betatron motion in the picture of normalised coordinates  $\{\xi_1, \xi_2\}$  is equivalent to the dynamics of a harmonic oscillator. Hence, the mapping of the normalised phase space vector  $\vec{\xi}$  (112) corresponds to the equation of motion presented in Expression (145). Here, the phase advance  $\psi$  is referenced to the domain  $\psi \in [0, 2\pi Q_o]$ .

$$\frac{d^2\xi_1}{d\psi^2} + \xi_1 \stackrel{(112)}{=} 0 \quad (145)$$

If non-linear elements are installed into the lattice the circle becomes distorted. As an example, if sextupoles are considered at sufficiently large betatron amplitude then the circle approaches a triangular shape, as indicated in Figure (22). Consequently, it is desirable to aim for a canonical transformation from  $\{J, \psi\}$  to a new set of canonical variables  $\{\mathcal{I}, \phi\}$ , which maps the distorted phase space contour back to a pure circle. In order to determine the symplecticity conserving transformation a generating function must be found, so that Relation (146) is fulfilled.

$$\dot{\mathcal{I}} \stackrel{(9)}{=} -\frac{\partial \mathcal{H}}{\partial \phi} \stackrel{!}{=} 0 \quad (146)$$

Consequently, the Hamiltonian  $\mathcal{H}(\mathcal{I}, \phi)$  expressed in the new set of canonical variables does not explicitly depend on  $\phi$ . As a result, the angle  $\phi$  is cyclic and evolving linearly with time or with constant frequency  $\omega(\mathcal{I})$  respectively, as shown by equation (147). The evolution frequency  $\omega(\mathcal{I})$  depends on the betatron invariant  $\mathcal{I}$ , which gives rise to the effect of amplitude detuning [44].

$$\dot{\phi} \stackrel{(9)}{=} \frac{\partial \mathcal{H}}{\partial \mathcal{I}} = \omega(\mathcal{I}) = \text{const.} \Rightarrow \phi(t) = \omega(\mathcal{I}) \cdot t + \phi_0 \quad (147)$$

The required rotation invariance of the Hamiltonian regarding the phase  $\phi$  is equivalent to the constraint of a circular phase space contour. Hence, the equation of motion for a particle in a non-linear lattice is transformed to Equation (148), which corresponds to a harmonic oscillator.

$$\frac{d^2\xi_1}{d\phi^2} + \xi_1 = 0 \quad (148)$$

In general, a canonical transformation which generates a rotation invariant Hamiltonian  $\mathcal{H}$  is termed as a Floquet transformation  $\mathcal{F}$ . Or to put it differently, a canonical transformation which converts the equation of motion to the equation of a harmonic oscillator is defined as a Floquet transformation  $\mathcal{F}$ . In the case of a linear lattice, the Floquet transformation is simply a normalisation of the physical coordinates (112) but the canonical variables  $\{J, \psi\}$  remain unchanged. Hence, in the linear case it holds that  $\mathcal{I} = J$  and  $\phi = \psi$ . In general, the Floquet transformation in matrix representation (112) only applies in the linear case. If nonlinearities are considered, the Floquet transformation is composed of resonance driving terms (RDTs), which incorporate the nonlinear distortions into the transformation [26].

Hence, the triangular phase space distortion due to the impact of sextupoles can be compensated, as shown in Figure (22). Generally, the phase advance  $\psi$  and the phase  $\phi$  describe different physical pictures of the same dynamical system as depicted in Figure (20). In the physical phase space, the transfer map  $\mathcal{M}$  (46) describes the evolution of the state vector from  $\vec{\eta}_i$  to  $\vec{\eta}_f$ . The transfer map  $\mathcal{M}$  is characterized by the phase advance  $\psi$ . In the non-linear case, the Floquet transformation  $\mathcal{F}$  maps the distorted phase space contour onto a circle. The circle is parameterized by the phase  $\phi$  and the betatron invariant  $\mathcal{I}$ . The transfer map  $\mathcal{R}$  rotates the initial state vector  $\vec{\xi}_i$  to the final state  $\vec{\xi}_f$ . In the upper layer, the occupied phase space manifests as a distorted ellipse and in the lower layer as a circle.

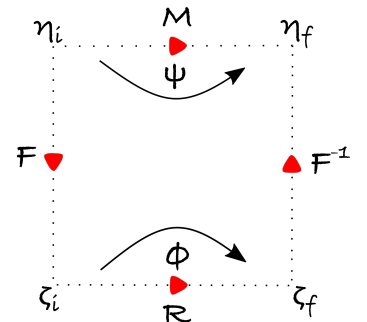


Figure 20: Visualisation of the relation between phase advance  $\psi$  and the Floquet transformed phase  $\phi$ .

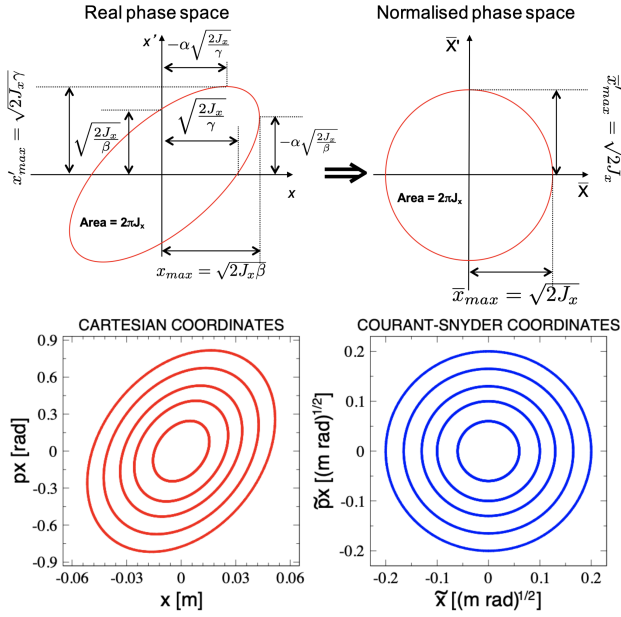


Figure 21: Physical phase space and normalised phase space in terms of the betatron action  $J$  [22]. As demonstrated in Equation (112), the contour in normalised phase space is characterised by a circle for linear lattices. Hence, the mapping of a state vector in normalised phase space is equivalent to a rotation [26].

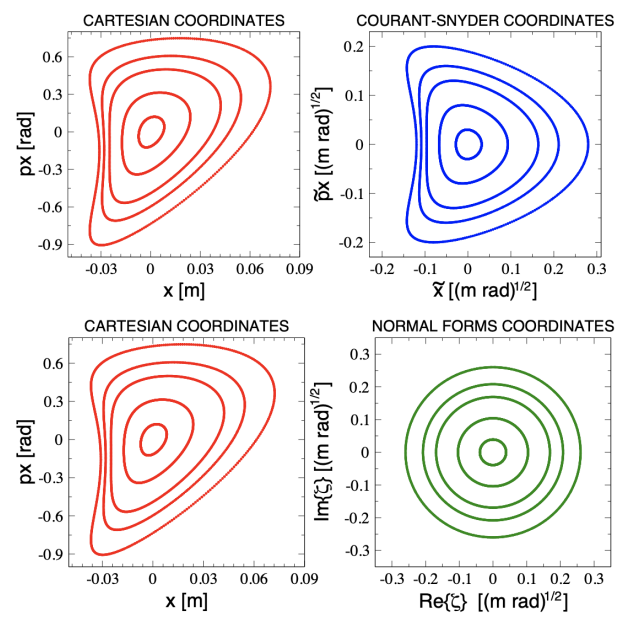


Figure 22: Visualisation of the difference between the linear Floquet transformation according to Equation (112) (upper left to upper right graph) and the nonlinear Floquet transformation using RDTs (lower left to lower right plot), which entirely incorporate the effects from non-linear perturbations [26].

### 7.3 Taylor representation of non-linear transfer maps

The effect of non-linear magnetic fields can be incorporated into the concept of transfer maps by utilising tensors of higher orders [23]. This approach is represented analytically by a Taylor expansion as shown in Equation (149).

$$\vec{\eta}_f[a] = \delta_a + \sum_b \mathcal{R}_{ab} \cdot \vec{\eta}_i[b] + \sum_{b,c} \mathcal{T}_{abc} \cdot \vec{\eta}_i[b] \cdot \vec{\eta}_i[c] + \sum_{b,c,d} \mathcal{U}_{abcd} \cdot \vec{\eta}_i[b] \cdot \vec{\eta}_i[c] \cdot \vec{\eta}_i[d] + \mathcal{O}(\vec{\eta}_i[x]^4) \quad (149)$$

Here,  $\delta$  accounts for kicks applied to the corresponding variable, for instance an orbit offset caused by a dipole distortion. In addition, the tensor of second order  $\mathcal{R}$  describes the linear motion of the particle due to drift sections, dipoles and quadrupoles. The tensor of third order  $\mathcal{T}$  accounts for sextupoles in the lattice. Furthermore, the tensor of fourth order  $\mathcal{U}$  covers the impact of octupoles on the particle beam. As higher the order of the magnetic field contribution as higher the dimensionality of the required tensor to catch all correlations. The indices of the tensors are characterised by the components of the phase space vector, as presented in Table (3).

| Index     | 1   | 2    | 3   | 4    | 5   | 6        |
|-----------|-----|------|-----|------|-----|----------|
| Component | $x$ | $x'$ | $y$ | $y'$ | $z$ | $\delta$ |

Table 3: Indices of the tensors in the Taylor-series representation.

The Taylor-series (149) for non-linear sections are symplectic but also infinite [21]. In order to make this approach applicable, the Taylor-series has to be truncated at some point. Consequently, the required

property of symplecticity has to be sacrificed in order to make the computational costs affordable. This major drawback in real life gives rise to aim for an alternative formalism.

## 7.4 Lie representation of non-linear transfer maps

In accelerator physics, the longitudinal position  $s$  serves as an independent variable in the Hamilton formalism. Hence, the derivative of any function  $f$  with respect to  $s$  can be written in terms of the canonical variables  $\{q_i, p_i\}$ , as presented in Equation (150).

$$\frac{df}{ds} = \sum_{n=1}^d \frac{\partial f}{\partial q_i} \frac{\partial q_i}{\partial s} + \frac{\partial f}{\partial p_i} \frac{\partial p_i}{\partial s} \stackrel{(140)}{=} \left[ \sum_{n=1}^N \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \right] \times f \stackrel{(151)}{=} :H: f \quad (150)$$

Here,  $d$  labels the dimension of the system. Furthermore, the definition of the Lie operator (151) was adapted in order to condense the equation of motion. The Lie operator may be viewed as a Poisson bracket waiting to happen [23].

$$:H: \equiv \sum_{k=1}^d \frac{\partial H}{\partial q_k} \frac{\partial}{\partial p_k} - \frac{\partial H}{\partial p_k} \frac{\partial}{\partial q_k} \Rightarrow \begin{cases} :H: \vec{\eta}(q_k, p_k) \hat{=} \{H, \vec{\eta}(q_k, p_k)\} \\ [:H:]^2 \vec{\eta}(q_k, p_k) = \{H, \{H, \vec{\eta}(q_k, p_k)\}\} \end{cases} \quad (151)$$

The structure of the equation of motion (150) gives rise to the solution proposed in (152).

$$f(s_f) = e^{-\Delta s :H:} \times f(s_i) = \sum_{n=0}^{\infty} \frac{[-\Delta s :H:]^n}{n!} \times f(s_i) \quad (152)$$

As a result, an initial phase space vector  $\vec{\eta}_i$  can be propagated by the distance  $\Delta s$  to the final position  $s_f$ , by adopting the Lie transformation from Equation (152). The phase space mapping is illustrated by Equation (153), where the Lie operator has to be applied elementwise.

$$\vec{\eta}(s_f) = e^{-\Delta s :H:} \times \vec{\eta}(s_i) \equiv \mathcal{M}_{\text{Lie}} \times \vec{\eta}(s_i) \quad (153)$$

In general, any symplectic map  $\mathcal{M}$  can be factorized in terms of a set of homogeneous polynomials  $f_m$ , as presented in Equation (154).

$$\mathcal{M}_{\text{Lie}} = \mathcal{R} \cdot \prod_{m=3}^{\infty} \exp[:f_m:] = \mathcal{R} \cdot \prod_{m=3}^{\infty} \sum_{n=0}^{\infty} \frac{[:f_m:]^n}{n!} \quad (154)$$

Here,  $\mathcal{R}$  denotes the linear transfer map of the lattice. The homogeneous polynomials  $f_m$  characterise higher order effects and they establish an orthogonal basis  $\langle f_m, f_k \rangle = 0$  (for  $m \neq k$ ) for the function space of lattice non-linearities. In the terminology of accelerator physics the homogeneous polynomials  $f_m$  are termed as resonance driving terms (RDTs). In detail,  $f_3$  corresponds to second order effects as caused by sextupoles. Utilising Lie algebra for non-linear transfer maps provides a way to obey symplecticity even if higher order contributions are truncated [25]. The product of exponentials containing Lie operators can be concatenated by applying the Baker-Campbell-Hausdorff (BCH) theorem (155). Generally, the exact solution of a truncated or full Hamiltonian is symplectic. In contrast, if the solution of a Hamiltonian (initially exact or approximated) is truncated then the symplecticity condition will be violated unless the truncation corresponds to a solution of a simplified Hamiltonian [47].

$$\exp[:f_1:] \cdot \exp[:f_2:] = \exp \left[ :f_1: + :f_2: + \frac{1}{2} \{ :f_1:, :f_2: \} + \mathcal{O}(\{ :f_1:, \{ :f_1:, :f_2: \} \}) \right] \quad (155)$$

## 8 Lattice function measurements with BPMs

For the LHC, parallel-plate beam position monitors (BPMs) are used in order to measure the beam dynamics. In general, a BPM consists of two horizontal plates and two vertical plates. If a particle bunch is passing the plates a charge is induced. The charge signal will be stronger at those plates, which are closer to the bunch centroid. The voltage difference  $\Delta U$  between the plates characterises the displacement from the reference orbit in the corresponding plane. In detail, the voltage difference normalised by the accumulated potential  $\Sigma U$  of both plates quantifies the displacement  $x$  if multiplied by the calibration factor  $k$ . The calibration factor  $k$  specifies the sensitivity of the BPM on the normalised potential difference. Furthermore,  $x_{\text{off}}$  accounts for BPMs, which are aligned off-axis. In addition, it is possible that the plates of the BPM have a different zero level potential. Hence, a zero level shift in the normalised voltage difference must be considered eventually. This measurement technique inherently double counts the displacement  $x$  from the reference orbit because the normalised voltage difference is impacted by the rise at plate  $A$  as well as the drop at plate  $B$  simultaneously. In other words,  $\Delta U$  is two times more sensitive to changes in the displacement of the beam centroid  $x$  than the accumulated potential signal  $\Sigma U$ . As a consequence, an additional calibration factor for the potential ratio is defined, which accounts for the systematic bias of the measurement technique. This systematic bias is constant and equal to two. Mathematically, the displacement  $x$  of the bunch centroid from the reference orbit is given by Equation (156).

$$x(\text{BPM}) = \frac{k}{\frac{\Delta U}{\Sigma U}|_{\text{cal}}} \cdot \left( \frac{\Delta U}{\Sigma U}|_{\text{meas}} - \frac{\Delta U}{\Sigma U}|_{\text{zero}} \right) - x_{\text{off}} = \frac{k}{2} \cdot \left( \frac{\Delta U}{\Sigma U}|_{\text{meas}} - \frac{\Delta U}{\Sigma U}|_{\text{zero}} \right) - x_{\text{off}} \quad (156)$$

### 8.1 N-BPM method

In general, beam position monitors (BPMs) measure the orbit of the excited beam turn-by-turn. Applying a harmonic analysis to the turn-by-turn data will result in the local phases  $\phi$  at the BPM locations. With the model transfer matrix between three BPMs and the measurements of the fractional part of the phase advances  $\psi$  between those BPMs, the  $\beta$ - and  $\alpha$ -function can be deduced from the turn-by-turn orbit measurements. Assuming three BPMs  $\{i, j, k\}$  the model transfer matrices can be combined according to Equation (157).

$$\left. \begin{aligned} \mathcal{M}_{11}(j|i) &\stackrel{(46)}{=} \sqrt{\frac{\beta_j}{\beta_i}} \cdot (\cos \psi_{ji} + \alpha_i \sin \psi_{ji}) \\ \mathcal{M}_{12}(j|i) &\stackrel{(46)}{=} \sqrt{\beta_j \beta_i} \cdot \sin \psi_{ji} \end{aligned} \right\} \Rightarrow \frac{\mathcal{M}_{11}(j|i)}{\mathcal{M}_{12}(j|i)} = \frac{1}{\beta_i} \cdot \cot \psi_{ji} + \frac{\alpha_i}{\beta_i} \quad (1) \quad (157)$$

$$\left. \begin{aligned} \mathcal{M}_{11}(k|i) &\stackrel{(46)}{=} \sqrt{\frac{\beta_k}{\beta_i}} \cdot (\cos \psi_{ki} + \alpha_i \sin \psi_{ki}) \\ \mathcal{M}_{12}(k|i) &\stackrel{(46)}{=} \sqrt{\beta_k \beta_i} \cdot \sin \psi_{ki} \end{aligned} \right\} \Rightarrow \frac{\mathcal{M}_{11}(k|i)}{\mathcal{M}_{12}(k|i)} = \frac{1}{\beta_i} \cdot \cot \psi_{ki} + \frac{\alpha_i}{\beta_i} \quad (2)$$

The ratios of transfer matrices can be used to determine  $\beta_i$ , as presented by Expression (158).

$$\begin{aligned} (1) - (2) : \quad \frac{\mathcal{M}_{11}(j|i)}{\mathcal{M}_{12}(j|i)} - \frac{\mathcal{M}_{11}(k|i)}{\mathcal{M}_{12}(k|i)} &= \frac{1}{\beta_i} \cdot (\cot \psi_{ji} - \cot \psi_{ki}) \\ \Leftrightarrow \beta_i &= \frac{\cot \psi_{ji} - \cot \psi_{ki}}{\frac{\mathcal{M}_{11}(j|i)}{\mathcal{M}_{12}(j|i)} - \frac{\mathcal{M}_{11}(k|i)}{\mathcal{M}_{12}(k|i)}} \stackrel{(46)}{=} \frac{\cot \psi_{ji} - \cot \psi_{ki}}{\cot \psi_{ji}^m - \cot \psi_{ki}^m} \cdot \beta_i^m \end{aligned} \quad (158)$$

Mathematically, Equation (158) is a rule of three for  $\beta$ -functions. Here, the index  $m$  corresponds to

the model values. Equivalently, an expression for the  $\alpha$ -function at  $\text{BPM}_i$  can be derived (159).

$$\begin{aligned}
(2): \quad \alpha_i &= \beta_i \cdot \frac{\mathcal{M}_{11}(k|i)}{\mathcal{M}_{12}(k|i)} - \cot \psi_{ki} \stackrel{(158)}{=} \frac{\frac{\mathcal{M}_{11}(k|i)}{\mathcal{M}_{12}(k|i)}}{\frac{\mathcal{M}_{11}(j|i)}{\mathcal{M}_{12}(j|i)} - \frac{\mathcal{M}_{11}(k|i)}{\mathcal{M}_{12}(k|i)}} \cdot (\cot \psi_{ji} - \cot \psi_{ki}) - \cot \psi_{ki} \\
&\Leftrightarrow \alpha_i \stackrel{(157)}{=} \frac{\cot \psi_{ki}^m + \alpha_i^m}{\cot \psi_{ji}^m - \cot \psi_{ki}^m} \cdot (\cot \psi_{ji} - \cot \psi_{ki}) - \cot \psi_{ki} \\
&\Leftrightarrow \alpha_i = \frac{\cot \psi_{ki}^m \cdot \cot \psi_{ji} - \cot \psi_{ki} \cdot \cot \psi_{ji}^m + \alpha_i^m \cdot (\cot \psi_{ji} - \cot \psi_{ki})}{\cot \psi_{ji}^m - \cot \psi_{ki}^m}
\end{aligned} \tag{159}$$

The N-BPM method [28] calculates  $\beta_i$  and  $\alpha_i$  for  $n$  different combinations of three BPMs. Subsequently, the set of  $\beta_i$  and  $\alpha_i$  is weighted by the variance for the respective BPM configuration. In detail, an upper limit  $q$  is defined up to which a neighbouring  $\text{BPM}_q$  is considered acceptable for the calculation of  $\beta_i$  and  $\alpha_i$  at the location of  $\text{BPM}_i$ . Hence,  $n$  and  $q$  are related according to Equation (160), which follows from the binomial coefficient with replacement and two lag parameters for characterising the neighbouring BPMs of  $\text{BPM}_i$ .

$$n = \frac{(q+1)!}{2(q-1)!} \tag{160}$$

For all simulations in this thesis the upper limit for the BPM neighbourhood was set to  $q = 5$ . Consequently, fifteen combinations of BPMs are evaluated to determine the weighted average according to Equation (161).

$$\langle \beta_i \rangle \equiv \sum_{h=1}^{15} w_{(h)} \cdot \beta_i^{(h)} \quad , \quad \langle \alpha_i \rangle \equiv \sum_{h=1}^{15} w_{(h)} \cdot \alpha_i^{(h)} \tag{161}$$

The weights  $w$  depend on the element-wise sum of the covariance matrices for the systematic and statistical errors [28]. The weights are maximised and the corresponding variance is minimised for BPM combinations that fulfill condition (162) [28].

$$\psi_{ji} = \frac{\pi}{4} + n_1 \frac{\pi}{2} \quad \wedge \quad \psi_{ki} = \frac{\pi}{4} + (2n_2 + 1 - n_1) \frac{\pi}{2} \quad , \quad \text{with: } n_1, n_2 \in \mathbb{Z} \tag{162}$$

Condition (162) minimises the variance for  $\beta_i^{(h)}$  and  $\alpha_i^{(h)}$  because the sensitivity coefficients, which relate  $\sigma_\beta$  and  $\sigma_\alpha$  to  $\sigma_\psi$ , are minimised. Hence, the transmission of errors is minimised for BPM combinations fulfilling condition (162), which makes those combinations more robust and less sensitive to errors in the phase measurement. As a consequence, the weights  $w$  can be redefined in terms of the distance to the next optimal phase advance. In detail, the normalised squared inverse of the phase residuum is used for this thesis as the weights of the BPM combinations in Equation (161).

## 8.2 Measurements at ELENA

As a proof of principle, the measurement process described for  $\beta$  and  $\alpha$  is demonstrated for the Extra Low ENergy Antiproton synchrotron (ELENA). ELENA is a deceleration ring that is coupled with the Antiproton Decelerator. The circumference of the ring is 30 metres [37]. The purpose of ELENA is to reduce the energy of antiprotons by a factor of 50, from 5.3 MeV to merely 0.1 MeV, in order to provide cold antiprotons to the antimatter experiments [46]. The parameter settings of the BPMs



installed in the ELENA decelerator are characterised by Expression (163).

$$\left\{ k = 35 \text{ mm} , \frac{\Delta U}{\Sigma U} \Big|_{\text{zero}} = 0 , x_{\text{off}} = 0 \right\} \longrightarrow x(\text{BPM}) \stackrel{(156)}{=} 17.5 \text{ mm} \times \frac{\Delta U}{\Sigma U} \Big|_{\text{meas}} \quad (163)$$

The lattice layout for ELENA is presented in Figure (23). In general, there are twelve plate-type BPMs installed in the ring and the revolution frequency at injection is  $f_o = 1044.3 \text{ kHz}$  [37]. The BPM data acquisition system samples with a sampling rate of 104 measurements per turn. Due to the internal processing system of the analog to digital converter (ADC) the data acquisition is limited to 250 turns. For the coherent beam excitation a chirp was used as an AC-dipole equivalent. A chirp is a signal generator which increases or decreases the driving frequency with time. The accelerator-physics equivalent is a sinusoidally powered dipole, where the frequency of the alternating current is ramped up/down. In order to apply the AC-dipole formalism derived in Chapter 6, the chirp frequency range has to be restricted to the narrowest frequency bandwidth possible. The corresponding knob entitles ELENA users to shrink the frequency bandwidth up to 0.001 in tune units. Unfortunately, there is no automatized routine for an adiabatic ramp up of the excitation amplitude of the chirp. Hence, the adiabaticity constraint elaborated in Chapter 6.3.3 is generally not met for the excitations applied. As a consequence, an emittance growth has to be accepted for the beam excitation in ELENA. Therefore, the conducted lattice function measurements at ELENA were destructive measurements.

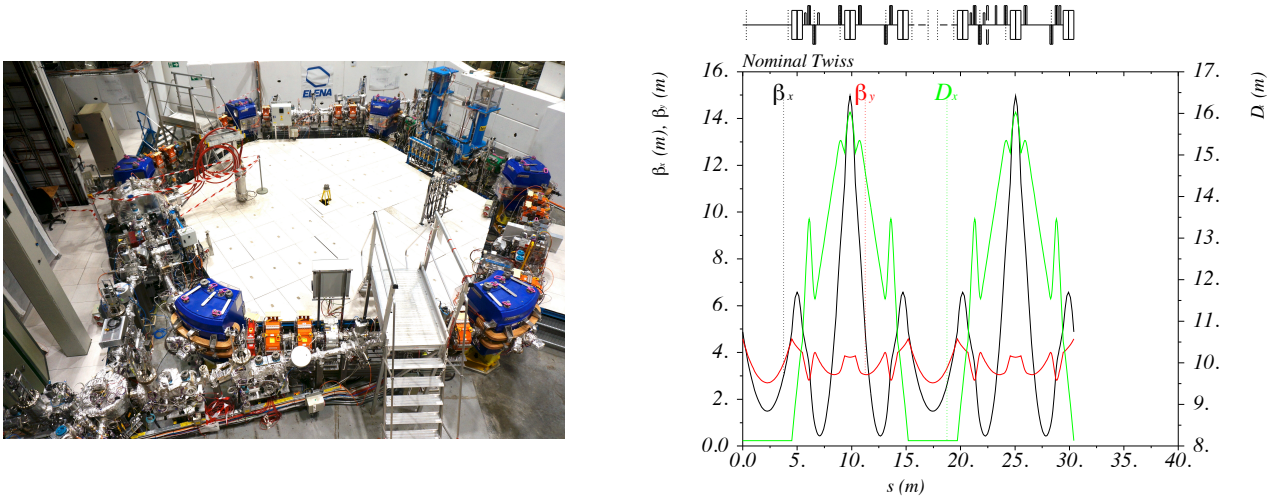


Figure 23: The technical layout of ELENA (left) and the corresponding lattice model (right) is depicted. On top of the right plot, the placement of the lattice elements is indicated. Here, the vertical dotted lines indicate the positions of the plate-type BPMs.

For the measurement setup, the electron cooler was turned off and the ring was injected with a proton beam from the local source. The cycle length was roughly 4 s. As an example, the horizontal orbit measurements from BPMEA.0240.H at 17:34:43 on 04.12.2020 (CET) are visualised in Figure (24) and Figure (25) for a horizontal excitation frequency of  $Q_d = 0.437$ . In order to extract the signal of the induced charge when the beam is passing the BPM plates, a comb function was used as a filter. In detail, the maximum of each single  $\Sigma U$  peak was determined within the comb window and the corresponding timestamp was utilised to select the datapoint of the  $\Delta U$  signal. The potential measurements were then used as an input for Equation (156) under the settings listed in Expression (163). At this point it has to be emphasized, that the procedure described above resulted in an observation of asynchronicity between the  $\Sigma U$  and  $\Delta U$  signal. The  $\Delta U$  datapoint, which corresponds to the timestamp of the selected  $\Sigma U$  measurement, is not equal to the absolute maximum of the  $\Delta U$

signal within the comb window. The opposite is expected because the absolute potential difference between the plates should be maximized if a bunch is passing the BPM. As a consequence, the data acquisition routine was modified such that the maximum of the absolute  $\Delta U$  signal is selected within the corresponding comb window for the evaluation of the orbit  $x$ . The distribution of the time delay between  $\max(\Sigma U)$  and  $\max|\Delta U|$  did not reveal a systematic timing offset. Hence, it has to be further investigated what causes the sporadic asynchronicity between the  $\Sigma U$  and the  $\Delta U$  signal.

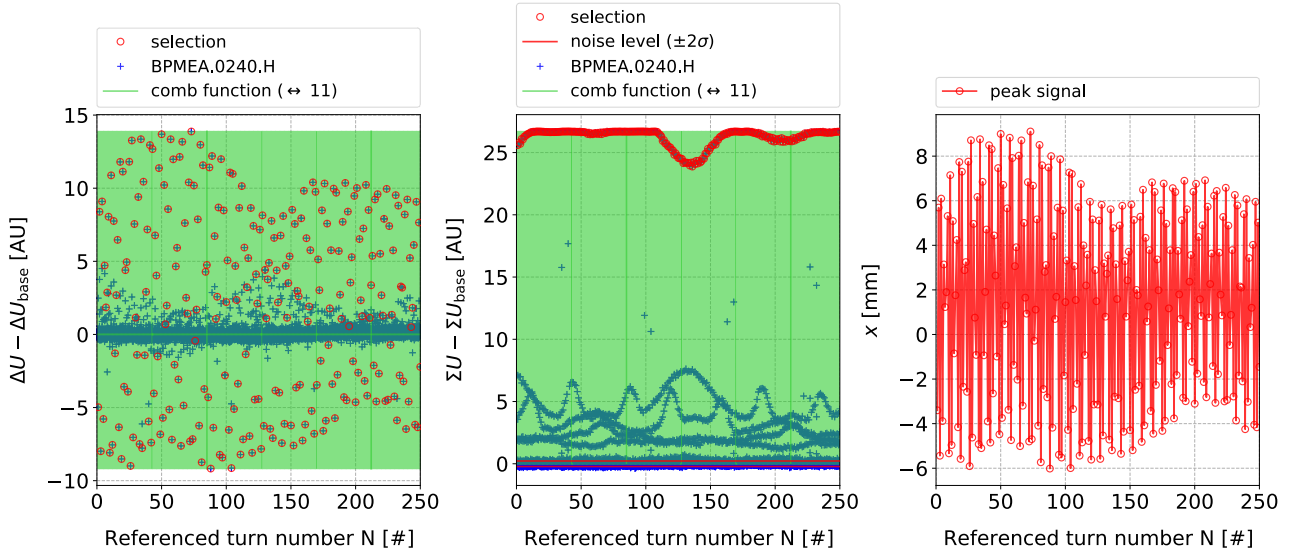


Figure 24: Turn-by-turn measurement data from BPMEA.0240.H of the ELENA decelerator, performed at 17:34:43 on 04.12.2020 (CET), is presented. On the left side, the potential difference between the plates of the BPM is plotted as a function of the turn number. The turn number was referenced to the first measurement taken. In the middle graph, the accumulated potential of both plates is shown as a function of the referenced turn number. On the right side, the orbit resulting from Equation (163) is visualized. In blue the raw measurement data is shown and in green the comb windows from the filter are visualized. The datapoints highlighted by red circles are those selected for the orbit calculation.

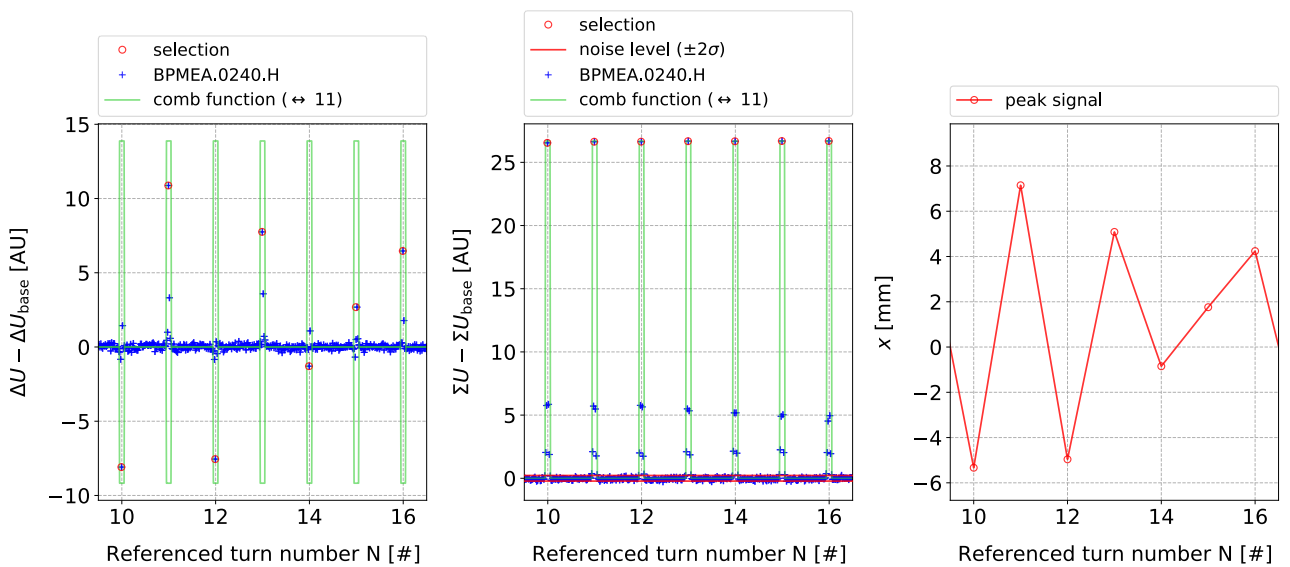


Figure 25: Turn-by-turn measurement data from BPMEA.0240.H of the ELENA decelerator, performed at 17:34:43 on 04.12.2020 (CET), is presented. The plots shown are equivalent to those in Figure (24) but restricted to  $N \in [10, 16]$ .

In general, three tune scans have been performed. The excitation frequency was varied for different excitation amplitudes as listed in Figure (26).

|          | $Q_d(H)$ | $Q_d(V)$ | timestamp |          | $Q_d(H)$ | $Q_d(V)$ | timestamp |          | $Q_d(H)$ | $Q_d(V)$ | timestamp |
|----------|----------|----------|-----------|----------|----------|----------|-----------|----------|----------|----------|-----------|
|          | [1]      | [1]      | [CET]     |          | [1]      | [1]      | [CET]     |          | [1]      | [1]      | [CET]     |
| $i_1$    | 0.443    | 0.0      | 17:18:13  | $i_1$    | 0.0      | 0.428    | 18:14:43  | $i_1$    | 0.443    | 0.0      | 17:16:36  |
| $i_2$    | 0.442    | 0.0      | 17:13:26  | $i_2$    | 0.0      | 0.427    | 18:10:18  | $i_2$    | 0.442    | 0.0      | 17:15:03  |
| $i_3$    | 0.441    | 0.0      | 17:23:48  | $i_3$    | 0.0      | 0.426    | 18:11:54  | $i_3$    | 0.441    | 0.0      | 17:26:12  |
| $i_4$    | 0.440    | 0.0      | 17:27:32  | $i_4$    | 0.0      | 0.418    | 17:56:54  | $i_4$    | 0.440    | 0.0      | 17:28:44  |
| $i_5$    | 0.439    | 0.0      | 17:31:18  | $i_5$    | 0.0      | 0.417    | 17:58:20  | $i_5$    | 0.439    | 0.0      | 17:30:18  |
| $i_6$    | 0.438    | 0.0      | 17:32:24  | $i_6$    | 0.0      | 0.416    | 17:59:31  | $i_6$    | 0.438    | 0.0      | 17:33:54  |
| $i_7$    | 0.437    | 0.0      | 17:35:31  | $i_7$    | 0.0      | 0.415    | 18:00:42  | $i_7$    | 0.437    | 0.0      | 17:34:43  |
| $i_8$    | 0.432    | 0.0      | 16:32:59  | $i_8$    | 0.0      | 0.414    | 18:02:18  | $i_8$    | 0.432    | 0.0      | 17:21:48  |
| $i_9$    | 0.422    | 0.0      | 16:37:18  | $i_9$    | 0.0      | 0.413    | 18:04:02  | $i_9$    | 0.422    | 0.0      | 16:39:12  |
| $i_{10}$ | 0.412    | 0.0      | 17:09:48  | $i_{10}$ | 0.0      | 0.412    | 18:05:30  | $i_{10}$ | 0.412    | 0.0      | 17:11:25  |
|          |          |          |           | $i_{11}$ | 0.0      | 0.411    | 18:06:56  |          |          |          |           |
|          |          |          |           | $i_{12}$ | 0.0      | 0.410    | 18:07:54  |          |          |          |           |

(a) MSC1:  $\theta_d(H)/\theta_{d_{\max}} = 50\%$ .      (b) MSC2:  $\theta_d(V)/\theta_{d_{\max}} = 50\%$ .      (c) MSC3:  $\theta_d(H)/\theta_{d_{\max}} = 100\%$ .

Figure 26: Tune scans performed at ELENA on December 4<sup>th</sup>, 2020 for different kick amplitudes  $\theta_d$ .

The harmonic analysis of the turn-by-turn data for the orbit  $x$  is shown in spectrum (27) for  $i_4$  of measurement set MSC1. In detail, spectrum (27) is an accumulation of all spectra for the individual BPMs. Furthermore, the complete scan of the maximum in the normalised spectral amplitude is presented as a function of the driven tune for MSC1. First of all, it has to be noticed that the horizontal excitation frequency of  $Q_d = 0.44$  does not match with the frequency of the maximum in the spectrum at 0.438. At the first glance, this is nothing suspicious because the natural tune could be at  $Q_o = 0.438$ . Nevertheless, the amplitude of the spectrum is normalised to the maximum amplitude observed during the tune scan. Consequently, the amplitude in the spectrum should only be one if the driven tune is equal to the natural tune  $Q_d \stackrel{!}{=} Q_o$ . This points to a failure of the beam excitation at  $Q_d = 0.438$  for MSC1. This hypothesis is underpinned by the tune scan MSC3, which indicates that the horizontal natural tune is  $Q_o(H) = 0.437 \pm 0.001$ . Alternatively, a tune jump [38] could have been observed during the tune scan MSC1. Do to the limitation in the maximum number of turns that can be acquired, the resolution in the frequency domain is highly limited. For the spectral analysis, zero padding was used to artificially increase the number of samples in the spectrum. Zero padding will reveal details in the frequency domain but not increase the spectral resolution, since adding zeros does not change the information content or the Shannon entropy respectively. As a consequence, the width of the frequency peak will be invariant under zero padding. The cardinality of the zero pad was set to one million. In the spectrum the driven frequency clearly outstands but a lot of sidebands are noticeable in addition. Those sidebands are theoretically expected because no windowing has been used in the discrete Fourier transformation (DFT). A single Dirac-Delta impulse in the frequency domain is only expected for an infinite sinusoidal time signal. In contrast, time limited signals implicitly equal rectangularly windowed signals. Hence, the sinusoidal betatron motion is implicitly convoluted by a rectangular window function, which results in a Sinc-function centered at the driven frequency  $Q_d$ . As a result, the sidebands in the spectrum are expected artefacts from the implicit windowing of the turn-by-turn data.

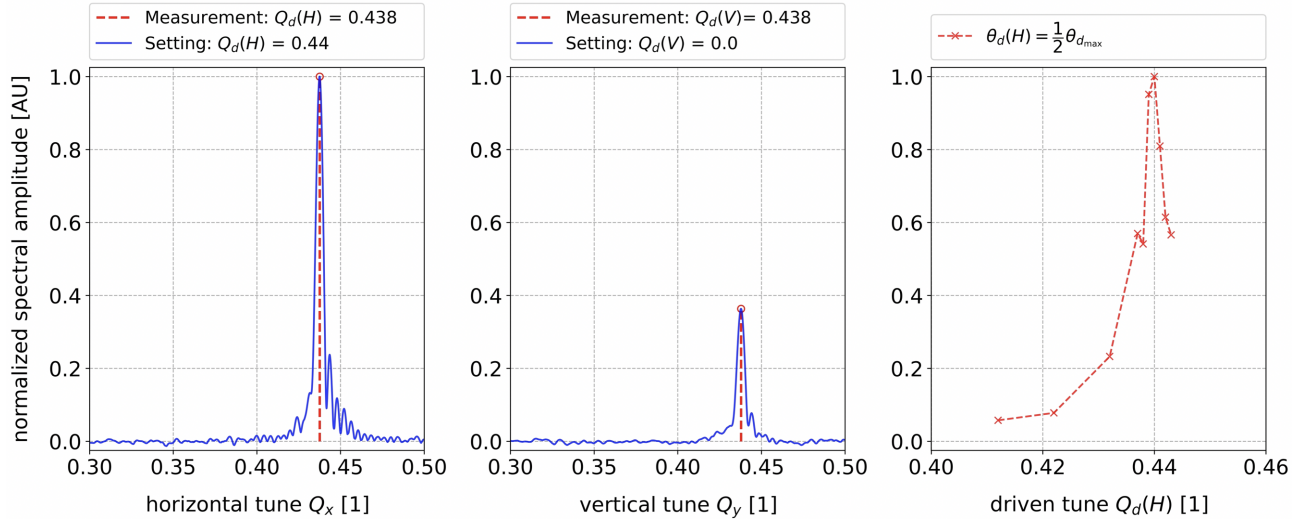


Figure 27: Frequency spectrum of the turn-by-turn data from  $i_4$  of MSC1. On the left side, the accumulated spectrum for all twelve horizontal BPMs is visualised. In the middle graph, the accumulated spectrum for all twelve vertical BPMs is shown. The frequency at maximum amplitude is highlighted by a red dashed line. On the right side, the normalised spectral amplitude is presented as a function of the horizontal driven tune for MSC1.

Figure (28) presents the results of the spectral analysis for  $i_6$  of MSC2. The vertical tune scan indicates a natural tune of  $Q_o(V) = 0.416 \pm 0.001$ . For all measurement series, significant coupling between the horizontal and vertical plane is observed. Even though the chirp excitation is only applied in a single plane at a time, an excitation in the perpendicular plane is observed at the same frequency. The question could arise, if this effect might be caused by a single BPM which is skewed. This question is easily answered by visualizing the spectrum of the turn-by-turn data for every BPM separately. From this check it was concluded that the coupling is observed by all twelve BPMs. Hence, there is a source of coupling in the ELENA synchrotron.

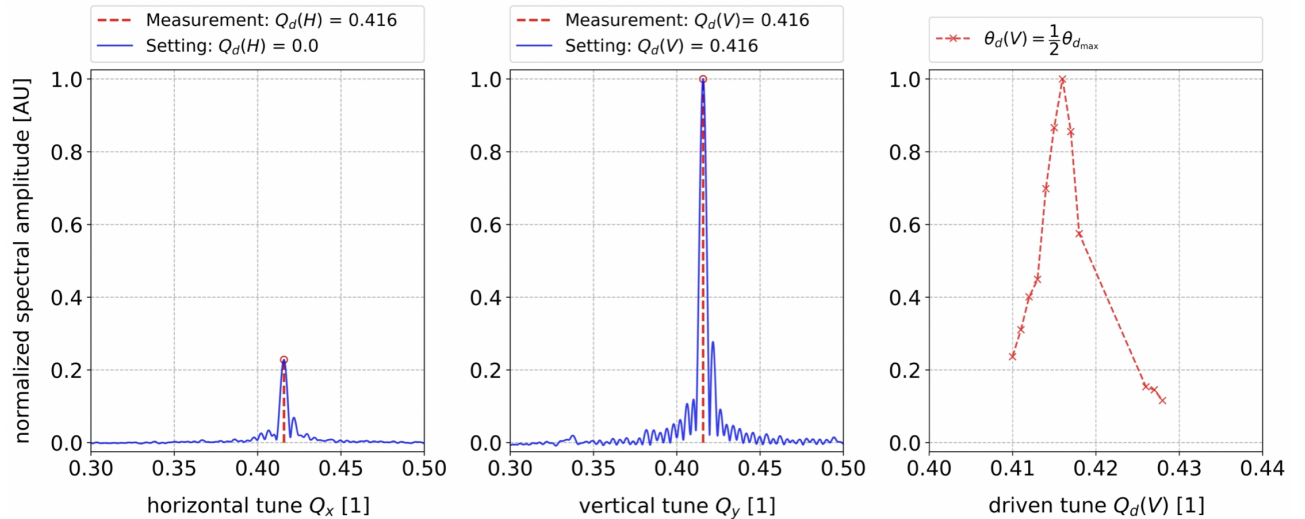


Figure 28: Frequency spectrum of the turn-by-turn data from  $i_6$  of MSC2. On the left side, the accumulated spectrum for all twelve horizontal BPMs is visualised. In the middle graph, the accumulated spectrum for all twelve vertical BPMs is shown. The frequency at maximum amplitude is highlighted by a red dashed line. On the right side, the normalised spectral amplitude is presented as a function of the vertical driven tune for MSC2.



For future machine diagnostic studies at ELENA, the observed coupling should be minimized before new measurements are taken. Figure (29) visualizes the spectrum for  $i_7$  of MSC3. The kick amplitude for MSC3 was set twice as high as for MSC1. As discussed, the tune scans for MSC3 and MSC1 differ because of a malfunctioning in the excitation system for MSC1.

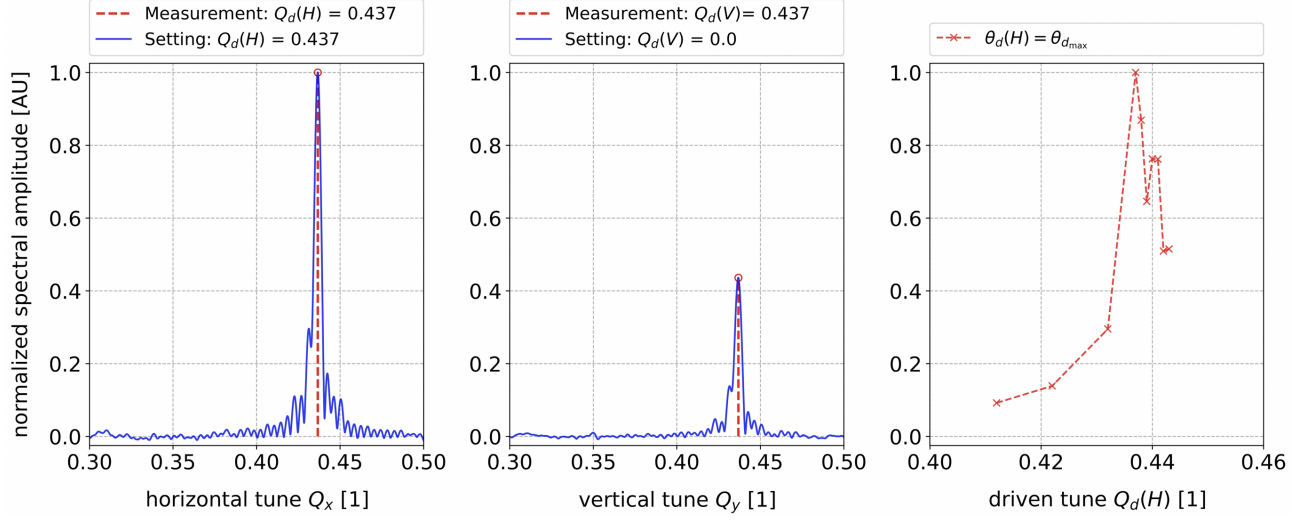


Figure 29: Frequency spectrum of the turn-by-turn data from  $i_7$  of MSC3. On the left side, the accumulated spectrum for all twelve horizontal BPMs is visualised. In the middle graph, the accumulated spectrum for all twelve vertical BPMs is shown. The frequency at maximum amplitude is highlighted by a red dashed line. On the right side, the normalised spectral amplitude is presented as a function of the horizontal driven tune for MSC3.

From the complex amplitude  $\hat{A}_i$  at the driven frequency  $Q_d$  the local phase  $\phi$  can be obtained for each BPM $_i$ , as presented by Equation (164).

$$\phi_i = \arctan_2 \left( \frac{\text{Im} [\hat{A}_i \{Q_d\}]}{\text{Re} [\hat{A}_i \{Q_d\}]} \right) \quad (164)$$

The local phase  $\phi$  for subsequent elements determines the fractional part of the phase advance  $\psi$  between them. The measured fractional phase advance  $\psi_{Q_{AC}}$  can then be compared to the model fractional phase advance  $\psi^{(m)}$ , as illustrated in the left graph of Figure (30) for  $i_7$  of MSC3.

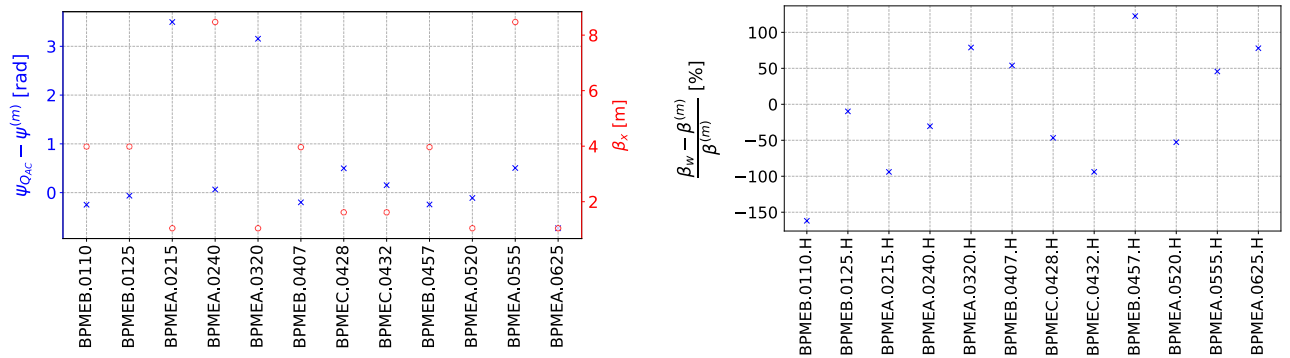


Figure 30: On the left side, the difference between measurement and model for the fractional part of the phase advance for subsequent elements is shown. On the right side, the deviation of the reconstructed  $\beta$ -function from the model is visualized. The reconstruction is based on Equation (158) utilising the N-BPM method.

Most of the BPMs show a phase deviation in the order of one rad except of those which are influenced by high  $\beta$ -values. This is reasonable, since the  $\beta$ -function serves as a sensitivity coefficient for lattice imperfections. The phase measurements can be utilised to determine the lattice function according to the N-BPM method. Applying Equation (158) results in the right plot of Figure (30). The huge deviation of the reconstructed  $\beta$ -function from the model reveals the fact, that optics and beam were unmatched. Nevertheless, there is a huge discrepancy in the phase measurement, which in first place deteriorates the results for the reconstructed  $\beta$ -function. In order to improve the accuracy of the phase measurement, the data acquisition system at ELENA has to be upgraded so that more turns can be extracted and the spectral resolution in the frequency domain increases.

Nevertheless, the measurements can be used to determine the AC-dipole invariant  $\mathcal{I}$ . The AC-dipole invariant  $\mathcal{I}$  is the excitation equivalent to the emittance  $\epsilon$  of the unforced betatron motion. Hence, equation (38) can be utilised to determine  $\mathcal{I}$  for the driven motion. As a sight check, the absolute value of the maximum spectral amplitude is plotted (31) against the position of the corresponding BPM in the ring. If the artificial sidebands from the Sinc-function are neglected, the maximum spectral amplitude equals half the amplitude of the betatron motion. Consequently, the maximum spectral amplitude should correlate with the model value of  $\sqrt{\beta}$ . This correlation can be quantified by calculating the slope of the graph  $\hat{A} = f\{\sqrt{\beta}\}$ , as visualized in the right plot of Figure (31). In detail, the slope of the graph equals the quadrature of the AC-dipole invariant  $\mathcal{I}$ . For a driven frequency of  $Q_d = 0.437$  the ACD invariant is determined as  $\mathcal{I}_x = 2.99 \times (1 \pm 0.17) \mu\text{m}$ . The uncertainty of the slope from the line fit, with the constraint  $\hat{A}(0) = 0$ , was transformed to an uncertainty in  $\mathcal{I}$  by applying Gaussian error propagation. As highlighted earlier, the analysis method used to calculate the ACD invariant implicitly assumes a Dirac-Delta impulse in the frequency domain for the driven frequency  $Q_d$ . This assumption corresponds physically to a pure AC-dipole excitation. Due to the fact, that a restricted chirp is used instead of an AC-dipole, the assumption of a Dirac-Delta frequency distribution is highly questionable. This sets the agenda, the results for the ACD invariant under the Dirac-Delta assumption have to be compared to a second analysis method, which is based on a high-pass filter.

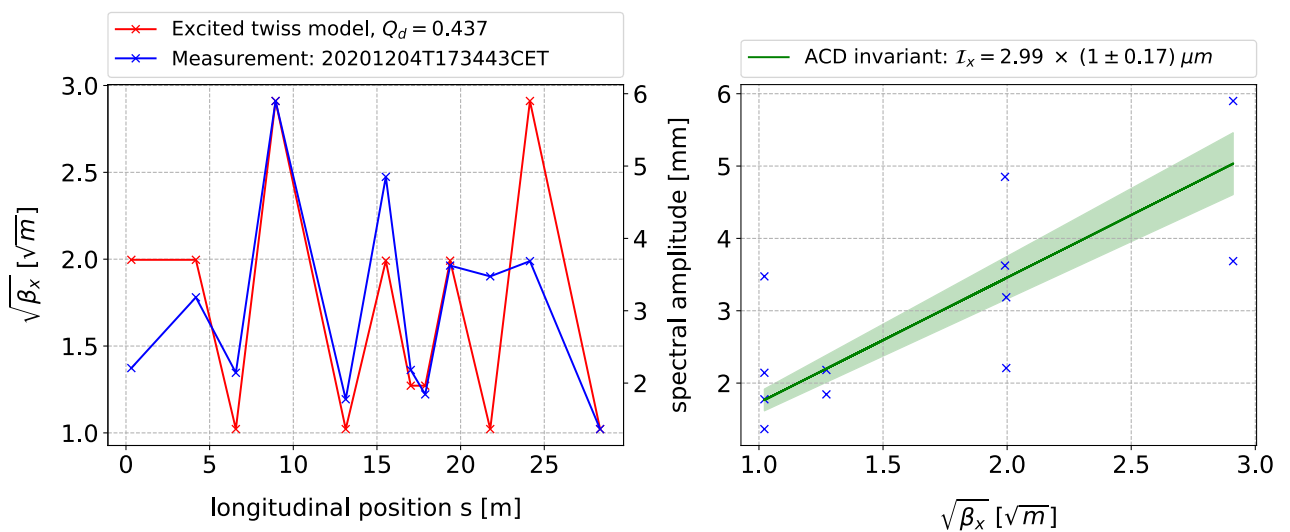


Figure 31: On the left side, the square root of the  $\beta$ -function  $\sqrt{\beta_x}$  as well as the absolute value of the maximum spectral amplitude  $|\hat{A}|$  is plotted for each BPM location of  $i_7$  in MSC3. On the right side, the correlation between  $|\hat{A}|$  and  $\sqrt{\beta_x}$  is visualised. The line of best fit under the constraint  $\hat{A}(0) = 0$  is plotted in green. The corresponding confidence band on a confidence level of 68% is visualized in light green. Measurement  $i_7$  indicates an ACD invariant of  $\mathcal{I}_x = 2.99 \times (1 \pm 0.17) \mu\text{m}$  for  $Q_d = 0.437$ .

In so doing, a conclusion can be drawn whether the restricted chirp excitation has AC-dipole character or not. If all measurement sets from Figure (26) are analysed under the Dirac-Delta assumption, the results presented in Figure (32) will be obtained. Figure (32) shows the evolution of the ACD invariant when the excitation frequency is changed.

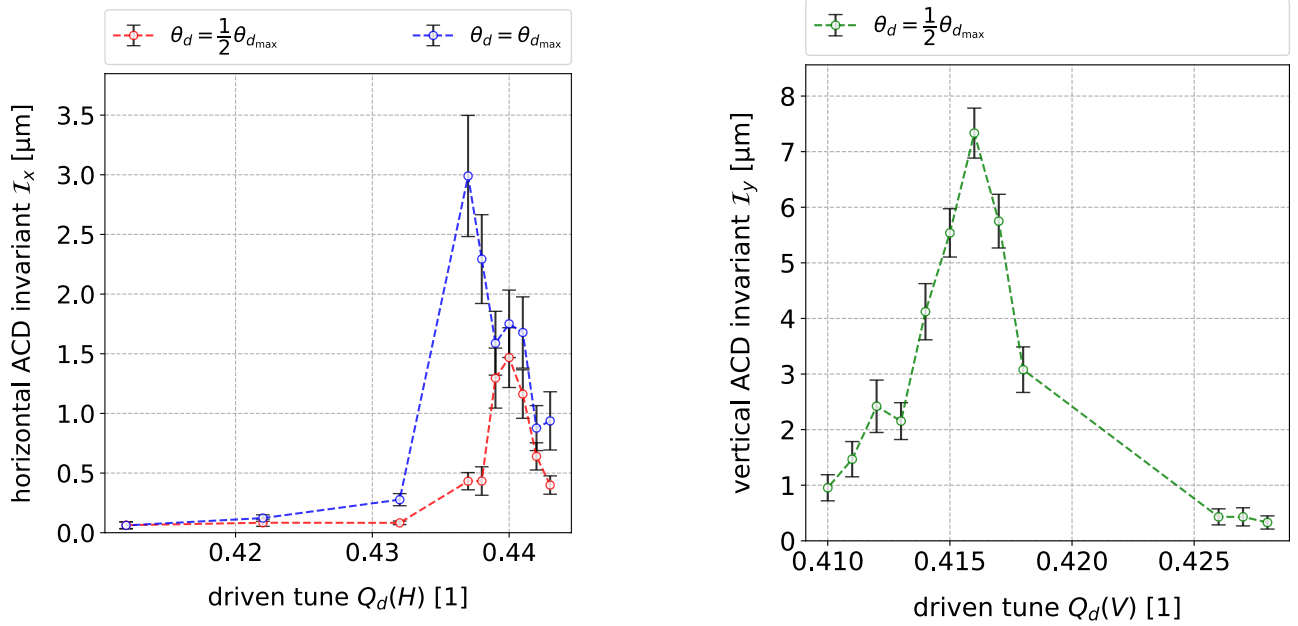


Figure 32: The horizontal ACD invariant  $\mathcal{I}_x$  is visualised as a function of the driven tune  $Q_d(H)$  (left). Additionally, the vertical ACD invariant  $\mathcal{I}_y$  is plotted versus the driven tune  $Q_d(V)$  (right). The ACD invariant was determined under the Dirac-Delta impulse assumption in the frequency domain. The curves corresponding to MSC1 and MSC3 differ by a factor of two in amplitude because of the doubled excitation strength for MSC3 in comparison to MSC1. Furthermore, the peak shift in  $\hat{\mathcal{I}}_x$  from MSC1 too MSC3 can be traced back to a malfunctioning in the excitation system or a tune jump for MSC1.

As an alternative determination method for the ACD invariant, the amplitude  $\hat{x}$  of the turn-by-turn oscillations can directly be correlated to the square root of the  $\beta$ -function. The slope of the line of best fit, with the constraint of  $\hat{x}(0) = 0$ , will correspond to the quadrature of the ACD invariant according to Equation (38). The lines of best fit for MSC3 are presented in Figure (33). For the model  $\beta$ -function the  $\beta$ -beating contribution from the chirp has to be considered, as elaborated in Chapter 6. Methodologically, using the amplitude of the turn-by-turn data for the analysis is equivalent to a high-pass filter. The reason for this is that the amplitude of the spatial betatron oscillation around the closed orbit is influenced by the entire frequency distribution around  $Q_d$ . Hence, all frequency components of the chirp affecting the amplitude of the betatron motion will be included into the calculation of the ACD invariant, which is tantamount to a high-pass filter (HPF). The induced  $\beta$ -beating by the chirp is in the order of several percents, as visualized in the right plot of Figure (33). As the driven tune  $Q_d$  gets closer and closer to the natural tune  $Q_o$ , the  $\beta$ -beating of the chirp decreases because the imposed tune shift reduces. In contrast, the ACD invariant increase if the beam is excited close to the natural tune of the lattice because of the resonance condition. Figure (34) concludes the studies on the ACD invariant analysis utilising the high-pass filter approach. By comparing Figure (32) with Figure (34) it can be concluded, that the results of both analysis methods are in good agreement within the measurement uncertainties. Consequently, the restricted chirp excitation does not comprise a long tail frequency distribution. As a result, the restricted chirp excitation can be considered equivalent to the AC-dipole excitation.



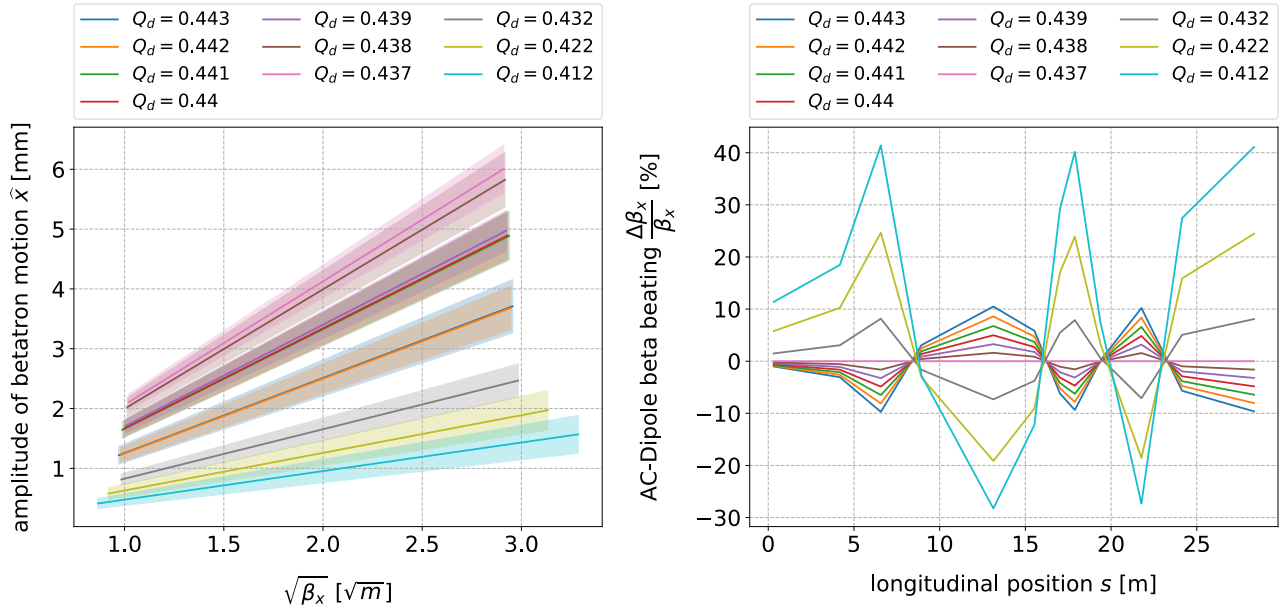


Figure 33: On the left side, the amplitude of the betatron motion  $\hat{x}$  is plotted versus the square root of the  $\beta$ -function. In detail, the lines of best fit under the constraint  $\hat{x}(0) = 0$  are visualized with the corresponding confidence band on a confidence level of 68%. As the driven tune  $Q_d$  increases, the slopes are rising as well. On the right side, the induced  $\beta$ -beating from the chirp is presented as a function of the longitudinal position.

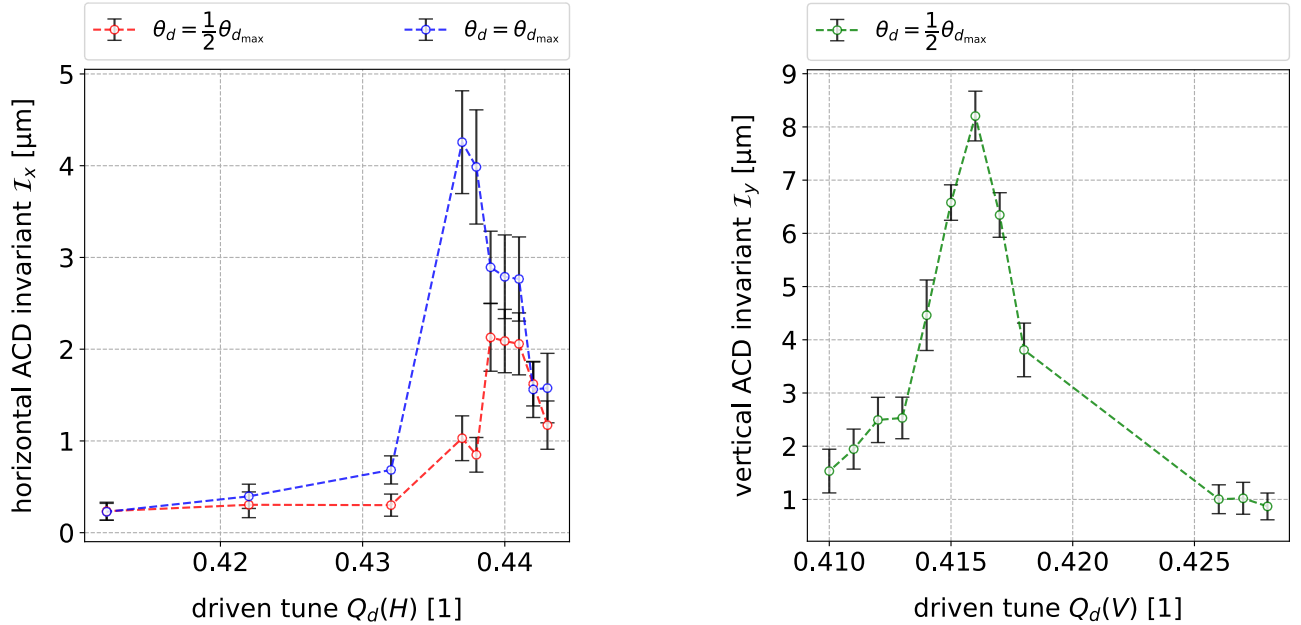


Figure 34: On the left side, the horizontal ACD invariant  $\mathcal{I}_x$  is visualised as a function of the driven tune  $Q_d(H)$ . On the right side, the vertical ACD invariant  $\mathcal{I}_y$  is plotted versus the driven tune  $Q_d(V)$ . The ACD invariant was determined by utilising the high-pass filter method. The curves corresponding to MSC1 and MSC3 differ by a factor of two in amplitude because of the doubled excitation strength for MSC3 in comparison to MSC1. Furthermore, the peak shift in  $\hat{\mathcal{I}}_x$  from MSC1 too MSC3 can be traced back to a malfunctioning in the excitation system or a tune jump for MSC1 respectively.

## 9 Simulation method

In Chapter 8, the measurement principle for the lattice functions was presented. Furthermore, orbit measurements at the ELENA decelerator at CERN were conducted in order to demonstrate the feasibility of the measurement technique. This chapter will focus on the correction method for quadrupolar imperfections in the arcs. In detail, closed orbit bumps will be constructed to generate quadrupolar feed-downs from sextupoles in order to tare the lattice. First of all, the conceptual idea is set out and the correction method is defined mathematically by the response matrix approach. Secondly, the implementation of the correction method will be discussed including several critical considerations regarding the bump neighbourhood. Finally, the scope of the implemented correction method will be determined and the method will be validated by testing the correction robustness against noise and measurement uncertainties. The simulations conducted within this section will focus on minimizing the residual RMS-beating in all lattice quantities. Since the correction of serial powered quadrupoles in the telescopic squeezing region around IP5 (CMS) and IP1 (ATLAS) is matter of particular interest, the simulations will mainly focus on systematic errors within arc45 and arc56, see Figure 10.

### 9.1 Conceptual idea

The flowchart in Figure 35 visualizes the conceptual idea of the correction strategy. In general, the proposed strategy belongs to the class of Mikado correction schemes. In blue the given models are indicated. The nominal lattice model is needed as well as the excited model, in order to account for the inherent beta beating of the AC-dipole (labelled as  $\Delta\beta_{AC}$ ). In orange the unknown perturbed model is illustrated, which results from the underlying error distribution. If the beam motion is synchronised by AC-dipole excitations, turn-by-turn measurements with the BPMs can be taken. The local phase of the betatron motion at the BPM position can be obtained by a spectral analysis of the turn-by-turn data. Afterwards, the N-BPM method [29] can be applied to extract the beta-function at the BPM location. Those beta-values are given in the perturbed frame, wherefore the betas have to be corrected by the expected beta beating from the AC-dipole.

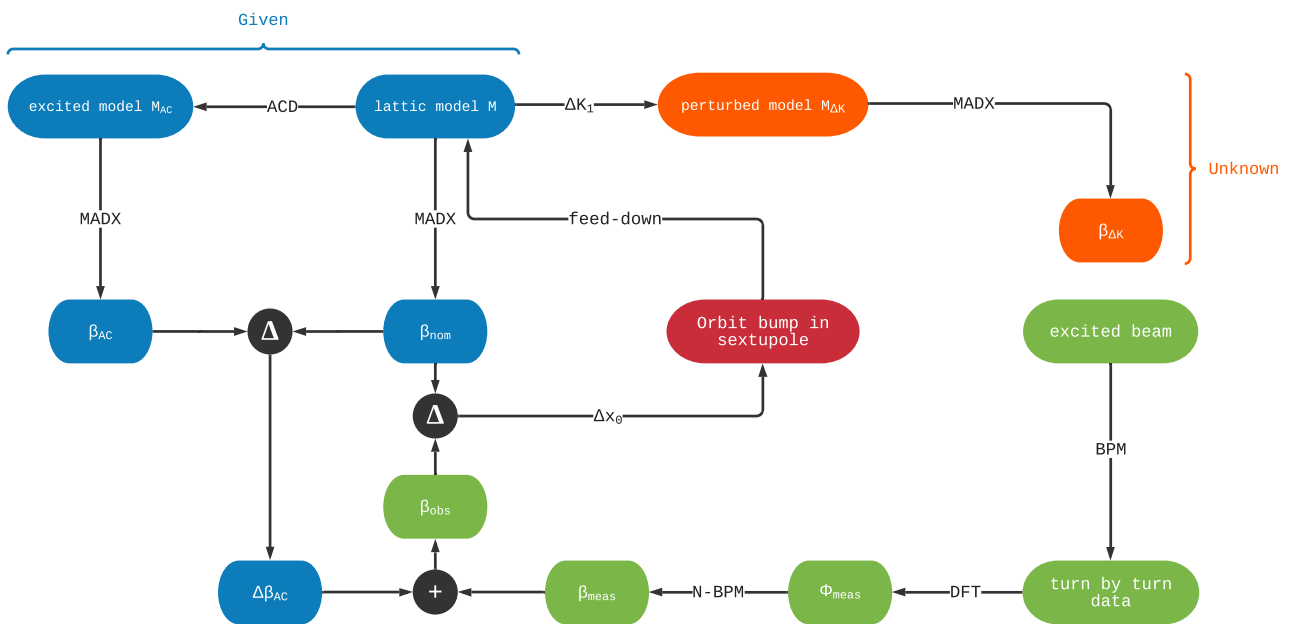


Figure 35: Flowchart of simulation strategy.

Finally, the measured beta-function at the BPMs location can be compared to the nominal values. If a significant beta beating contribution from a quadrupole field error in the arcs is noticed, an orbit bump around a suitable sextupole is implemented for compensation. Here, the desired closed orbit offset in the sextupole serves as a knob for the correction process. The magnitude of the knob is determined by Equation (167) using the response matrix approach.

## 9.2 Response matrix approach

Assume a linear system  $\mathbf{S}_1$  which is determined by a set of measurable observables  $\mathcal{O} \in \mathbb{R}^m$  and a set of input parameters  $\mathcal{P} \in \mathbb{R}^n$ . If the system is linear then the observables can be calculated directly by the response matrix  $\mathbf{R}$  of the system, as presented in Equation (165).

$$\mathbf{S}_1 : \quad \vec{\mathcal{O}} = \mathbf{R} \cdot \vec{\mathcal{P}} \quad \text{with:} \quad \mathbf{R} \equiv \begin{pmatrix} \frac{\partial \mathcal{O}_1}{\partial \mathcal{P}_1} & \frac{\partial \mathcal{O}_1}{\partial \mathcal{P}_2} & \cdots & \frac{\partial \mathcal{O}_1}{\partial \mathcal{P}_n} \\ \frac{\partial \mathcal{O}_2}{\partial \mathcal{P}_1} & \frac{\partial \mathcal{O}_2}{\partial \mathcal{P}_2} & \cdots & \frac{\partial \mathcal{O}_2}{\partial \mathcal{P}_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \mathcal{O}_m}{\partial \mathcal{P}_1} & \frac{\partial \mathcal{O}_m}{\partial \mathcal{P}_2} & \cdots & \frac{\partial \mathcal{O}_m}{\partial \mathcal{P}_n} \end{pmatrix} \in \mathbb{R}^{m \times n} \quad (165)$$

Equation (165) becomes invalid if a non-linear system  $\mathbf{S}_2$  is considered. Nevertheless, deviations from the expected parameter vector  $\vec{\mathcal{P}}$  can be propagated with the response matrix  $\mathbf{R}$  in first order approximation, as indicated by Equation (166).

$$\mathbf{S}_2 : \quad \overrightarrow{\Delta \mathcal{O}} = \mathbf{R} \cdot \overrightarrow{\Delta \mathcal{P}} + \mathcal{O}(\Delta^2) \quad (166)$$

In practice, deviations of the observables with respect to the model values are measured and it is desired to correct them as much as possible. Hence, the inversion of Equation (166) is needed as indicated in Expression (167). The idea is to adjust the parameters or the controllable knobs in order to equalise the deviations in the observables.

$$\mathbf{S}_2 : \quad \overrightarrow{\Delta \mathcal{P}}_{\text{knobs}} \approx \mathbf{R}^{-1} \cdot \overrightarrow{\Delta \mathcal{O}}_{\text{meas}} \quad (167)$$

Mathematically, the inversion of the response matrix  $\mathbf{R}$  is feasible if  $\mathbf{R}$  is a square matrix with full rank. Due to the fact that the number of observables  $m$  often exceeds the number of adjustable knobs  $n$ , this condition is often not met. As a consequence, the response matrix  $\mathbf{R}$  has to be pseudo-inverted, which can be realized by using singular value decomposition (SVD) [36].

For the purpose of this thesis, the observable vector  $\vec{\mathcal{O}}$  is constructed by the measured beating pattern in the phase advance  $\Delta\psi$ , segment by segment, as well as the normalised dispersion  $D_n \equiv D/\sqrt{\beta}$ . The N-BPM method can be utilised in order to determine the  $\beta$ - and  $\alpha$ -function. Consequently, the beating pattern of four lattice quantities is observable. Combining several beating patterns will increase the spatial resolution of the correction method because potentially degenerated beating waves will become undegenerated in the vector space  $\mathbb{R}_{\mathcal{O}} \equiv \mathbb{R}_{\Delta\beta} \times \mathbb{R}_{\Delta\alpha} \times \mathbb{R}_{\Delta\psi} \times \mathbb{R}_{\Delta D}$ . Hence, the Synergy effect of combining different beating patterns will boost the localization power of the method for lattice imperfections. The mathematical description of the beating pattern has been derived in Chapter (5) and is condensed in the observable vector  $\vec{\mathcal{O}} \in \mathbb{R}^{4N_{\text{BPM}}}$ , specified by Expression (168).

$$\vec{\mathcal{O}} \equiv \begin{pmatrix} \Delta\beta(s_1 = \text{BPM}_1) \\ \vdots \\ \Delta\alpha(s_1 = \text{BPM}_1) \\ \vdots \\ \Delta\psi(s_1 = \text{BPM}_1) \\ \vdots \\ \frac{\Delta D(s_1 = \text{BPM}_1)}{\sqrt{\beta(s_1 = \text{BPM}_1)}} \\ \vdots \\ \frac{\Delta D(s_N = \text{BPM}_N)}{\sqrt{\beta(s_N = \text{BPM}_N)}} \end{pmatrix} = \begin{cases} \Delta\beta(s_1) \stackrel{(82)}{=} -\frac{\beta_d(s_d) \cdot \beta(s_1)}{2f_\epsilon \sin(2\pi Q_o)} \cdot \cos\left(2|\psi(s_1) - \psi(s_d)| - 2\pi Q_o\right) \\ \vdots \\ \Delta\alpha(s_1) \stackrel{(87)}{=} \alpha(s_1) \cdot \frac{\Delta\beta}{\beta}(s_1) - \theta(s_1 - s_d) \cdot \frac{\beta_d}{2f_\epsilon \sin \psi_o} \cdot \sin\left(2|\psi(s_1) - \psi_d| - \psi_o\right) \\ \vdots \\ \Delta\psi_{fi} \stackrel{(93)}{=} \frac{\beta_d}{2f_\epsilon \sin(2\pi Q_o)} \cdot \sin \psi_{fi} \cdot \cos\left(2\pi Q_o - |\psi_f - \psi_d| - |\psi_i - \psi_d|\right) \\ \vdots \\ \frac{\Delta D(s_1)}{\sqrt{\beta(s_1)}} \stackrel{(111)}{=} -\frac{\sqrt{\beta(s_d)} \cdot D(s_d)}{2f_\epsilon \sin(\pi Q_o)} \cdot \cos\left(|\psi(s_1) - \psi(s_d)| - \pi Q_o\right) \\ \vdots \\ \frac{\Delta D(s_N)}{\sqrt{\beta(s_N)}} \stackrel{(111)}{=} -\frac{\sqrt{\beta(s_d)} \cdot D(s_d)}{2f_\epsilon \sin(\pi Q_o)} \cdot \cos\left(|\psi(s_N) - \psi(s_d)| - \pi Q_o\right) \end{cases} \quad (168)$$

### 9.3 Bump design

The response matrix approach requires a precise definition of the knobs. In general the partial derivative of the observable beating vector  $\vec{\mathcal{O}}$  (168) with respect to the set of sextupolar knob parameters  $f_\epsilon$  constructs the response matrix  $\vec{\mathcal{R}}$ , as defined by Equation (169).

$$\vec{\mathcal{R}} \equiv \left[ \vec{\nabla}_{f_\epsilon: \text{MS.i.B1}} \otimes \vec{\mathcal{O}}^T \right]^T \in \mathbb{R}^{4N_{\text{BPM}} \times N_{\text{bumps}}} \quad (169)$$

Each knob quantifies the closed orbit offset  $\Delta x_o$  inside a suitable sextupole (MS) in the arc, where a closed orbit bump is possible.

A closed orbit bump is constructed by three horizontal kickers, which are basically dipoles that do not change the reference orbit. The kick strengths are determined by three constraints. First of all, the elongation from the closed orbit  $\Delta x_o$  has to be accomplished at the location of the target sextupole. Secondly, the orbit bump has to be closed, which results in the constraints that  $x(\text{last kicker}) = 0$  and  $x'(\text{last kicker}) = 0$ . Figure (36) visualises a closed orbit bump with 1 mm amplitude. In general, there are 169 possible bumps and 392 FODO-cell quadrupoles in the arc, which are powered in series. The factor of roughly two comes from the fact, that every second sextupole in the arc is equipped with a horizontal kicker and the other half is equipped with vertical kickers.

Figure (36) indicates that the middle kicker is almost negligible in terms of the kicking strength. This is expected because the periodic lattice structure in the arc imposes a reflection symmetry of the closed

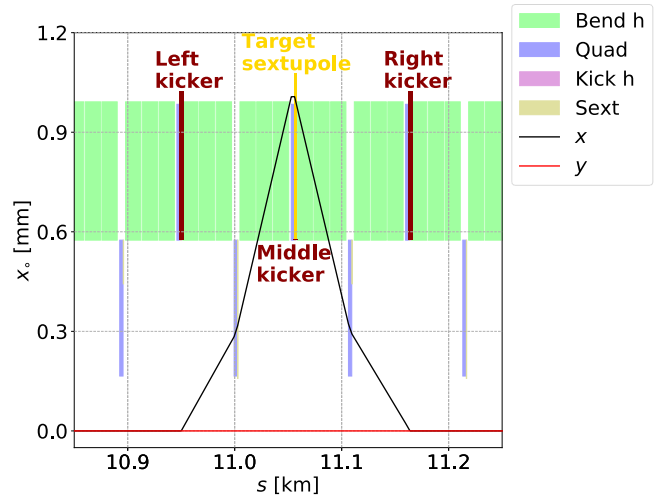


Figure 36: Exemplary horizontal closed orbit bump within arc56. The closed orbit offset is plotted versus the longitudinal beamline position and the lattice elements are indicated as colored blocks. The elements highlighted are actively involved in generating the quadrupolar feed-down.

orbit bump with respect to the target sextupole. As a consequence, the number of constraints reduces to two for perfect reflection symmetry of the bump. Since this condition is not met for all possible bumps, the bump is generally constructed by three horizontal kickers.

### 9.3.1 Bump neighbourhood

In Figure (36) the general layout of a closed orbit bump has been presented. Due to the alternating structure of horizontal and vertical dipole kickers, multiple sextupoles are typically located within a single bump. As a consequence, an accumulated quadrupolar feed-down will be generated by a single knob. The importance of distinguishing between the accumulated feed-down and the target feed-down is visualized in Figure (37). For this simulation the sextupole MS.33L5.B1 was targeted and the beta- and phase beating pattern from the feed-down has been studied. In general, sextupole MS.33L5.B1 was selected as a demonstration example but is explicitly mentioned to guarantee for the reproducibility of the simulation results. The corresponding location of the sextupole can be deduced from the label as elaborated in Chapter 4.6.1.1. In blue, the expected feed-down from sextupole MS.33L5.B1 according to Equation (82) is shown. In green, the calculated feed-down from the MAD-X twiss module is visualized. In red, the expected accumulated feed-down from sextupole MS.33R4.B1 and MS.33L5.B1 according to Equation (82) is indicated. For the calculation of the closed orbit offset at the location of MS.33R4.B1 and MS.34L5.B1 a free drift between kicker and sextupole is assumed. It has to be emphasized, that this is only an approximation since there is always a quadrupole between kicker and sextupole, which will deflect the beam and generally break the free drift approximation. Nevertheless, the approximation is valid for this case because the sextupoles are stacked on top of the quadrupoles. Therefore, the deflection of the beam is negligible on such small distances. Overall, the accumulation of the beating patterns (168) under free drift approximation is in good agreement with the results obtained by MAD-X, which validates the physical description of the bumps.

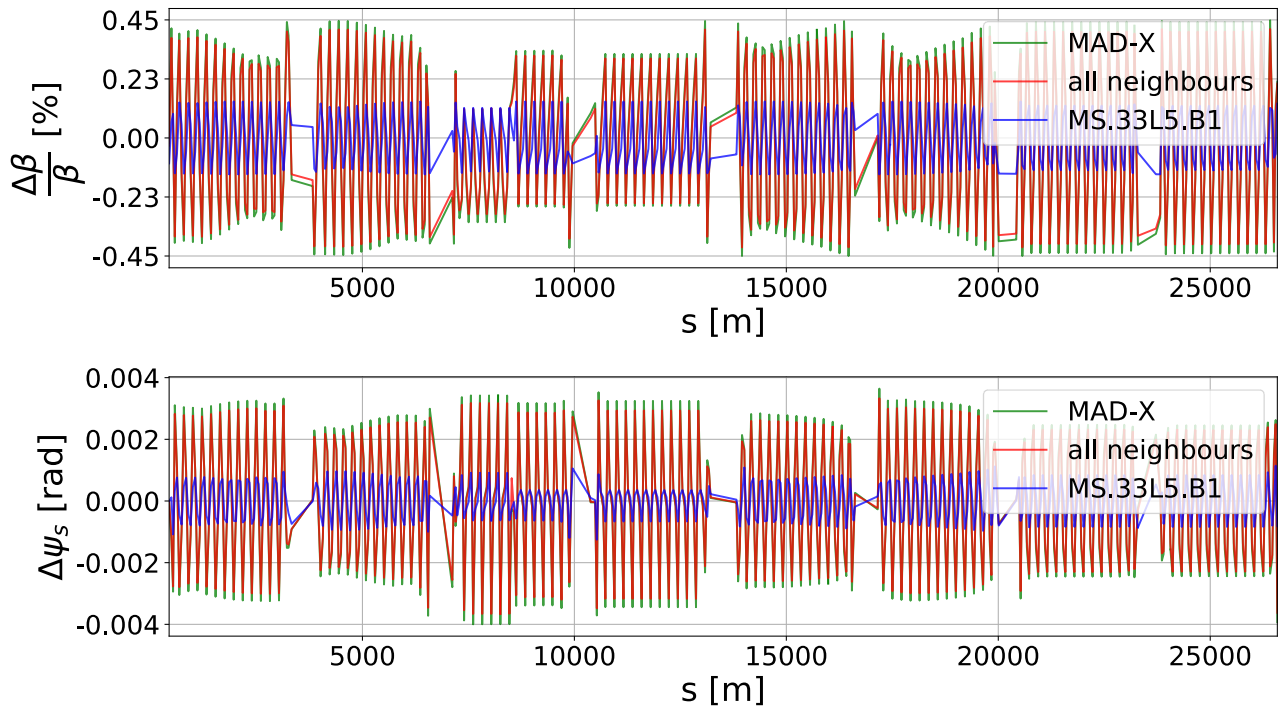


Figure 37: Beating pattern of feed-down from sextupole MS.33L5.B1 versus accumulated beating pattern from sextupole MS.33R4.B1 and MS.33L5.B1. The physical description of the bumps (red) is in good agreement with the expected physical impact according to MAD-X (green).

### 9.3.2 Bump scaling

As elaborated in Chapter 9.3.1, a closed orbit bump consists of several feed-down components from sextupoles. Even though the feed-down components are added linearly and the components itself scale linearly with the closed orbit offset, it has to be validated that the correction method exactly shows this behaviour. Hence, a case study was conducted which included a quadrupolar perturbation of MQ.17L6.B1 and MQ.18R4.B1. Firstly, a parallel ramp up of the systematic errors was simulated. Secondly, the perturbation of MQ.17L6.B1 was kept constant, whereas the distortion of MQ.18R4.B1 was ramped up. The results are shown in Figure 38 and prove the linear independence of the bumps.

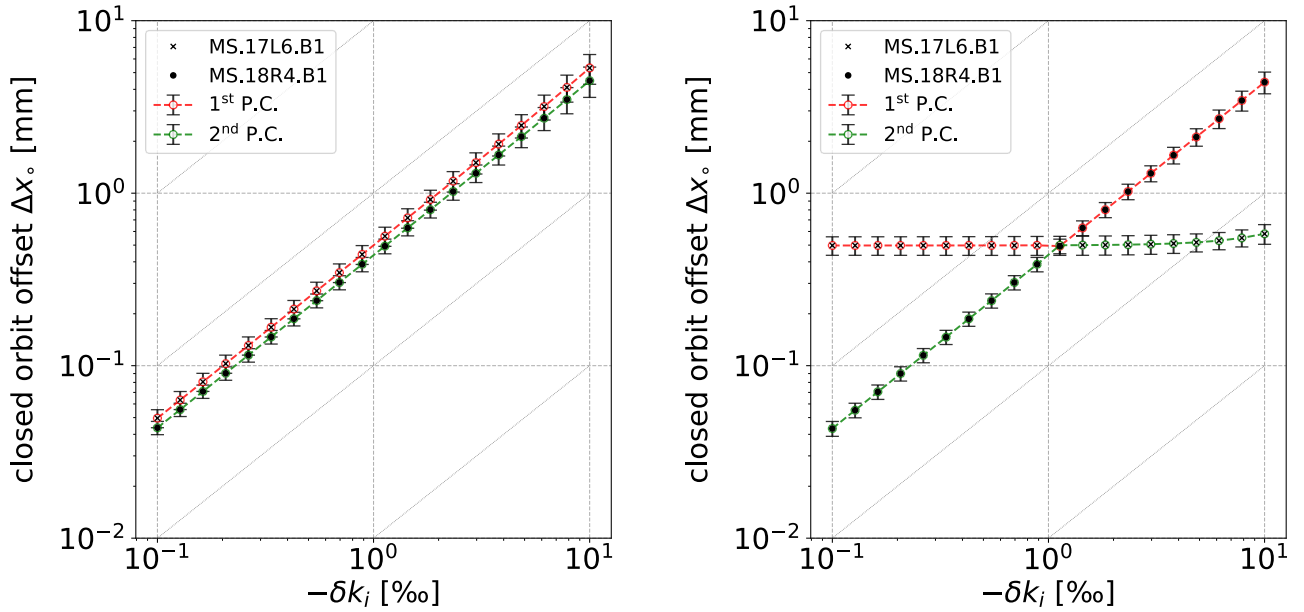


Figure 38: On the left side, the parallel ramp up of a systematic error in quadrupole MQ.17L6.B1 and MQ.18R4.B1 is shown. The ordinate quantifies the necessary closed orbit offset in the sextupoles MS.17L6.B1 and MS.18R4.B1 respectively in order to correct for the quadrupolar distortions. If the magnitude of the imperfections are of the same order the distortion in MQ.17L6.B1 desires always a larger correction bump, since the beta function at MQ.17L6.B1 is slightly larger than the beta function at the location of MQ.18R4.B1. On the right side, the ramp up of a systematic error in quadrupole MQ.18R4.B1, under a constant perturbation of  $\delta k_1 = -1‰$  in quadrupole MQ.17L6.B1, is shown. As expected, the closed orbit offset in MS.17L6.B1 remains almost constant, whereas the amplitude of the correction bump in MS.18R4.B1 scales up linearly. The crossing point at  $\delta k_1 = -1‰$  indicates the change of the first principle component from MS.17L6.B1 to MS.18R4.B1.

## 9.4 Systematic errors

As a proof of principle the correction method presented in Chapter 9.2 is applied in a test case. The test scenario is based on the ATS lattice layout 4.6.1 and includes two systematic quadrupolar perturbations of  $+1‰$  in MQ.17L6.B1 and in MQ.18R4.B1. Figure 39 visualizes the impact of correcting the lattice by the first two principle components of the response matrix approach. The proposed correction includes a closed orbit offset in MS.17L6.B1 of  $\Delta x_o = -0.49 \text{ mm} \pm 0.06 \text{ mm}$  and a closed orbit offset in MS.18R4.B1 of  $\Delta x_o = -0.44 \text{ mm} \pm 0.04 \text{ mm}$ . As a result, the observed RMS beating was reduced in all lattice quantities by roughly one magnitude. Additionally, the correction method was tested on the LHC injection optics using the same setup for the quadrupole distortions and the results underpin the outcome of the ATS-optics correction. In a nutshell, the implemented correction method effectively corrects systematic quadrupolar errors, independent of the lattice model.



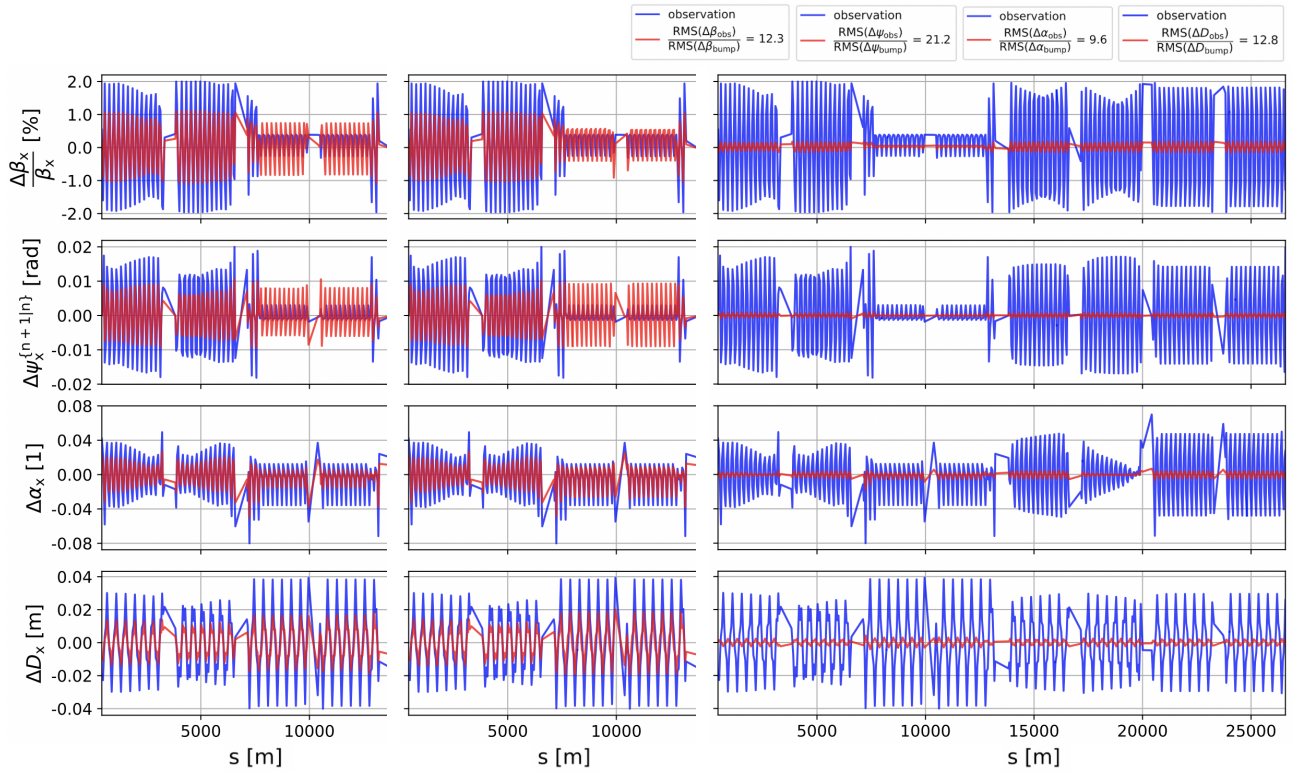


Figure 39: The beating wave created by lattice imperfections is shown in blue for all four observables. In red the residual beating wave after applying the correction is visualised. The plot on the left side corresponds to the first principle component and the middle plot visualises the impact of the second principle component. On the right side, the residual beating wave after applying both corrections is shown. The RMS  $\beta$ -beating was corrected by a factor of 12.3 which underpins the performance of the response matrix approach.

#### 9.4.1 Multiple systematic errors in the quadrupole strength

If the number of systematic lattice imperfections in an arc is increased it could potentially become more challenging for the correction method to find the right adjustments for the knobs. As a consequence, another case study was conducted where all quadrupoles within arc45 and arc56 were perturbed by a random Gaussian error of  $\pm 2\%$ . The obtained values for the knobs were used to calculate the integrated strength of the feed down according to Equation (86). Figure (40) visualizes the correlation between assigned perturbation and integrated strength of the feed-down. The graph clearly shows that the response matrix approach is capable of correcting multiple lattice imperfections at once.

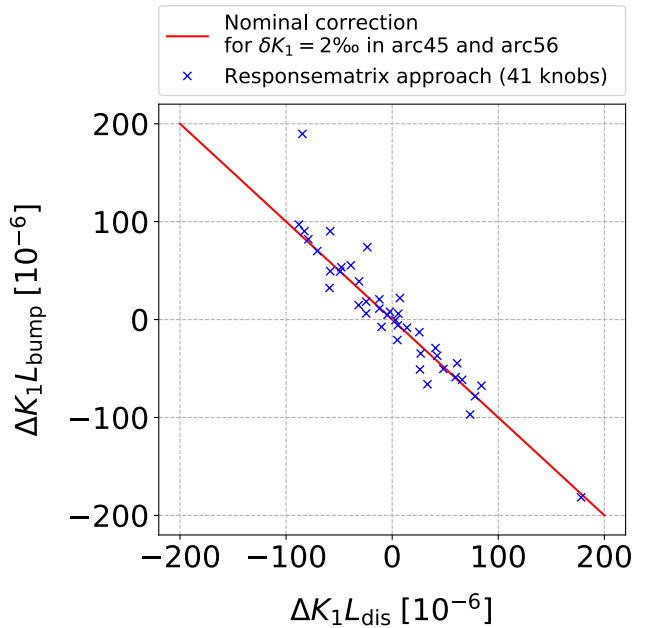


Figure 40: The correlation between integrated strength of the assigned perturbation and the integrated strength of the obtained feed-down is shown.



If the simulation underlying Figure (40) is repeated several times a distribution of knob settings will be obtained for each distortion. In order to avoid outliers in the strength of the quadrupole distortions, a uniform distribution is used for the perturbation. The half-width of the uniform distribution is labelled as  $\pm\sigma$  [100%]. The adjusted simulation underlying Figure (40) is performed for 100 subsamples and the largest correction knob obtained (in absolute terms) for every subsample is kept. The distribution of the largest knob correction settings for a given perturbation magnitude is visualized in a boxplot (41). The blue box indicates the range between first quantile Q1 and third quantile Q3 of the knob distribution. Furthermore, the median of the distribution is highlighted as a red line within the box. The range of the whiskers of the blue box corresponds to a confidence interval of  $\pm 2.7\sigma$  [99.3%] for a standard normal Gaussian distribution. This convention has been introduced by Tukey (40). For non-Gaussian datasets the definition translates into the determination specification  $Q1 - 1.5 \times (Q3 - Q1)$  for the lower end of the whisker and  $Q3 + 1.5 \times (Q3 - Q1)$  for the upper end of the whisker. Data points outside the domain of the whisker are marked as outliers. Figure (41) clearly indicates that the distributions for  $\max(|\Delta x_o|)$  and for the residual RMS are skewed. In comparison to a Gaussian distribution, long tails towards larger values are observed, similar to a Landau-distribution. If the maximum correction knob exceeds 2 mm at injection or 5 mm at flattop energy under ATS-optics configuration, then the aperture limit of the LHC is violated by the closed orbit bump (41) and the beam will be lost. Hence, correction bumps of this magnitude are unrealizable and have to be rejected. If the threshold is crossed it indicates an upper limit for the scope of the correction method. In case of systematic errors in the quadrupole strengths, the upper threshold for correction is approximately  $\sigma_k = 0.6\text{‰}$  for injection and  $\sigma_k = 1.6\text{‰}$  at flattop energy, according to Figure (41). If the observable vector  $\vec{\mathcal{O}}$  (168) is restricted to a single beating attribute, the upper limit for the scope of the correction method varies. The corresponding results of the single input simulations are attached in the appendix, see Figure (58) and (59).

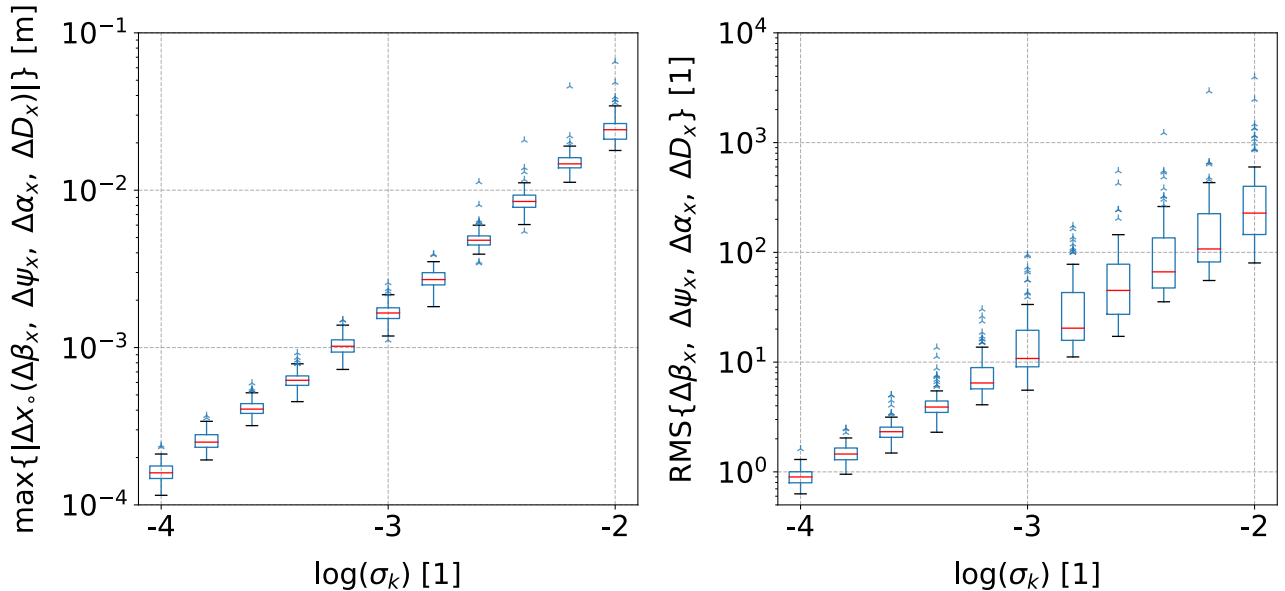


Figure 41: On the left side, the maximum correction knob is plotted versus the logarithm of the error distribution half-width  $\sigma_k$ . All quadrupole strengths  $k_1$  in arc45 and arc56 have been perturbed and 100 subsamples were generated to establish a distribution in  $\max(|\Delta x_o|)$ . On the right side, the corresponding residual RMS value after correction is presented as a function of the perturbation magnitude. The boxplots visualize the characteristics of every distribution at constant perturbation magnitude. For the calculation of the correction knobs, the unified beating pattern was exploited, according to Equation (168).

### 9.4.2 Multiple systematic errors in the longitudinal placement of quadrupoles

In first order approximation, the issue of a longitudinally displaced quadrupole can be translated into a problem with two quadrupolar strength distortions. If a quadrupole acting on the domain  $[-L, L]$  with  $k_1$  is offset by  $\Delta s$ , then this will create a quadrupole field perturbation in the domain  $[L, L + \Delta s]$  with magnitude  $\delta k_1 = +k_1$  and a second field perturbation in the domain  $[-L, -L + \Delta s]$  with magnitude  $\delta k_1 = -k_1$ . As a consequence, the correction method presented in Chapter 9.1 should also be applicable to longitudinal displacements of quadrupoles. As a proof of concept, several case studies were conducted including several longitudinal displaced quadrupoles. In Figure 42 the scaling of the orbit bumps is plotted as a function of the displacement for quadrupole MQ.17L6.B1. As expected, two bumps in sextupoles are required to compensate a single longitudinally displaced quadrupole.

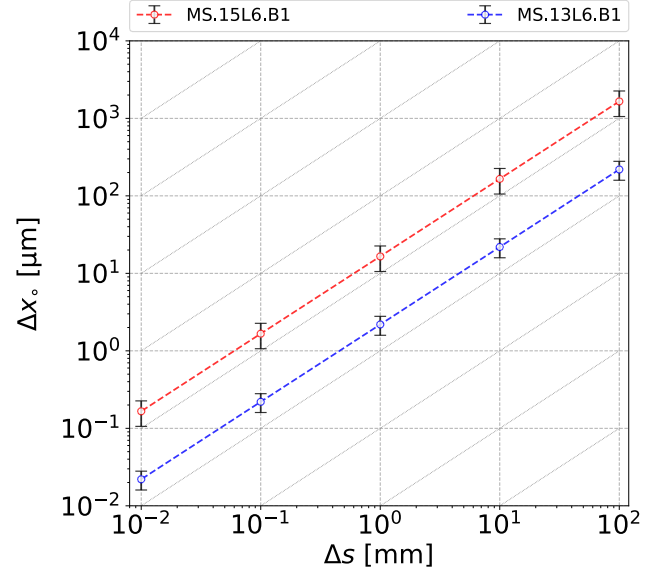


Figure 42: The closed orbit offset  $\Delta x_o$  is plotted as a function of the displacement  $\Delta s$  for quadrupole MQ.17L6.B1. An orbit bump in sextupole MS.15L6.B1 and in sextupole MS.13L6.B1 is required in order to correct for the longitudinal displacement of quadrupole MQ.17L6.B1.

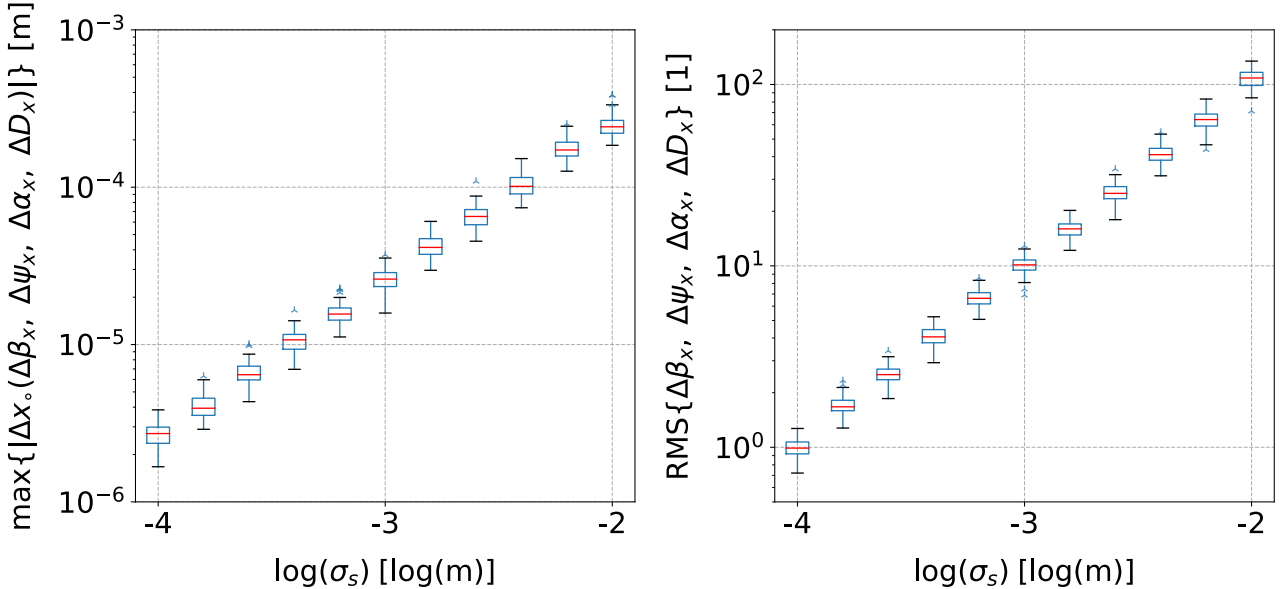


Figure 43: On the left side, the maximum correction knob is plotted versus the logarithm of the error distribution half-width  $\sigma_s$ . All quadrupoles in arc45 and arc56 have been displaced longitudinally and 100 subsamples were generated to establish a distribution in  $\max(|\Delta x_o|)$ . On the right side, the corresponding residual RMS value after correction is presented as a function of the perturbation magnitude. The boxplots visualize the characteristics of every distribution at constant perturbation magnitude. For the calculation of the correction knobs, the unified beating pattern was exploited, according to Equation (168).

Both bumps ramp up linearly in amplitude, which demonstrates the linear independence of the or-

bit bumps. Furthermore, the orthogonality of the bumps was demonstrated by considering multiple quadrupole displacement errors. In detail, one quadrupole displacement was kept constant, whereas a second displaced quadrupole was ramped up to larger offsets (Analogous to Figure (38)).

If all quadrupoles within arc45 and arc56 are displaced longitudinally, the results presented in Figure (43) are obtained. The simulation results clearly show a linear behaviour of the maximum orbit bump as a function of the displacement magnitude. This conclusion underpins the outcome of the scaling simulation for a single longitudinally displaced quadrupole (MQ.17L6.B1). From Figure (43) it can be extrapolated, that the upper limit for the scope of the correction method in case of longitudinally displaced quadrupoles is in the order of  $\sigma_s = 6$  cm for injection and  $\sigma_s = 10$  cm at flattop energy. If the observable vector  $\vec{\mathcal{O}}$  (168) is restricted to a single beating attribute, the upper limit for the scope of the correction method varies. The corresponding results of the single input simulations are attached in the appendix, see Figure (60) and (61).

### 9.4.3 Multiple systematic errors in the transversal placement of sextupoles

The underlying principle of the correction method, using orbit bumps in sextupoles, is based on quadrupolar feed-downs generated from sextupoles according to Equation (86). This effect can also change to the contrary if sextupoles are misaligned transversely in the first place. In order to study the correction capability of the response matrix approach for such imperfections, all sextupoles in arc45 and arc56 were displaced transversely according to a uniform error distribution with half-width  $\sigma_x$ . The results are shown in Figure (44) and indicate an upper limit for the scope of the correction method of  $\sigma_x = 0.6$  mm for injection and  $\sigma_x = 1$  mm at flattop energy. If the observable vector  $\vec{\mathcal{O}}$  (168) is restricted to a single beating attribute, the upper limit for the scope of the correction method varies. The results of the single input simulations are attached in the appendix, see Figure (62) and (63).

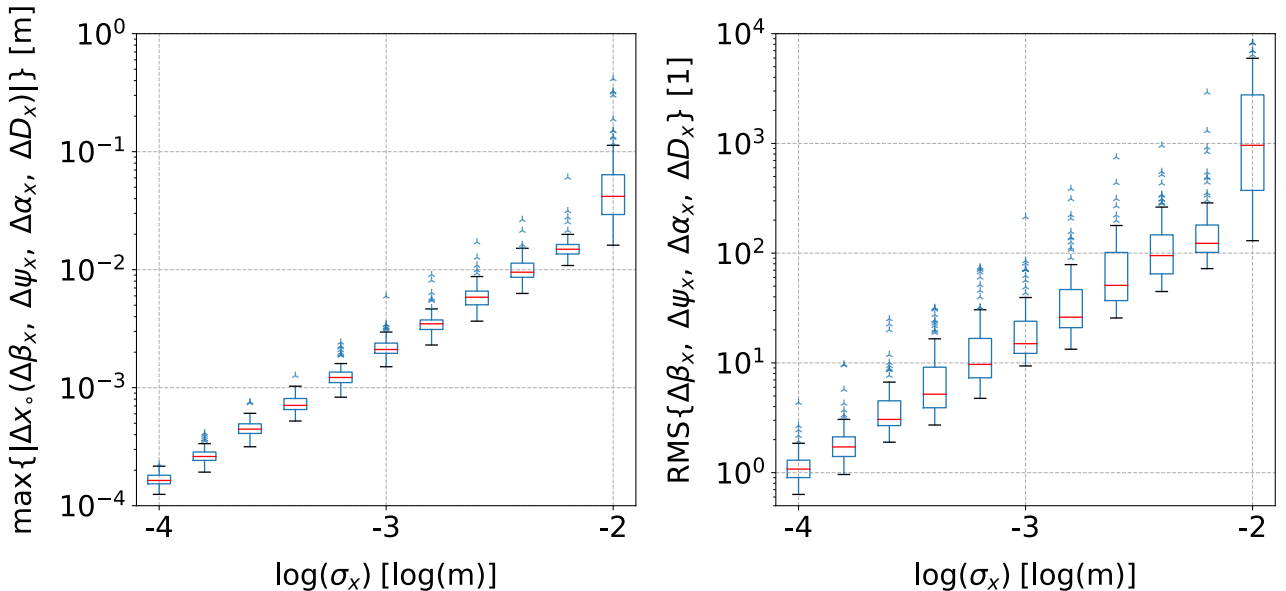


Figure 44: On the left side, the maximum correction knob is plotted versus the logarithm of the error distribution half-width  $\sigma_x$ . All sextupoles within arc45 and arc56 have been displaced transversely and 100 subsamples were generated to establish a distribution in  $\max(|\Delta x_o|)$ . On the right side, the corresponding residual RMS value after correction is presented as a function of the perturbation magnitude. The boxplots visualize the characteristics of every distribution at constant perturbation magnitude. For the calculation of the correction knobs, the unified beating pattern was exploited, according to Equation (168).

#### 9.4.4 Degeneracy of orbit bumps

Even though the idea of correcting quadrupolar errors in the arcs by orbit bumps in sextupoles has already been demonstrated experimentally [27] for the ATS optics, some issues remained unsolved. In detail, the correction obtained by the response matrix approach, matching the phase measurement, pointed to a horizontal orbit bump of 5.2 mm in the sextupole MS.26R4.B1 [27]. In order to avoid such a large orbit bump, the correction was spread to the sextupoles MS.18R4.B1 and MS.22R4.B1. This was assumed to be equivalent, since those sextupoles have a phase advance of  $\pi$  and would therefore generate a congruent beating pattern. As a consequence, the question came up if the correction obtained for quadrupole MQ.26R4.B1 is itself an accumulation of degenerated beating waves. If the hypothesis of degenerated beating waves is true, then this might be a good explanation for the large magnitude ( $\approx -1\%$ , if compared with Figure (38)) of the MQ.26R4.B1 imperfection. Hence, it is worth to investigate whether a degeneracy of the beating wave is expected or not.

For that reason, the difference in the beta beating pattern between periodically spaced quadrupoles was studied. In detail, the beta beating wave of MQ.27R5.B1 and MQ.17L6.B1 was referenced to the beta beating wave generated by a distortion in MQ.23R5.B1. The systematic error of MQ.23R5.B1 is assigned with a positive sign and the errors for MQ.27R5.B1 and MQ.17L6.B1 are negated. The design phase advance between MQ.23R5.B1 and MQ.27R5.B1 is exactly  $\pi$  and the phase advance between MQ.23R5.B1 and MQ.17L6.B1 is equal to  $7\pi$ . Figure (45) visualizes a unique beating bump between the quadrupoles MQ.23R5.B1 and MQ.27R5.B1 (in red). In the opposite domain the beating wave completely vanishes due to destructive interference.

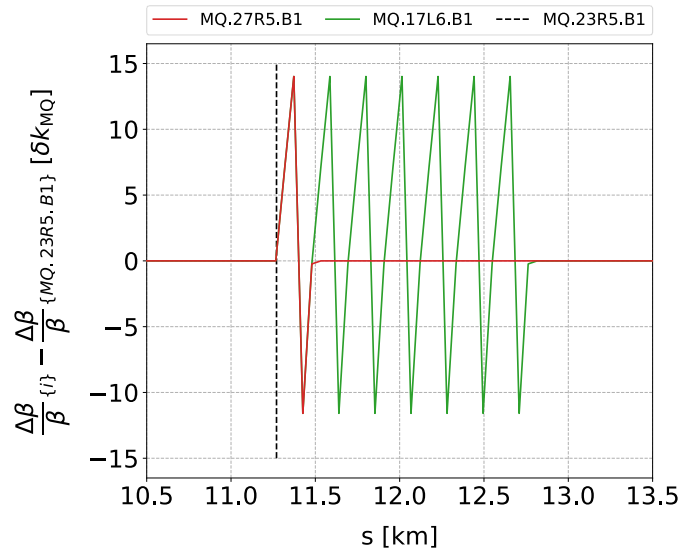


Figure 45: Difference in beating wave between periodically spaced quadrupoles. The dashed black line indicates the location of the reference distortion at MQ.23R5.B1.

This behaviour is theoretically expected if the beta beating Equation (82) is reconsidered. In general, the beating wave depends on  $|\psi(s) - \psi(s_d)|$ . Consequently, the beating wave generated by  $MQ_k$  depends on  $|\psi(s) - \psi_k|$  and the beating wave form  $MQ_{k+4n}$  depends on  $|\psi(s) - \psi_k - n\pi|$ . Both waves generate the same pattern if one of the conditions in Expression (170) is met.

$$\psi(s) - \psi_k - n\pi \geq 0 \quad \vee \quad \psi(s) - \psi_k \leq 0 \quad (170)$$

On the contrary, the beating wave of  $MQ_{k+4n}$  differs from the wave of  $MQ_k$  if condition (171) is met.

$$\psi_k < \psi(s) < \psi_k + n\pi = \psi_{k+4n} \quad (171)$$

In a nutshell, periodically spaced quadrupoles are theoretically not degenerated. The reason for the unique beating bump lies in the case differentiation between downstream (78) and upstream (79) sce-

nario. Hence, causality teaches us uniqueness. Furthermore, the phase jump at the location of the quadrupolar distortion (75) already indicates a local signature.

The problem is, that only three BPMs are installed between periodically placed quadrupoles. Hence, it is quit difficult to measure the phase jump generated by a quadrupolar distortion. The same applies to the beating bump. Therefore, the question arises if a degeneracy of the bumps is induced by the measurement uncertainty. In order to tackle this question a few studies have been undertaken.

#### 9.4.4.1 Broken FODO-cell periodicity

The ATS optics layout (4.6.1) assigns a phase advance of  $90^\circ$  to the FODO-cells in the arcs. Hence, quadrupoles every second FODO-cell show a phase advance of  $\pi$  and could potentially become degenerated if measurement uncertainties are considered. In order to break this degeneracy the quadrupole currents of the FODO-cells are slightly varied. Figure (46) presents the design of three different configurations for arc45/arc56. In detail, the circuits are perturbed in arc45 by a relative error of  $\delta k_{\text{arc}}$  and in arc56 by a relative error of  $-\delta k_{\text{arc}}$ . If the circuits are perturb by one percent the periodicity of the quadrupoles will be broken by 2.3% or  $4^\circ$  respectively.

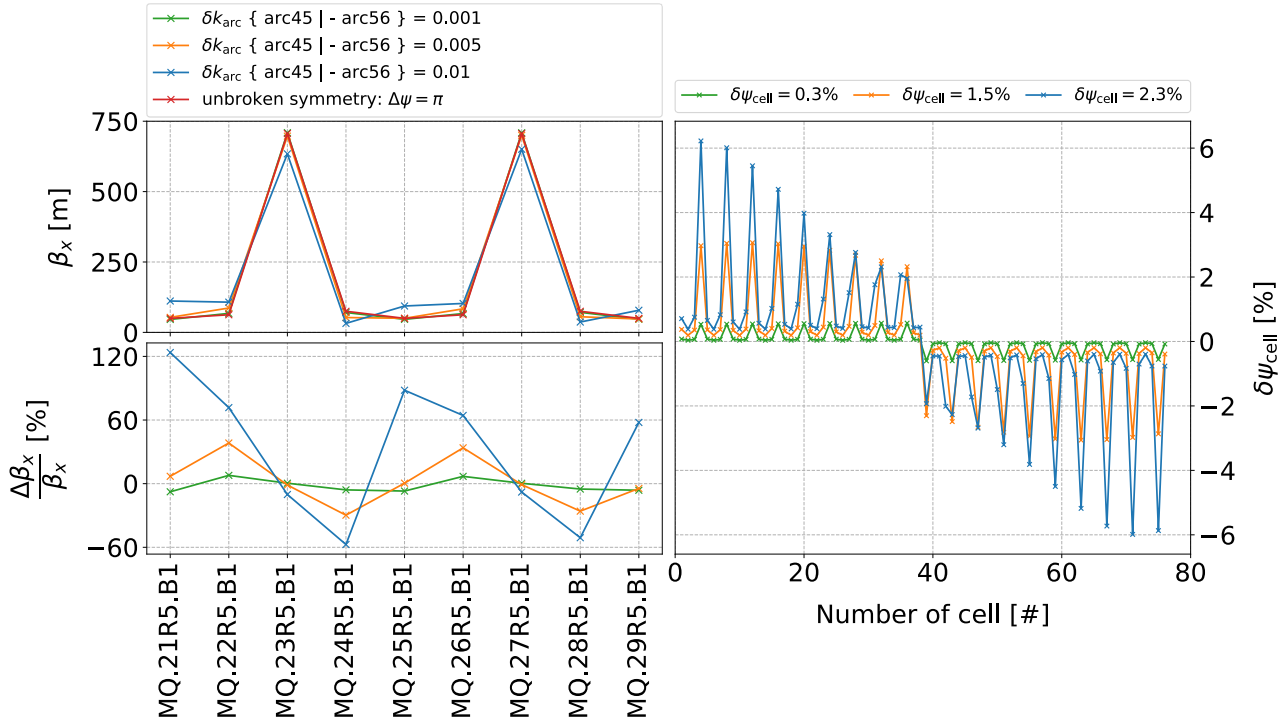


Figure 46: On the upper left side, the beta function for a short section in arc56 is shown. In the plot on the lower left side, the induced beta beating wave is visualized for three different configurations of the FODO-cell phase advance. On the right side, the relative phase deviation from  $\pi$  is plotted for every fourth quadrupole. The RMS phase deviation for the arc configuration  $\delta k_{\text{arc}} = 1\%$  equals 0.3% or  $0.5^\circ$  respectively.

The benefit of breaking the periodicity in the arcs is visualized by Figure (47). Breaking the periodicity will add an artificial beating pattern, which potentially suppresses the degeneracy. Nevertheless, since the unique beta beating bump will lose in amplitude, the local information will spread out. Hence, by breaking the periodicity local information is deteriorated and an artificial beating pattern in the domain of (170) is generated.

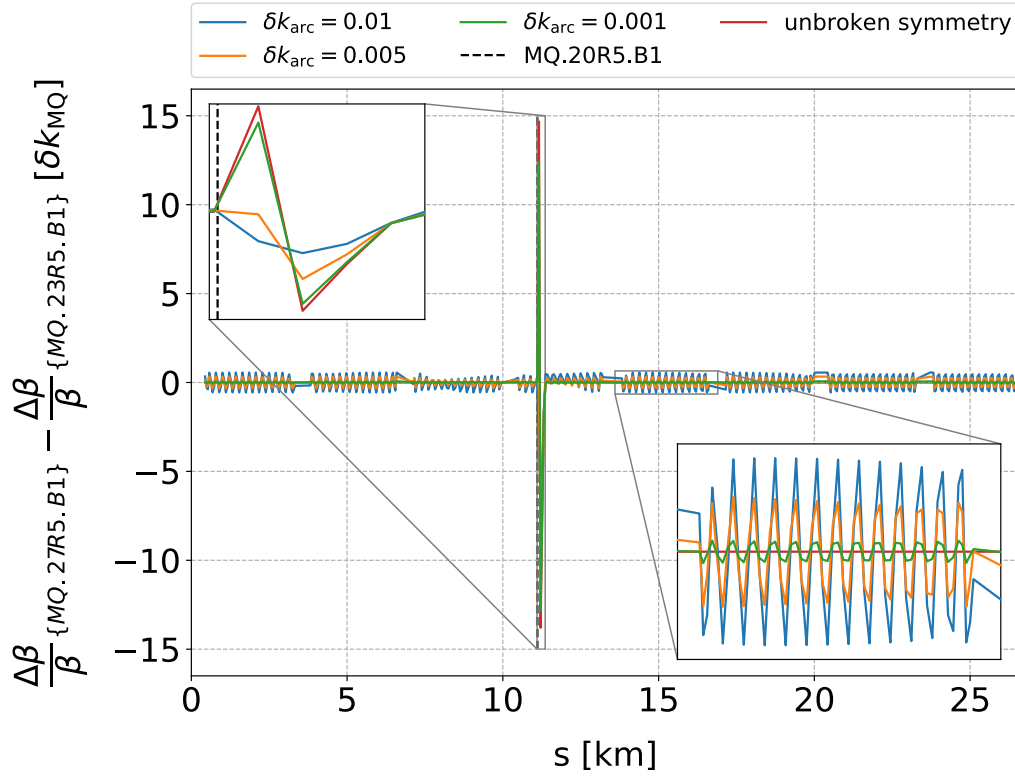


Figure 47: The difference in the beta beating pattern for an error in MQ.27R5.B1 in comparison to a perturbation of MQ.23R5.B1 is visualized for different FODO-cell configurations. The error in MQ.27R5.B1 is a negated in relation to MQ.23R5.B1. In the magnification plot on the top left side, the impact of the phase advance configuration on the local beta beating bump is highlighted. The bump shrinks in amplitude as the phase advance adjustment is increased. The magnification plot on the lower right side indicates the artificially generated beta beating pattern in the domain of (170).

If the approach of breaking the phase advance of  $\pi$  is really beneficial then the residual beating RMS should be smaller than without breaking the periodicity. Therefore, a simulation was created, which determines the residual beating RMS after correction for the four quantities of interest, depending on the FODO-cell configuration. For the simulation setup, a perturbation of +1‰ in MQ.23R5.B1 and a distortion of MQ.27R5.B1 with -1‰ was assumed. This setup should be most sensitive to the effect of degeneracy because those quadrupoles are nearest neighbours in terms of periodicity. Consequently, the beta beating bump will only be observable by three out of 439 BPMs in the LHC. The results are presented by Figure (48) and provide no evidence for the hypothesis of a significant improvement of the correction method under broken periodicity. This could be explained by the phenomenon seen in Figure (47). The benefit of an artificial beating pattern in the domain (170) comes hand in hand with a loss of local information on the beating bump in the domain (171).

The simulation underlying Figure (48) focused on the general issue of degeneracy. But it could be the case, that the beating bump in the domain (171) interferes with other beating waves and becomes degenerated if the number of distortion increases. As consequence, it has been investigated if the residual RMS beating behaviours differently for a broken periodicity configuration if the number of quadrupolar imperfections within arc45/arc56 increases. The results of those studies are presented by Figure (49). The residual RMS beating for the unbroken symmetry configuration always outperforms the alternative parameterizations for the FODO-cell phase advance. The line of best fit with the constraint  $RMS(N=0) \stackrel{!}{=} 0$  underpins this observation. Due to the fact that the confidence bands of



the regression lines overlap it can not be made a clear statement about the best configuration. But the tendency indicates that breaking the phase advance increases the residual RMS beating after correction. Anyhow, the hypothesis of an improvement due to a suppressed degeneracy is not supported by the results and there is no evidence for this effect.

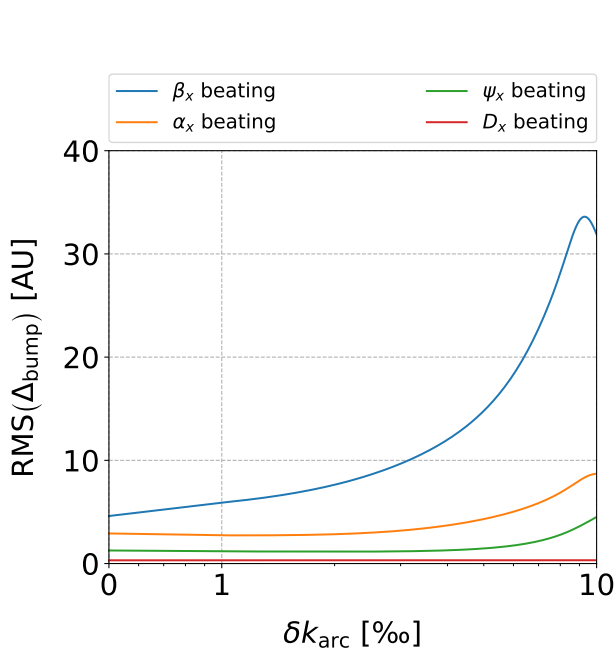


Figure 48: The plot visualizes the residual RMS beating after correction with the response matrix method as a function of the magnitude of the circuit perturbation in arc45/arc56. As expected, the residual RMS beating shows the same behaviour on different scales for all lattice quantities.

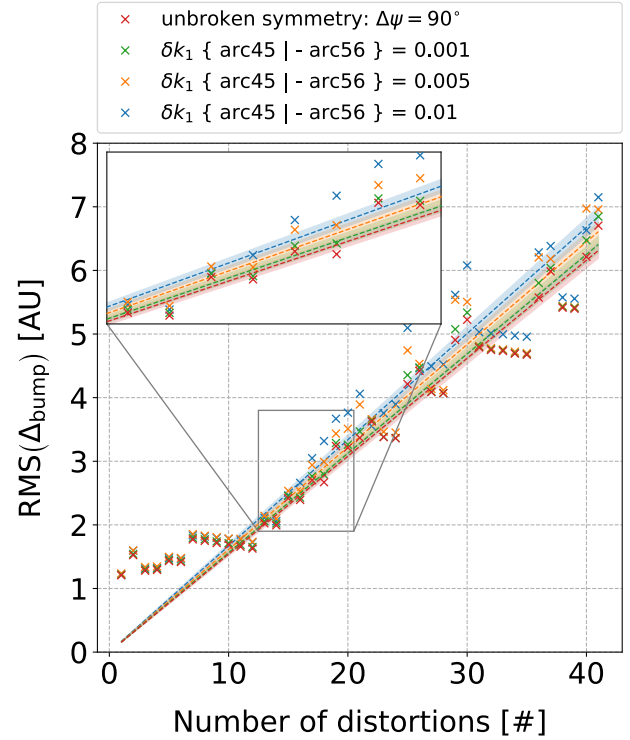


Figure 49: The graph shows the residual RMS beating after correction with the response matrix method as a function of the number of distortions in arc45/arc56. In order to guarantee an uniform impact on the lattice functions, the magnitude of the distortions is weighted by the beta function  $\beta_d$  at the location of the distortion. The weighted strength of a distortion is referenced to an imperfection of 1‰ in MQ.23R5.B1.

So far, the analyses were based on the residual RMS beating after applying the correction obtained by the response matrix method. From a mathematical point of view, the response matrix entirely characterises the responding qualities of the lattice on quadrupolar perturbations. Hence, a mathematical analysis of the matrix itself could reveal an improvement of the correction method under different FODO-cell configurations. For that reason, the condition number and the singular value spectrum has been calculated for every configuration. The results are shown in Figure (50). The condition number drops by roughly 15% if the periodicity in the arcs is broken. In general, the condition number determines the sensitivity coefficient for relative changes in the observable vector  $\vec{\mathcal{O}}$  if the parameter vector  $\vec{\mathcal{P}}$  is varied. If the condition number of a system B is by several orders smaller than the condition number of a system A, then system B will be more stable than system A. On the contrary, a difference in the condition number in the order of a few percent does not justify a stability conclusion. The spectrum of singular values does also not reveal a significant difference between the FODO-cell configurations. Theoretically, a system A would show a more uniform spectral distribution than the system B. In conclusion, there is no evidence that breaking the periodicity in the arcs will improve the performance of the response matrix method.



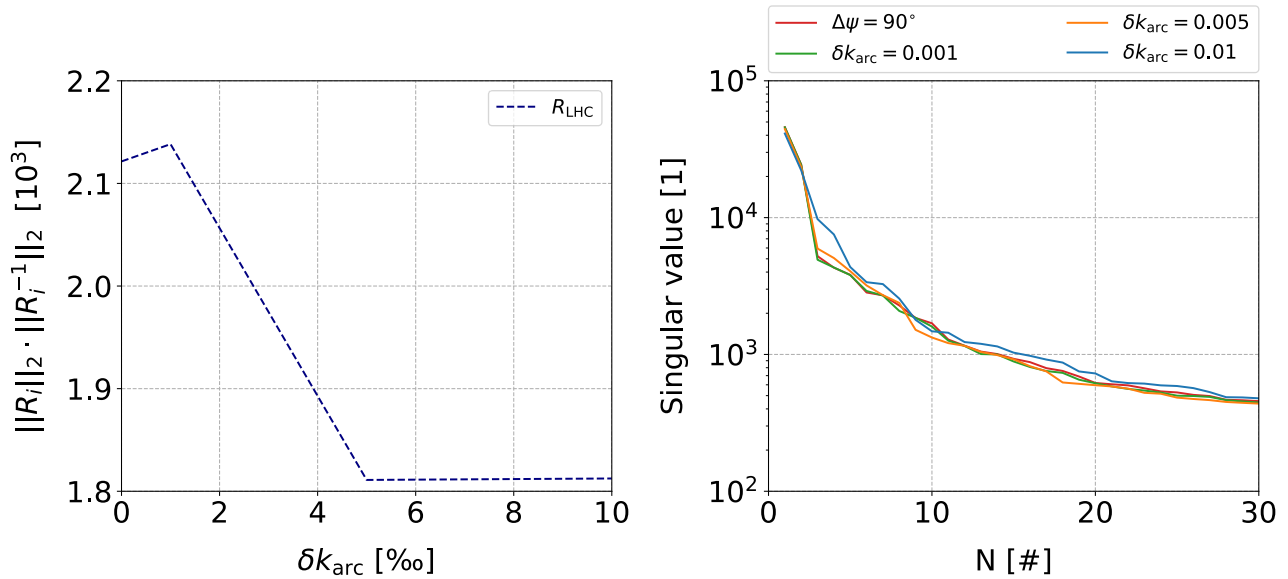


Figure 50: On the left side, the condition number of the response matrix  $\mathcal{R}$  is plotted as a function of the perturbation magnitude of the FODO-cell circuits. For the calculation of the condition number the spectral norm was utilised. On the right side, the singular value spectrum of the response matrix  $\mathcal{R}$  is visualised for different configurations.

## 9.5 Statistical errors

In Chapter 9.4.4.1 it was shown that breaking the periodicity of  $\pi$  between every fourth quadrupole in the arc does not improve the performance of the correction method. This proves, that the error of the analysis process, visualised in Figure 35 as the green path, is small enough to resolve the unique beating bump between periodically placed quadrupoles. Consequently, no degeneracy of the bumps was observed as theoretically expected. Nevertheless, there are additional error contributions which have not been studied so far. If the measurement uncertainty of the orbit  $x$  or the uncertainty in the local phase information  $\phi$  is considered, it might be sufficient to deteriorate the information on the local beating bump. Therefore, Monte-Carlo simulations have to be performed in order to account for statistical errors in the generated measurement data. In general, the simulations are based on the ATS-lattice layout (see Chapter 4.6.1) including a perturbation of quadrupole MQ.23R5.B1, with  $\delta k_1 = +1\text{‰}$ , and a distortion of MQ.27R5.B1, with  $\delta k_1 = -1\text{‰}$ . This configuration should be most sensitive to potentially degenerated bumps as explained in Chapter 9.4.4. In order to ensure the comparability of the results, the residual RMS beating after correction has been normalised by the observed (uncorrected) RMS beating without noise for the respective FODO-cell configuration, as precisely specified in Expression 172. Figure 51 summarizes the results of the Monte-Carlo studies.

$$\langle \text{RMS}(\Delta_{\text{bump}} \mid \Delta\psi_{\text{FODO}}) \rangle \equiv \frac{\text{RMS}(\text{corrected} \mid \text{systematic} \wedge \text{statistical errors}, \Delta\psi_{\text{FODO}})}{\text{RMS}(\text{uncorrected} \mid \text{systematic errors}, \Delta\psi_{\text{FODO}})} \quad (172)$$

If the normalised residual RMS beating 172 exceeds one, then the correction applied fails to suppress the beating under the uncorrected RMS level. Hence, the systematic quadrupole distortions are getting dominated by the statistical measurement uncertainties. Consequently, the measurement uncertainty level corresponding to a normalised residual RMS beating of one quantifies an upper noise limit for the beneficial application of the correction method. It is an upper limit because all 169 knobs are used for the correction. Therefore, statistical errors will also be interpreted as systematic errors and are corrected in the second step of the routine. This implicitly biases the simulation to larger noise

limits. In order to avoid the bias, the number of singular values used for the correction could have been restricted or the correction should have been applied iteratively until convergence is reached for every Monte-Carlo step. Monte-Carlo simulations are very time consuming and the setup for the following studies include 100 subsamples with 10k turns for 439 BPMs. As a consequence, a single Monte-Carlo simulation takes roughly 12 h if executed on 12 CPU cores in parallel. Since the main focus of the Monte-Carlo studies lies in the investigation of the degeneracy of the orbit bumps and not in the specification of the measurement uncertainty limits, an upper noise limit is deemed acceptable.

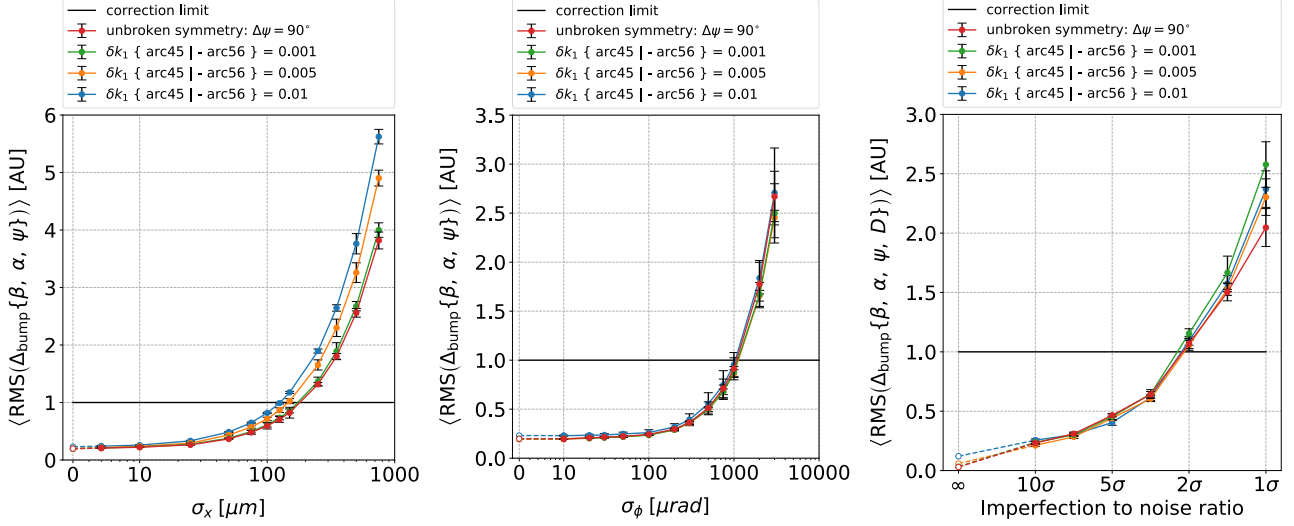


Figure 51: The normalised residual RMS beating after correction with the response matrix method is shown for different FODO-cell configurations as a function of the magnitude of: (left) the measurement uncertainty in  $x$  [68%], (middle) the measurement uncertainty in  $\phi$  [68%], (right) the noise level  $\sigma_{k_{\text{arc}}}$  in the arc circuits. The observable vector  $\vec{\mathcal{O}}$  (168) was restricted to the lattice functions  $\beta$ ,  $\alpha$  and  $\psi$  for the measurement precision studies because the dispersion is not measurable at a single beam energy.

### 9.5.1 Measurement uncertainty of the orbit

As described in Chapter 8 the excited beam motion is tracked by plate beam position monitors (BPMs). The orbit  $x$  of the coherent particle oscillations follows from Equation (156). In order to study the impact of BPM measurement uncertainties a Monte-Carlo simulation was conducted and the results are visualised by Figure (52). For this simulation, turn-by-turn data was acquired with the PTC tracking module of MAD-X, using 10k turns, and the local phase information was retrieved by a spectral analysis of the turn-by-turn data. Subsequently, the segment by segment phase advance  $\psi(\text{BPM}_{i+1}|\text{BPM}_i)$  is used to deduce the  $\beta$ - and  $\alpha$ -function by applying the N-BPM method from Chapter 8.1. The dispersion was not used as an input quantity because the dispersion requires another measurement setup at different beam energies. After determining the orbit, the measurement values for every BPM were perturbed by a random Gaussian error with standard deviation  $\sigma_x$ . None of the plots show an improvement in robustness by changing the phase advance of the FODO-cells. The most error robust configuration for all lattice quantities is the periodic configuration. As expected, the residual RMS variation is smallest for  $\psi$  and largest for the measurement of  $\alpha$ . This is theoretically predictable, since  $\beta$  (39) is related to the first derivative of the phase  $\psi$  and  $\alpha$  (42) is related to the second derivative of  $\psi$ . In addition, the  $\beta$ - and  $\alpha$ -functions are deduced from the phase measurement utilising the N-BPM method. From the Monte-Carlo study it can be concluded, that a measurement precision of  $\pm 200 \mu\text{m}$  [68%] is at least required for a beneficial application of the response matrix

method using orbit bumps. This holds if the unified beating pattern of  $\beta$ ,  $\alpha$  and  $\psi$  is exploited. There is no evidence for a degeneracy of the orbit bumps and the tendency of the curves indicates that it is unpromising to test larger phase deviations from the periodic configuration.

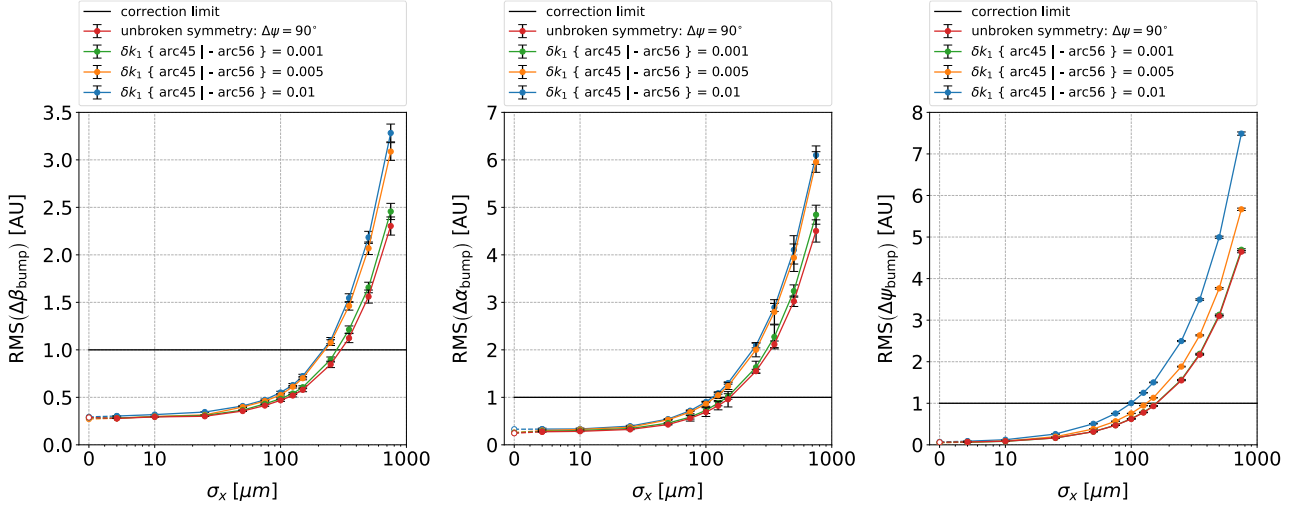


Figure 52: The normalised residual RMS beating after correction with the response matrix method is shown for different FODO-cell configurations as a function of the magnitude of the measurement uncertainty in the closed orbit displacement  $x$  [68%]. The observable vector  $\vec{\mathcal{O}}$  (168) was restricted: (left) to  $\beta$  as an input, (middle) to  $\alpha$  as an input, (right) to  $\psi$  as an input.

### 9.5.2 Measurement uncertainty of the local phase

If the N-BPM method is used to determine the lattice functions only measurement uncertainties regarding the orbit  $x$  have to be considered. But in practice, several cleaning algorithms filter the raw data from the BPMs in order to extract the local phase information. Hence, it is insightful to run a Monte-Carlo simulation, which assigns the statistical errors to the local phase measurement instead of the orbit. The results are presented by Figure (53). The curves of different configuration overlap within their uncertainties, wherefore a degeneracy of the bumps is not revealed.

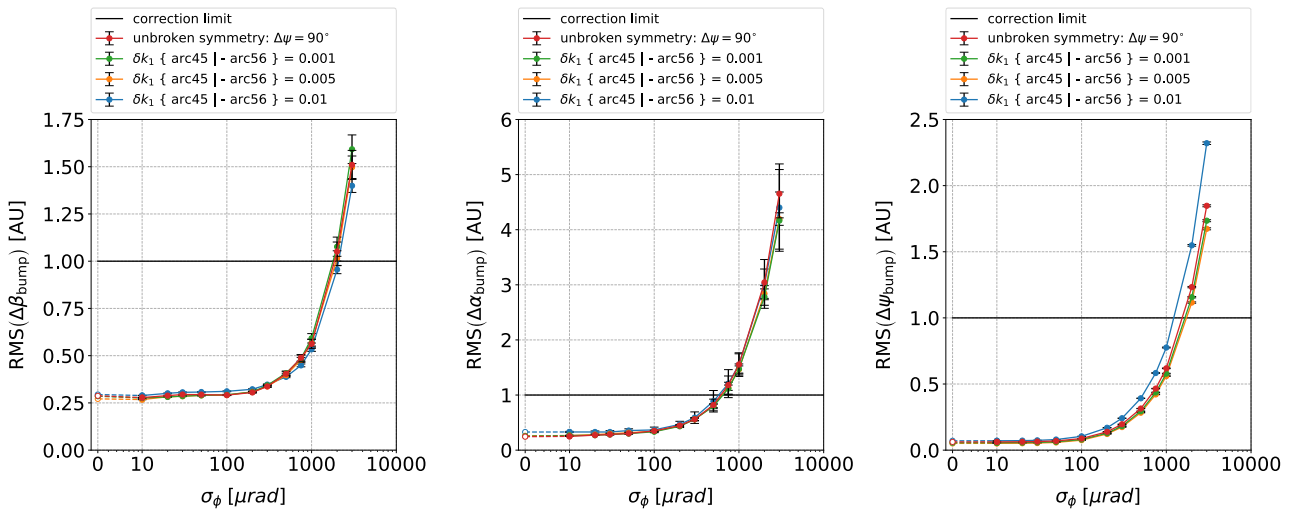


Figure 53: The normalised residual RMS beating after correction with the response matrix method is shown for different FODO-cell configurations as a function of the magnitude of the measurement uncertainty in the local phase  $\phi$  [68%]. The observable vector  $\vec{\mathcal{O}}$  (168) was restricted: (left) to  $\beta$  as an input, (middle) to  $\alpha$  as an input, (right) to  $\psi$  as an input.

The local phase simulation shows the same feature as depicted for the orbit simulation, namely that the noise robustness is largest if the phase beating is targeted for minimization and that the noise robustness is smallest if  $\alpha$ -beating is the figure of merit. In conclusion, Figure (53) validates the results presented in Figure (52). No improvement in robustness of the correction method is achieved by changing the phase advance of the FODO-cells. Nevertheless, a measurement precision for the local phase  $\phi$  of  $\pm 1$  mrad [68%] is at least required for a beneficial application of the response matrix method using orbit bumps. This holds if the unified beating pattern of  $\beta$ ,  $\alpha$  and  $\psi$  is exploited. If the main target is to suppress  $\alpha$ -beating, then a measurement precision for  $\phi$  of  $\pm 0.5$  mrad [68%] is at least required. Generally,  $\alpha$ -beating is most sensitive to noise since  $\alpha$  depends on the second derivative of the phase measurement. The local phase measurement is directly related to the driven tune  $Q_d$ . Hence, a precision requirement for the local phase  $\phi$  can be translated into a precision requirement for the driven tune  $Q_d$ . In order to do so, the corresponding sensitivity coefficient has to be determined. For this reason, tracking data for an excited beam under ATS-lattice configuration was generated by MAD-X, as presented in Figure (54). The local phase information is extracted by a spectral analysis of the orbit measurements. The sensitivity coefficient for the driven tune jitter  $\sigma_{Q_d}$  on the local phase precision  $\sigma_\phi$  is quantified by a Monte-Carlo simulation. In order to reveal the dependency, a distribution of local phase deviations  $\Delta\phi_0$  is generated under constant magnitude of the driven tune jitter  $\sigma_{Q_d}$ , as presented in Figure (55) for  $\sigma_{Q_d} = 10$  ppm. If the Monte-Carlo simulation is repeated several times as a function of  $\sigma_{Q_d}$  then the sensitivity coefficient can be determined. In detail, the sensitivity coefficient is quantified by the slope of the graph  $\sigma_{Q_d} = f(\sigma_\phi)$ . The corresponding line of best fit indicates a sensitivity coefficient of  $\partial_{\sigma_\phi} \sigma_{Q_d} = (9.9 \pm 0.1$  [95%]) [ppm/mrad]. As a consequence, a stability requirement for the driven tune  $Q_d$  can be deduced, as presented in Expression (173).

$$\sigma_{Q_d} = \left| \frac{\partial \sigma_{Q_d}}{\partial \sigma_\phi} \right| \cdot \sigma_\phi \Rightarrow \sigma_{Q_d} \approx 10 \text{ ppm [68\%]} \quad , \quad \text{with: } \sigma_\phi = 1 \text{ mrad [68\%]} \quad (173)$$

Resultantly, the waveform generator which excites the beam has to ensure a driven tune stability of at least  $\pm 10$  ppm for  $Q_d = 0.270$ . In other words, the driven tune jitter has to be smaller than  $\pm 10$  ppm for a beneficial application of the response matrix method using orbit bumps in arc sextupoles.

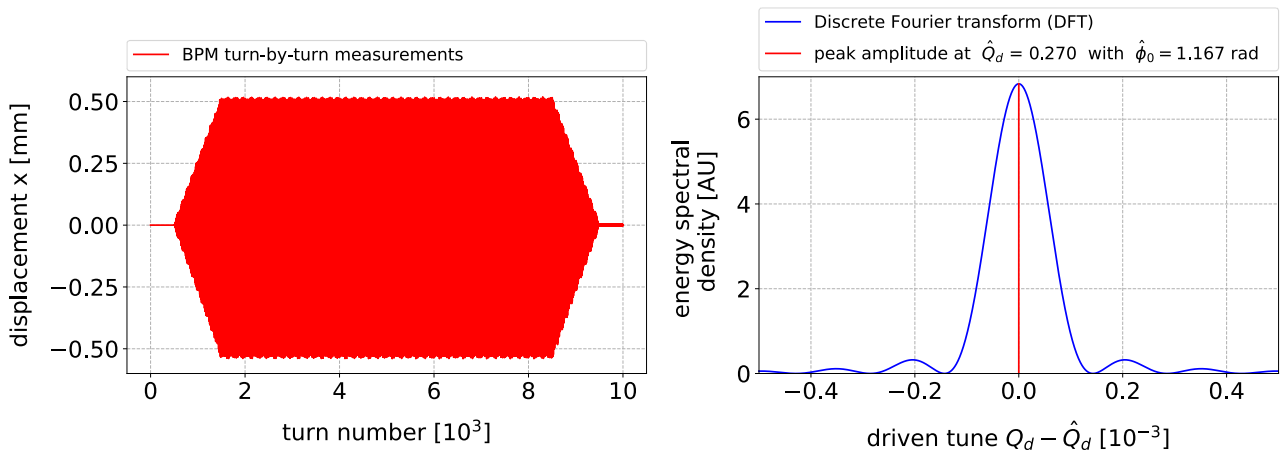


Figure 54: On the left side, turn-by-turn data from BPM.27L6.B1 is visualized in red. The tracking data was simulated by MAD-X using the ATS-optics layout. In addition, the beam was excited by an AC-dipole to enforce a coherent betatron oscillation. In order to obey adiabaticity the ramping and damping was spread over 1000 turns. Consequently, the second moment  $\langle x^2 \rangle$  is conserved in first order approximation. On the right side, the energy spectral density is plotted in blue versus the driven tune  $Q_d$ . The nominal driven tune was specified as  $\hat{Q}_d = 0.270$  and is indicated as a red line.

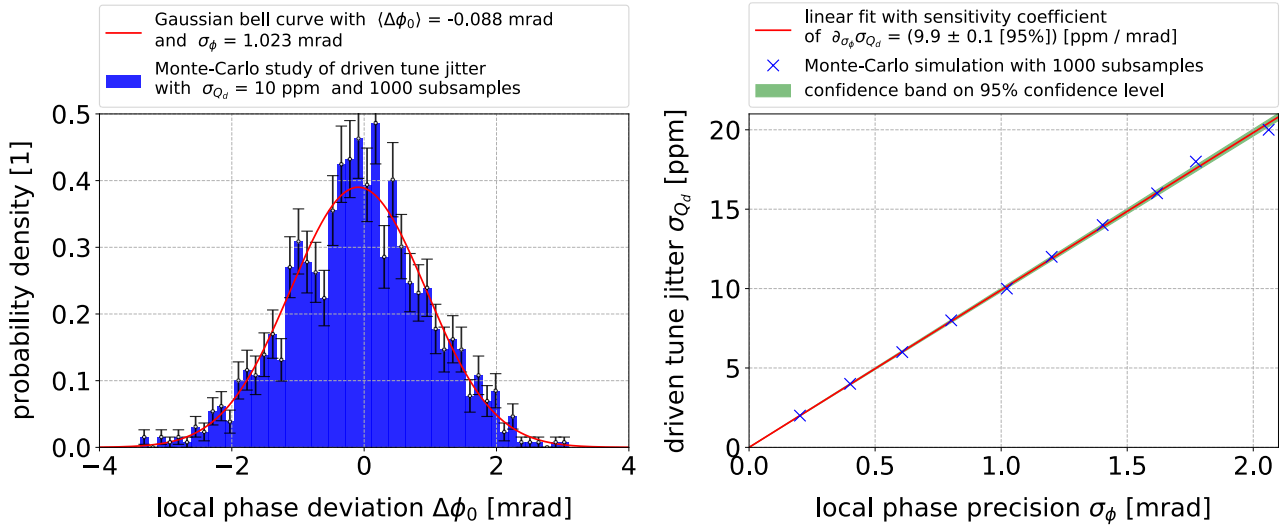


Figure 55: On the left side, the distribution of the local phase deviation  $\Delta\phi_0$  for a driven tune jitter of  $\sigma_{Q_d} = 10$  ppm is shown. The distribution was generated by a Monte-Carlo simulation with 1000 subsamples. On the right side, the driven tune jitter  $\sigma_{Q_d}$  is plotted versus the local phase precision  $\sigma_\phi$ . The slope of the curve quantifies the sensitivity coefficient for the driven tune jitter on the local phase precision.

Previous stability analyses on the natural tune jitter  $\sigma_{Q_o}$  [38] came to the conclusion that the natural tune under ATS-optics configuration obeys a stability level of  $\pm 30$  ppm. Since the natural tune  $Q_o$  of an accelerator depends on the entire lattice layout, each magnetic element imperfection of the lattice will influence  $Q_o$ . Hence, there are many potential sources of perturbations for  $Q_o$ . On the contrary, the driven tune  $Q_d$  is entirely enforced by the waveform generator, which supplies the AC-dipole. This leads to the assumption, that the driven tune jitter  $\sigma_{Q_d}$  should generally be smaller than the natural tune jitter  $\sigma_{Q_o}$ . For a precise quantification of the driven tune jitter, further studies have to be initiated but previous measurements [48] indicate that  $\sigma_{Q_d}$  is in the range of 0.5 ppm. Furthermore, the successful application of correction bumps in arc sextupoles during previous machine diagnostic sessions [27] implies, that the precision requirements were met. In conclusion, there is experimental evidence which supports the complexity argument of  $\sigma_{Q_d} \leq \sigma_{Q_o}$  and it can be inferred that the precision requirement for the local phase measurement of  $\pm 1$  mrad as well as the upper limit for the driven tune jitter of  $\sigma_{Q_d} < 10$  ppm is satisfied for the ATS-optics configuration.

### 9.5.3 Noise level of the arc circuits

In the previously presented Monte-Carlo simulations, turn-by-turn data from PTC was used as an input. The simulations imposed a precision requirement on the measurements of the closed orbit displacement  $x$  and the local phase  $\phi$ . The robustness of the correction method is not only a matter of the measurement uncertainties. In addition, statistical fluctuations of the magnet currents create a noise level in the lattice. Consequently, another Monte-Carlo study was setup, which uses the twiss module from MAD-X to generate noisy lattices. In detail, all quadrupoles powered in series are assigned with a Gaussian random error. Since the magnitude of the systematic errors was initialized with 1‰, the noise level is defined in relation to this magnitude. For example, a noise level of  $10\sigma$  corresponds to a statistical fluctuation of the arc circuits of  $\sigma_{k_1} = 0.1\%$  [68%]. The perturbed lattice functions from twiss are then used to calculate the knob corrections via the response matrix method. It is expected that the correction method starts to fail if a noise level of roughly  $1\sigma$  is reached. This reflects the fact,

that for  $1\sigma$  the statistical fluctuations in the quadrupole powering system are similar in magnitude as the systematic quadrupolar distortions. The results of the Monte-Carlo simulations are presented in Figure (56). If the unified beating pattern of  $\beta$ ,  $\alpha$ ,  $\psi$  and  $D$  is exploited, then the noise level for serially powered quadrupoles in the arcs should not exceed  $2\sigma$  for a beneficial application of the response matrix method using orbit bumps. At this point it has to be emphasized, that the simulation results regarding the  $\beta$ -beating pattern are not expected. In comparison to all other lattice quantities, the correction method seems to be least robust against quadrupolar noise in the arcs if applied on  $\beta$ . Typically,  $\alpha$ -beating is expected to be the most sensitive observable regarding noise. Nevertheless, the Monte-Carlo studies with noise in the arc quadrupole circuits did also not reveal an improvement of the correction method if the periodicity is broken. Hence, there is no evidence for a degeneracy of the correction bumps due to limitations in the precision of the data acquisition system. The upper noise limit of the arc circuits directly imposes a lower limit for the scope of the correction method if existing stability studies on the ATS-optics layout in the LHC are incorporated. Previous stability measurements regarding the quadrupole powering system in the arcs of the LHC indicate a noise level in the range of 230 ppm [39]. Hence, the correction method of using quadrupolar feed-downs from sextupoles in the arcs is capable of correcting systematic quadrupolar errors larger than 460 ppm if the unified beating pattern is exploited.

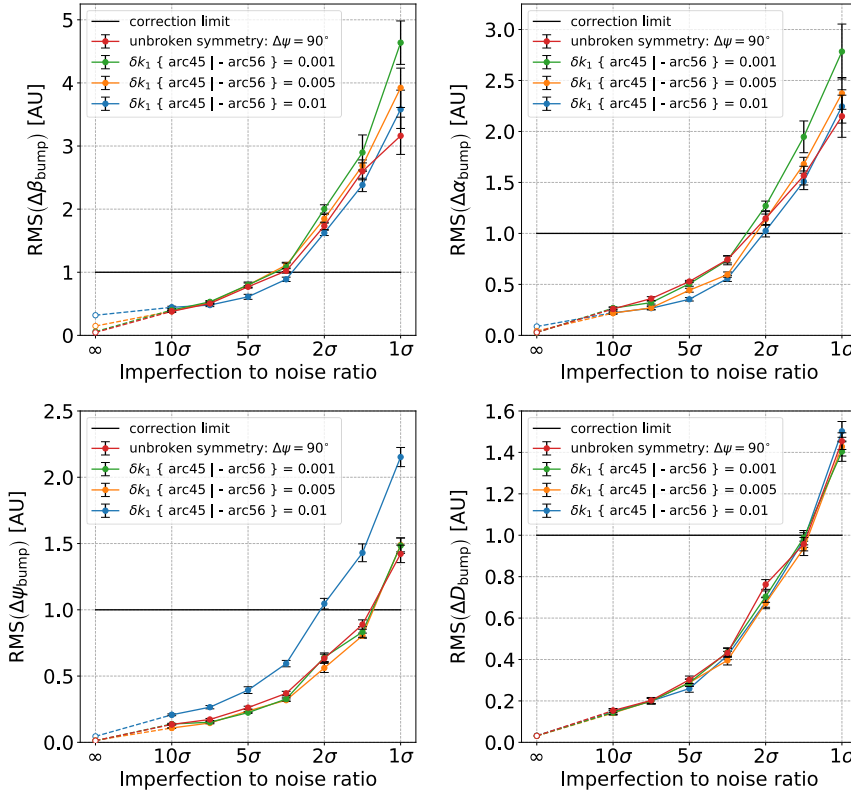


Figure 56: The normalised residual RMS beating after correction with the response matrix method is shown for different FODO-cell configurations as a function of the magnitude of the noise level in serially powered quadrupoles in the arcs. For the plot on the upper left side, the observable vector  $\vec{\mathcal{O}}$  (168) was restricted to the  $\beta$ -function as an input. Similarly, the observable vector  $\vec{\mathcal{O}}$  (168) was limited to  $\alpha$  for the upper right plot and restricted to  $\psi$  for the graph on the lower left side. Finally, the observable vector  $\vec{\mathcal{O}}$  (168) was limited to  $D$  as an input for the chart on the lower right side.



## 10 Conclusion

The theoretical studies of the response matrix correction method with orbit bumps in sextupoles from Chapter 9.4, as well as the Monte-Carlo studies from Chapter 9.5 did not reveal a degeneracy of the orbit bumps. Hence, it can be concluded that the uniqueness of the beating pattern is observable within the upper statistical dispersion limits condensed in Table (4).

| error source   | unified pattern   | $\Delta\beta$     | $\Delta\alpha$    | $\Delta\psi$      | $\Delta D$   |
|----------------|-------------------|-------------------|-------------------|-------------------|--------------|
| $\sigma_x$     | 200 $\mu\text{m}$ | 300 $\mu\text{m}$ | 150 $\mu\text{m}$ | 150 $\mu\text{m}$ | ...          |
| $\sigma_\phi$  | 1 mrad            | 2 mrad            | 0.5 mrad          | 1.5 mrad          | ...          |
| $\sigma_{Q_d}$ | 10 ppm            | 20 ppm            | 5 ppm             | 15 ppm            | ...          |
| $\sigma_k$     | 2 $\sigma$        | 3.5 $\sigma$      | 2 $\sigma$        | 1.5 $\sigma$      | 1.5 $\sigma$ |

Table 4: Upper statistical dispersion limits for the correction method.

Furthermore, the scope of the orbit bump correction method has been determined for different error sources. From the MC-studies in Chapter 9.4 the upper limits for the scope of the correction method are given by Table (5). In case of noise in the arc quadrupole circuits, the lower limit for the scope of the correction method is implicitly specified by the upper statistical dispersion limit from Table (4).

|           |              | $\sigma_k$ related to $\min\{L_2K_2 _{arc}\} / \sigma_k$ related to $\max\{L_2K_2 _{arc}\}$ |               |                |              |             |
|-----------|--------------|---------------------------------------------------------------------------------------------|---------------|----------------|--------------|-------------|
| mode      | error source | unified pattern                                                                             | $\Delta\beta$ | $\Delta\alpha$ | $\Delta\psi$ | $\Delta D$  |
| injection | $\sigma_k$   | 0.6 / 3.1 ‰                                                                                 | 0.6 / 3.1 ‰   | 0.6 / 3.1 ‰    | 0.6 / 3.1 ‰  | 0.6 / 3.1 ‰ |
|           | $\sigma_s$   | 6 cm                                                                                        | 1.6 cm        | 1.6 cm         | 2.5 cm       | 10 cm       |
|           | $\sigma_x$   | 0.6 mm                                                                                      | 0.4 mm        | 0.4 mm         | 0.4 mm       | 0.6 mm      |
| flattop   | $\sigma_k$   | 1.6 / 8 ‰                                                                                   | 1.6 / 8 ‰     | 1.0 / 5.2 ‰    | 1.6 / 8 ‰    | 1.6 / 8 ‰   |
|           | $\sigma_s$   | 10 cm                                                                                       | 4 cm          | 4 cm           | 6 cm         | 25 cm       |
|           | $\sigma_x$   | 1 mm                                                                                        | 0.6 mm        | 0.6 mm         | 0.6 mm       | 1.6 mm      |

Table 5: Upper limits for the scope of the correction method.

In general, the simulations from Chapter 9.4 were investigating the distribution of the maximum correction knob at constant perturbation magnitude. If the nominal quadrupole strength  $K_1$  is perturbed by  $\epsilon$ , the desired closed orbit offset inside the sextupole is determined by Equation (174). If the phase difference between the perturbed quadrupole and the bumped sextupole as well as the difference in the  $\beta$ -function at those locations is neglected, Equation (174) describes the relative perturbation  $\epsilon$  as a function of the closed orbit offset  $\Delta x_o$ .

$$\begin{aligned}
L_2K_2 \cdot \Delta x_o &\stackrel{!}{\approx} -\epsilon \cdot L_1K_1 \quad , \quad \text{with : } \epsilon \in \mathcal{U}(\pm\sigma_k) \\
&\Leftrightarrow \epsilon = -\frac{L_2K_2}{L_1K_1} \cdot \Delta x_o \\
&\Rightarrow \log(|\epsilon|) = \log\left(\left|\frac{L_2K_2}{L_1K_1}\right|\right) + \log(|\Delta x_o|)
\end{aligned} \tag{174}$$

The quadrupoles in the arcs are characterised by the same integrated strength in absolute terms because they are powered in series and are of equal length. Furthermore, the aperture limitation



for the closed orbit offset  $\Delta x_o$  applies to all possible bumps. Consequently, the error distribution half-width  $\sigma_k$  exclusively depends on the integrated strength of the bumped sextupole. In detail, the ordinate of Figure (41) depends on the normalised integrated strength of the bumped sextupole, which is not constant within the arc. Resultantly, the upper limits for the scope of the correction method from Table (5) scale with the normalised integrated strength of the bumped sextupole, according to Equation (175).

$$\stackrel{(174)}{\Rightarrow} \sigma_{k_i} [\text{MQ}_i] = \frac{|L_2 K_2| [\text{MS}_i]}{\min \{|L_2 K_2|_{\text{arc}}\}} \cdot \min \{\sigma_k\} \quad (175)$$

The boxplots in Figure (41) depict the distribution of maximum orbit bumps. For constant perturbation magnitude  $\sigma_k$ , the closed orbit offset  $\Delta x_o$  will be larger for weak sextupoles as for strong sextupoles. Hence, the deduced upper limits for the scope of the correction method from Figure (41) are implicitly related to the minimal integrated sextupole strength within the arc. This leads to a range of possible upper limits, according to Equation (175). Table (5) lists the bandwidth of upper limits for  $\sigma_k$ . In general, the minimal integrated sextupole strength in the arcs of the ATS-optics layout is  $\min \{|L_2 K_2|_{\text{arc}}\} \approx 222 \text{ cm}^2$  and the maximal strength is  $\max \{|L_2 K_2|_{\text{arc}}\} \approx 1159 \text{ cm}^2$ , which results into a ratio bandwidth for the upper limit of  $B_r = 5.2$ . The dependence of the orbit bump amplitude on the integrated strength of the sextupole also causes a different scaling in  $\sigma_k$  for Figure (38) and Figure (41). The theoretical results of this thesis can be used to interpret previous observations in ATS-optics measurements [27]. In detail, a closed orbit bump in MS.26R4.B1 of 5.2 mm was determined by the response matrix approach to minimize the global RMS beating. In order to obey the aperture constraints of the LHC, the bump was split into four corrections in neighbouring quadrupoles with  $\Delta\psi = n\pi$ . The allocation of the orbit bump comes hand in hand with a beating bump between those quadrupoles (45). Therefore, split orbit bumps will reduce the global RMS beating but increase the local beating at the same time. The studies conducted in this thesis did not reveal a degeneracy of the bumps due to limitations in the data acquisition system, wherefore a single quadrupolar distortion in MQ.26R4.B1 of  $-1.3\%$  (86) can be concluded as the error source. As a result, an investigation of quadrupole MQ.26R4.B1 should be initiated.

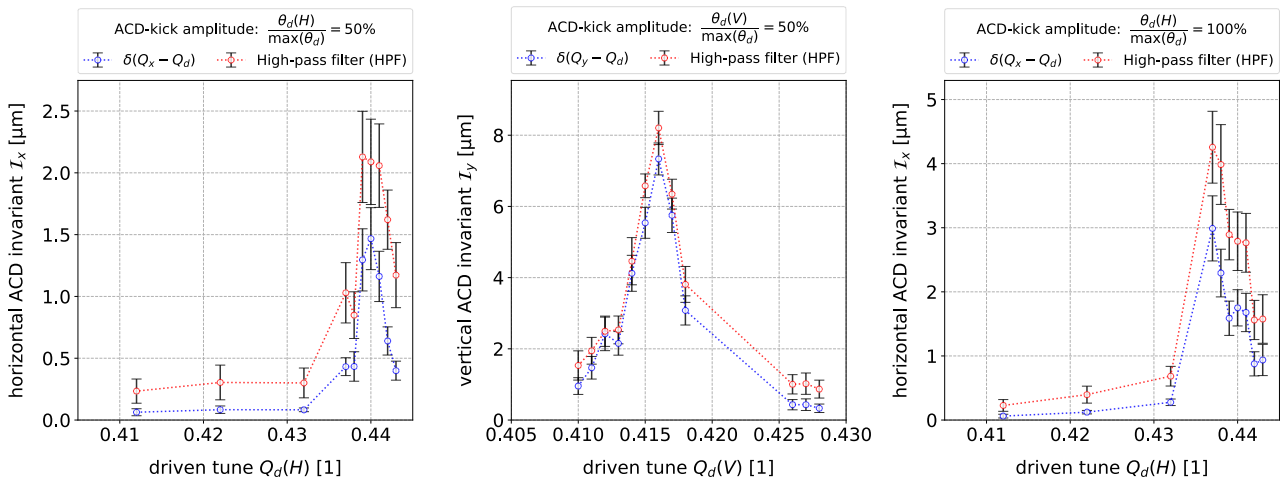


Figure 57: On the left side, the horizontal ACD invariant  $I_x$  is plotted versus the driven tune  $Q_d(H)$  for MSC1. In the middle graph, the vertical ACD invariant  $I_y$  is visualized as a function of the driven tune  $Q_d(V)$  for MSC2. On the right side, the horizontal ACD invariant  $I_x$  is presented depending on the driven tune  $Q_d(H)$  for MSC3. The blue curve is based on the Dirac-Delta impulse assumption for the driven frequency of the chirp. The red curve is utilising the high-pass filter method to determine the ACD invariant. The measurement uncertainties were determined by applying Gaussian error propagation. Both methods are in good agreement within the measurement uncertainties.

In general, the quadrupole powering system in the arcs of the LHC is specified by a stability level in the range of 230 ppm [39]. Hence, the correction method of using quadrupolar feed-downs from sextupoles in the arcs is capable of correcting systematic quadrupolar errors larger than 460 ppm if the unified beating pattern is exploited. In a nutshell, the scope of the correction method for quadrupolar imperfections is specified by  $\sigma_k \in [0.46, [0.6, 3.1]]\%$  for injection and by  $\sigma_k \in [0.46, [1.6, 8]]\%$  at flattop energy.

The orbit measurements at ELENA were presented in Section 8.2. The results of the single frequency analysis of the ACD invariant  $\mathcal{I}_x$  are in good agreement with the outcome of the betatron motion analysis in time domain, as graphically compared in Figure 57. As a result, the restricted chirp excitation can enforce a coherent betatron motion with an AC-dipole like excitation frequency range. Furthermore, it can be concluded that the data acquisition tool does not suffer from significant distortion artefacts. Due to the internal processing system of the analog to digital converter (ADC) at ELENA, the number of turns acquirable was limited to 250 turns. Resultantly, the N-BPM method for extracting the lattice functions suffers from a lack of precision in the natural tune measurement. During the measurements, a significant source of coupling was observed in ELENA. As a consequence, before further measurements are taken a coupling correction must be performed.

## 11 Appendix

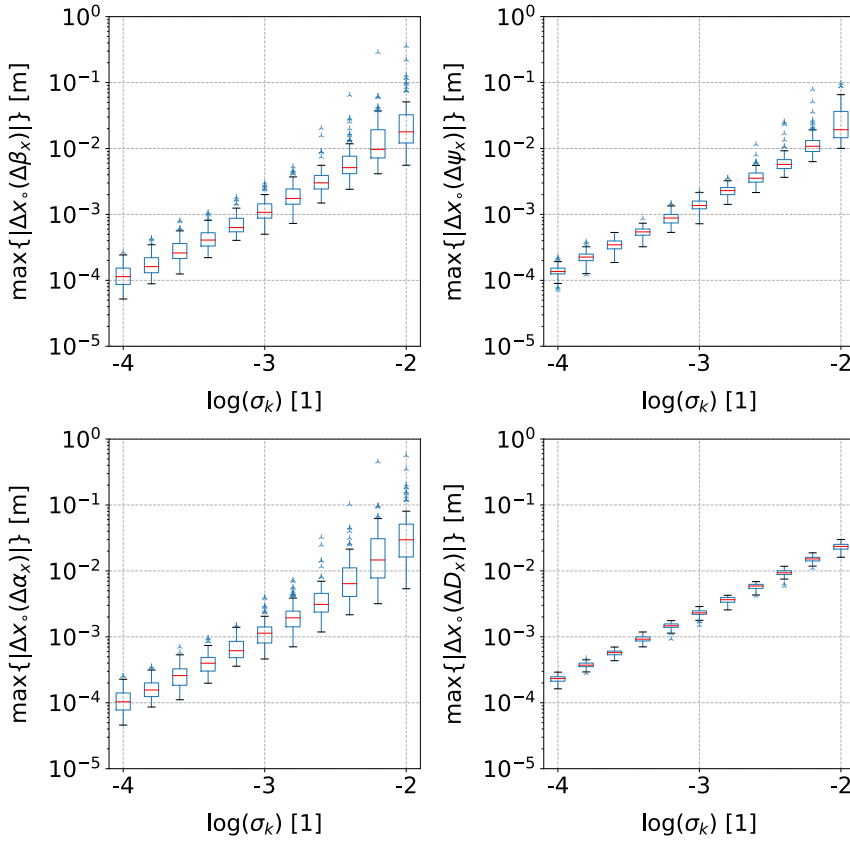


Figure 58: The maximum correction knob is plotted versus the logarithm of the error distribution half-width  $\sigma_k$ . All quadrupole strengths  $k_1$  in arc45 and arc56 have been perturbed and 100 subsamples were generated to establish a distribution in  $\max(|\Delta x_o|)$ . The boxplots visualize the characteristics of every distribution at constant perturbation magnitude. For the calculation of the correction knobs, a single attribute of the unified beating pattern was used. As a consequence, the observable vector (168) was restricted to: (upper left)  $\beta$ -beating, (upper right) phase-beating, (lower left)  $\alpha$ -beating, (lower right) dispersion-beating.

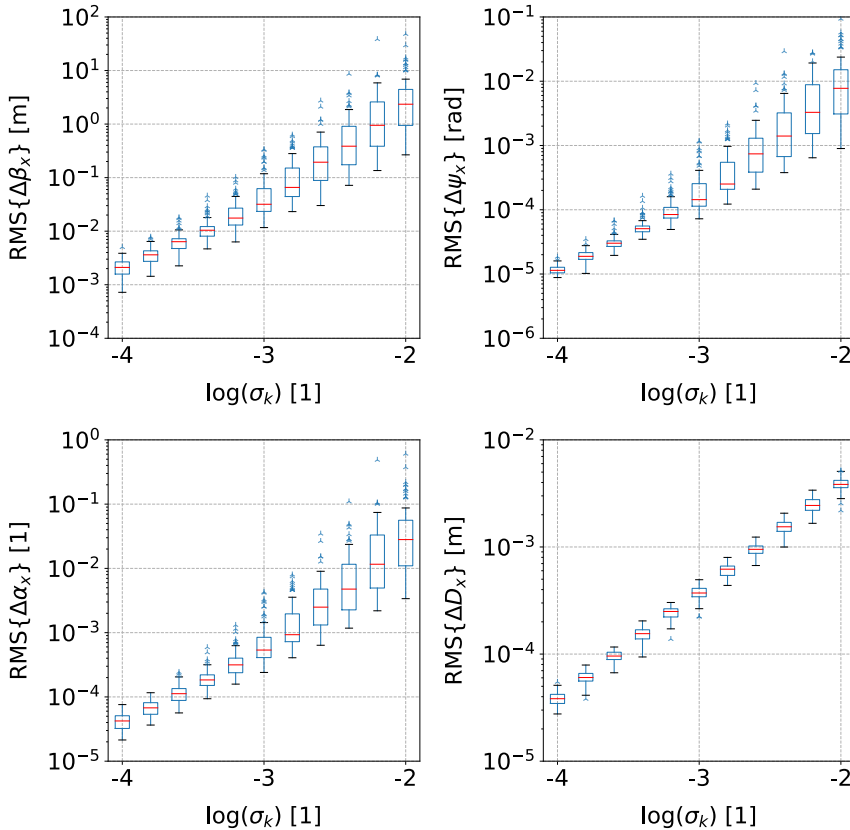


Figure 59: The residual RMS value after correction is presented as a function of the perturbation magnitude. All quadrupole strengths  $k_1$  in arc45 and arc56 have been perturbed and 100 subsamples were generated to establish a distribution for the residual RMS. The boxplots visualize the characteristics of every distribution at constant perturbation magnitude. For the calculation of the correction knobs, a single attribute of the unified beating pattern was used. As a consequence, the observable vector (168) was restricted to: (upper left)  $\beta$ -beating, (upper right) phase-beating, (lower left)  $\alpha$ -beating, (lower right) dispersion-beating.

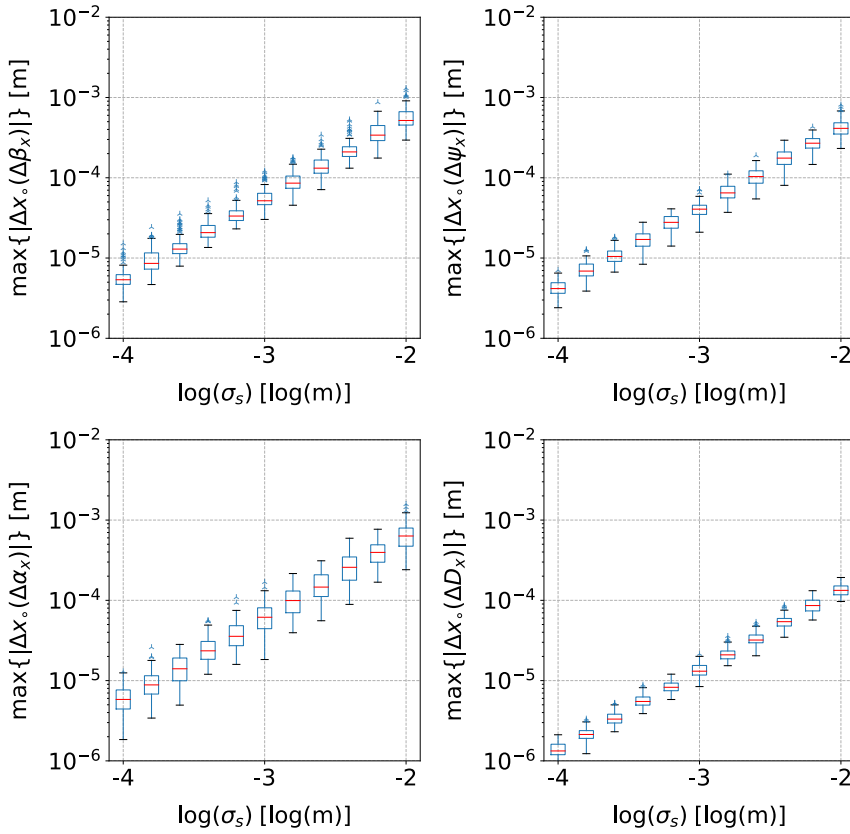


Figure 60: The maximum correction knob is plotted versus the logarithm of the error distribution half-width  $\sigma_s$ . All quadrupoles in arc45 and arc56 have been displaced longitudinally and 100 subsamples were generated to establish a distribution in  $\max(|\Delta x_{\circ}|)$ . The boxplots visualize the characteristics of every distribution at constant perturbation magnitude. For the calculation of the correction knobs, a single attribute of the unified beating pattern was used. As a consequence, the observable vector (168) was restricted to: (upper left)  $\beta$ -beating, (upper right) phase-beating, (lower left)  $\alpha$ -beating, (lower right) dispersion-beating.

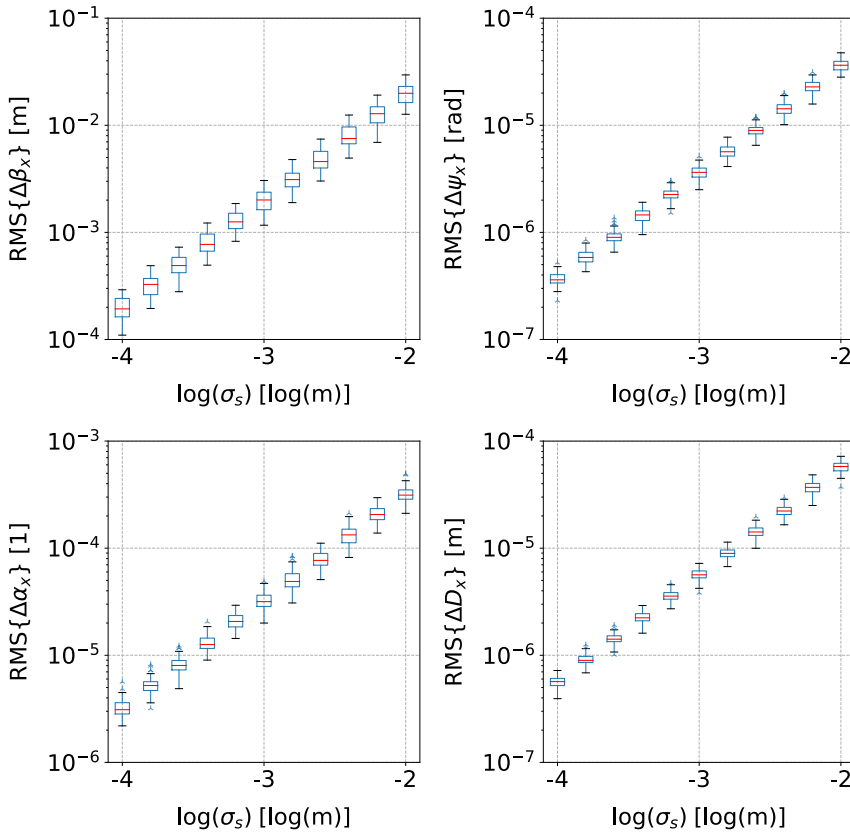


Figure 61: The residual RMS value after correction is presented as a function of the perturbation magnitude. All quadrupoles in arc45 and arc56 have been displaced longitudinally and 100 subsamples were generated to establish a distribution for the residual RMS. The boxplots visualize the characteristics of every distribution at constant perturbation magnitude. For the calculation of the correction knobs, a single attribute of the unified beating pattern was used. As a consequence, the observable vector (168) was restricted to: (upper left)  $\beta$ -beating, (upper right) phase-beating, (lower left)  $\alpha$ -beating, (lower right) dispersion-beating.

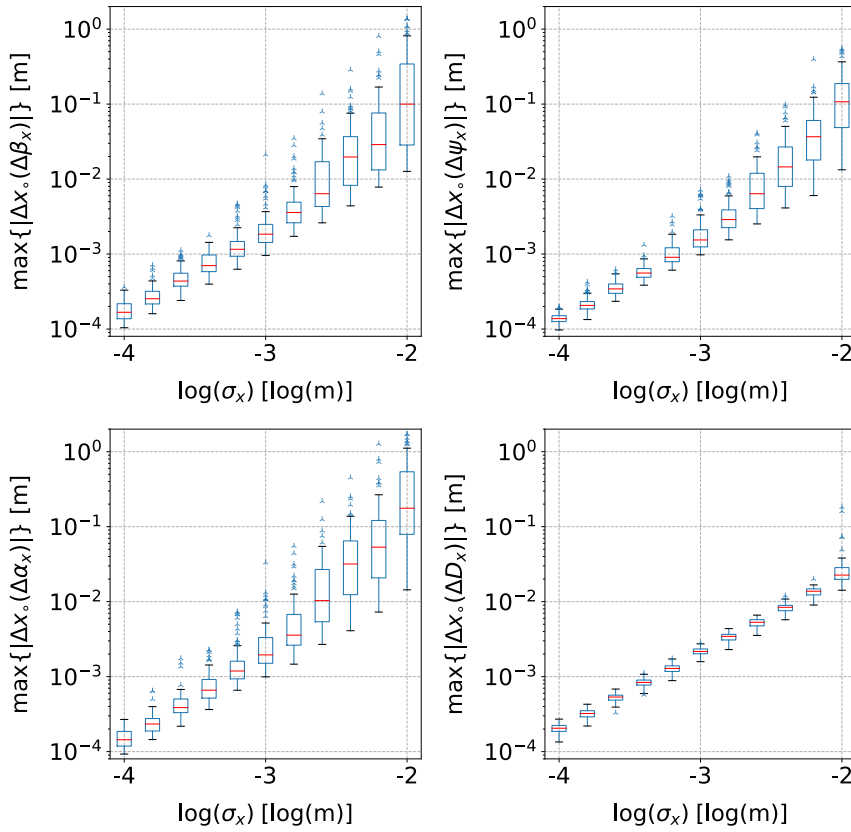


Figure 62: The maximum correction knob is plotted versus the logarithm of the error distribution half-width  $\sigma_x$ . All sextupoles within arc45 and arc56 have been displaced transversely and 100 subsamples were generated to establish a distribution in  $\max(|\Delta x_o|)$ . The boxplots visualize the characteristics of every distribution at constant perturbation magnitude. For the calculation of the correction knobs, a single attribute of the unified beating pattern was used. As a consequence, the observable vector (168) was restricted to: (upper left)  $\beta$ -beating, (upper right) phase-beating, (lower left)  $\alpha$ -beating, (lower right) dispersion-beating.

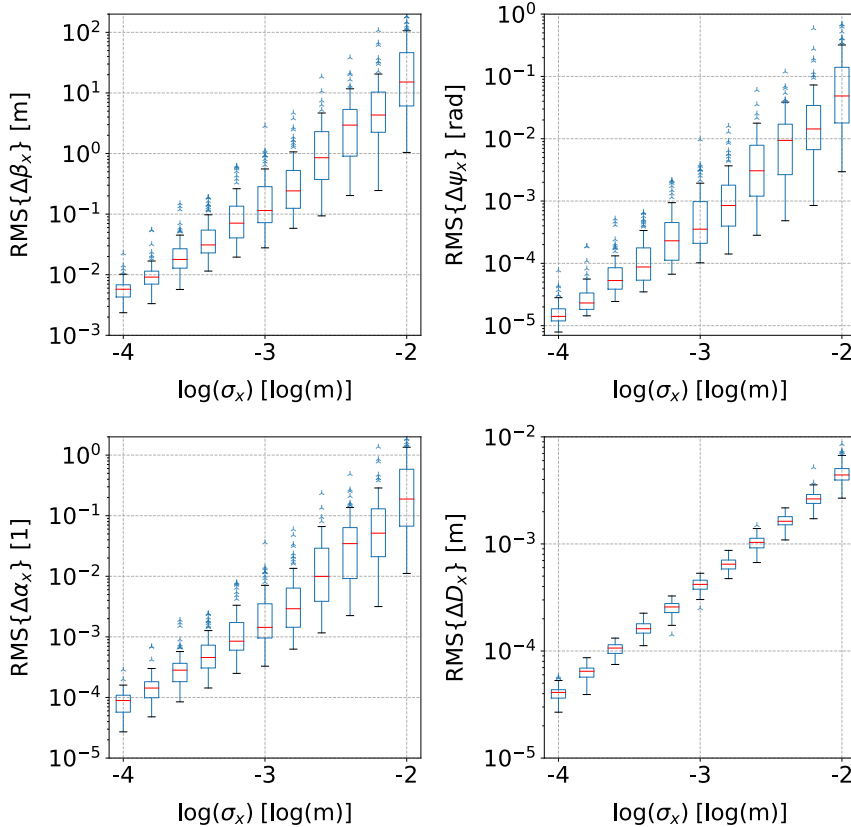


Figure 63: The residual RMS value after correction is presented as a function of the perturbation magnitude. All sextupoles within arc45 and arc56 have been displaced transversely and 100 subsamples were generated to establish a distribution for the residual RMS. The boxplots visualize the characteristics of every distribution at constant perturbation magnitude. For the calculation of the correction knobs, a single attribute of the unified beating pattern was used. As a consequence, the observable vector (168) was restricted to: (upper left)  $\beta$ -beating, (upper right) phase-beating, (lower left)  $\alpha$ -beating, (lower right) dispersion-beating.

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