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CLOSED EXPRESSIONS FOR $\int_0^1 t^{-1} \log^{n-1} t \log^p(1 - t)\, dt$

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CLOSED EXPRESSIONS FOR $\int_0^1 t^{-1} \log^{n-1} t \log^p(1 - t)\, dt$

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ABSTRACT

Closed expressions for the integral $\int_0^1 t^{-1} \log^{n-1} t \log^p(1 - t)\, dt$, whose general form is given elsewhere, are listed for n = 1(1)9, p = 1(1)9. These expressions were evaluated with the help of a computer.

1. INTRODUCTION

At the beginning of this century, Nielsen discussed, in a little-known monograph [1], properties of a family of functions

$$S_{np}(x) = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 t^{-1} \log^{n-1} t \log^p(1 - xt)\, dt \tag{1}$$

for positive integers n, p, and complex x. These functions include many special cases such as Euler's dilogarithm, Kummer's trilogarithm, the Spence functions and polylogarithms. As already proposed [2], it seems appropriate to call the family (1) Nielsen's generalized polylogarithms.

Although the monograph [1] contains quite a number of misprints and a few erroneous results, it does present a considerable amount of useful information, in particular transformation formulae relating $S_{n,p}(x)$ to $S_{n,p}(1/x)$ and $S_{n,p}(1-x)$. It is remarkable that these formulae, and consequently also those for $S_{n,p}(1/(1-x))$, $S_{n,p}((x-1)/x)$, and $S_{n,p}(x/(x-1))$ contain, apart from logarithms and constants, only functions $S_{\nu,\pi}(x)$. However, as far as the author knows, the important formulae of [1] have never found their way into any of the relevant handbooks.

Interest in these functions revived some time ago, at least for the case $p = 1$, in the context of more-dimensional integration of rational functions in quantum electrodynamics (see, for example [3], [4]). Their properties are also of interest in group theory and geometry [5]. The book of Lewin [6] gives many formulae and properties of $S_{n1}(x)$. A general discussion of Nielsen's monograph is given in [2].

2. THE VALUES $s_{n,p} = S_{n,p}(1)$

The purpose of this note is to give explicit expressions for the special values

$$s_{n,p} = S_{n,p}(1) = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 t^{-1} \log^{n-1} t \log^p(1 - t)\, dt , \tag{2}$$

at least for some n and p. It is easy to show that $s_{n,p} = s_{p,n}$, and hence we can restrict p to $n \geq p$. A closed expression for $s_{n,p}$ is given in [2] (in implicit form also in [6]), which reads

$$s_{n,p} = \sum_{k=1}^{p} \frac{(-1)^{k+1}}{k!} \sum_{m_i} \frac{H_p(m_1, \ldots, m_k)}{m_1 \ldots m_k} \zeta(m_1) \ldots \zeta(m_k) , \tag{3}$$

where

$$H_p(m_1, \ldots, m_k) = \sum_{p_i} \binom{m_1}{p_1} \ldots \binom{m_k}{p_k} . \tag{4}$$

The sum over m_i is to be taken over all sets of integers $\{m_i\}$ $(i = 1, \ldots, k)$ which satisfy

$$m_i \geq 2 , \quad \sum_{i=1}^{k} m_i = n + p \tag{5}$$

and the sum over p_i over all sets of integers $\{p_i\}$ $(i = 1, \ldots, k)$ which satisfy

$$1 \leq p_i \leq m_i - 1 , \quad \sum_{i=1}^{k} p_i = p . \tag{6}$$

The function

$$\zeta(m) = \sum_{k=1}^{\infty} k^{-m} \tag{7}$$

is the Riemann zeta function for integer argument. Nielsen remarked that the functions $S_{n,p}(x)$ are probably the simplest analytic functions which coincide with $\zeta(m)$ for special values of its arguments. He added that he was not able to use his theory of $S_{n,p}(x)$ to find expressions for $\zeta(2\mu+1)$ analogous to the known expressions for $\zeta(2\mu)$.

Nielsen [1] formulated a theorem about the structure of $s_{n,p}$ and gave the principle of the proof. He also calculated the cases $p \leq 3$. The case $p = 1$ is trivial, giving

$$\int_0^1 t^{-1} \log^{n-1} t \log (1 - t)\, dt = (-1)^n (n - 1)! s_{n,1} = (-1)^n (n - 1)! \zeta(n + 1) . \tag{8}$$

The case $p = 2$ can also be handled easily, but the $k = 3$ term in (3) for $p = 3$ is somewhat more involved, and Nielsen's final expression [1,§18(19),(20)] is incorrect. However, the expression for s_{73} given as an example in [1] differs from the correct expression only by a difference in the coefficient of $\zeta(2)\zeta^2(4)$ ($1/3$ instead of $1/2$), and this could be due to a misprint.

Writing (3) as

$$s_{n,p} = \sum_{k=1}^{p} \frac{(-1)^{k+1}}{k!} \alpha_k(n,p) , \tag{9}$$

it is easy to find from (4) the following expressions for $\alpha_k(n,p)$ in the case of some special values of p and k:

$$\alpha_1(n,p) = \frac{(n + p - 1)!}{n!p!} \zeta(n + p) \tag{10}$$

$$\alpha_2(n,2) = \sum_{\nu=2}^{n} \zeta(\nu)\zeta(n - \nu + 2) \tag{11}$$

$$\alpha_n(n,n) = \zeta^n(2) . \tag{12}$$

For $k = 2$, $p = 3$, we have from (4), for $\nu = 2, \ldots, n + 1$,

$$H_3(n - \nu + 3,\nu) = \varepsilon_{\nu,2} \binom{n - \nu + 3}{1}\binom{\nu}{2} + \varepsilon_{\nu,n+1} \binom{n - \nu + 3}{2}\binom{\nu}{1}$$

and

$$\frac{H_3(n - \nu + 3,\nu)}{(n - \nu + 3)\nu} = \begin{cases} \frac{1}{2} n & \text{if } \nu = 2,\ \nu = n + 1 \\ \frac{1}{2} (n + 1) & \text{if } \nu = 3, \ldots, n , \end{cases} \tag{13}$$

where $\varepsilon_{\nu,\mu} = 0$ for $\nu = \mu$ and $\varepsilon_{\nu,\mu} = 1$ for $\nu \neq \mu$, so that

$$\begin{aligned} \alpha_2(n,3) &= n\zeta(2)\zeta(n + 1) + \frac{1}{2} (n + 1) \sum_{\nu=3}^{n} \zeta(\nu)\zeta(n - \nu + 3) \\ &= \sum_{\nu=2}^{n} (n - \nu + 2)\zeta(\nu)\zeta(n - \nu + 3) . \end{aligned} \tag{14}$$

In the case $k = 2$, $p = 4$, one finds for $\nu = 2, \ldots, n + 2$,

$$H_4(n-\nu+4,\nu) = \varepsilon_{\nu,2}\varepsilon_{\nu,3}\binom{n-\nu+4}{1}\binom{\nu}{3} + \varepsilon_{\nu,2}\varepsilon_{\nu,n+1}\binom{n-\nu+4}{2}\binom{\nu}{2}$$
$$+ \varepsilon_{\nu,n+1}\varepsilon_{\nu,n+2}\binom{n-\nu+4}{3}\binom{\nu}{1} .$$

Thus

$$\frac{H_4(n-\nu+4,\nu)}{(n-\nu+4)\nu} = \begin{cases} \frac{1}{6}\, n\,(n+1) & \text{if } \nu = 2,\ \nu = n+2 \\ \frac{1}{6}\, n\,(n+2) & \text{if } \nu = 3,\ \nu = n+1 \\ \frac{1}{12}\left[\nu^2 - (n+4)\nu + 2n^2 + 7n + 7\right] & \text{if } \nu = 4,\ \ldots,\ n \end{cases} \tag{15}$$

and therefore

$$\alpha_2(n,4) = \frac{1}{3}\, n(n+1)\zeta(2)\zeta(n+2) + \frac{1}{3}\, n(n+2)\zeta(3)\zeta(n+1)$$
$$+ \frac{1}{12}\sum_{\nu=4}^{n}\left[\nu^2 - (n+4)\nu + 2n^2 + 7n + 7\right]\zeta(\nu)\zeta(n-\nu+4) . \tag{16}$$

For larger values of p, $\alpha_2(n,p)$ becomes more and more complicated.

For $k = p = 3$, we see that $p_1 = p_2 = p_3 = 1$ and $H_3(m_1, m_2, m_3) = m_1 m_2 m_3$. The sum over m_i in (3) therefore equals the sum over the products $\zeta(m_1)\zeta(m_2)\zeta(m_3)$ for all partitions $\{m_1, m_2, m_3\}$ of $n + 3$ satisfying $2 \le m_i \le [(n+3)/3]$, with a weight for possible permutations, where $[\xi]$ denotes the integer part of ξ. This leads to

$$\alpha_3(n,3) = \sum_{\mu=2}^{\mu^*}\zeta(\mu)\sum_{\nu=\mu}^{\nu^*}\omega(n;\mu,\nu)\zeta(\nu)\zeta(n+3-\nu-\mu) , \tag{17}$$

where $\mu^* = [(n+3)/3]$, $\nu^* = [(n-\mu+3)/2]$, and

$$\omega(n;\mu,\nu) = \begin{cases} 1 & \text{if } \mu = \nu \text{ and } 3\mu = n+3 \\ 3 & \text{if } \mu = \nu \text{ and } 3\mu \ne n+3 \text{ or} \\ & \text{if } \mu \ne \nu \text{ and } 2\mu + \nu = n+3 \text{ or} \\ & \text{if } \mu \ne \nu \text{ and } \mu + 2\nu = n+3 \\ 6 & \text{otherwise .} \end{cases} \tag{18}$$

From (1), (10), and (11) it follows that

$$\int_0^1 t^{-1}\log^{n-1} t\,\log^2(1-t)\,dt = 2(-1)^{n-1}(n-1)!\,s_{n,2}$$
$$= (-1)^{n-1}(n-1)!\left[(n+1)\zeta(n+2) - \sum_{\nu=2}^{n}\zeta(\nu)\zeta(n-\nu+2)\right], \tag{19}$$

and from (10), (14), and (17),

$$\int_0^1 t^{-1} \log^{n-1} t \log^3(1 - t)\, dt = 6(-1)^n(n - 1)!s_{n,3}$$

$$= (-1)^n(n - 1)! \left[(n + 1)(n + 2)\zeta(n + 3) - 3 \sum_{\nu=2}^{n} (n - \nu + 2)\zeta(\nu)\zeta(n - \nu + 3) \right.$$

$$\left. + \sum_{\mu=2}^{\mu^*} \zeta(\mu) \sum_{\nu=\mu}^{\nu^*} \omega(n;\mu,\nu)\zeta(\nu)\zeta(n + 3 - \nu - \mu) \right] . \qquad (20)$$

This last formula corrects formula [1,§18(19)] of Nielsen.

For arbitrary n and p, it is obvious that (3) can be reasonably evaluated only by means of a computer. The main task consists in constructing the sets $\{m_i\}$ and $\{p_i\}$. Because of the fact that all permutations have to be taken into account, the number of these sets grows rapidly with increasing values of $n + p$. We have constructed these sets up to $n = p = 9$ by means of a FORTRAN program. As an example, their number, disregarding the condition $1 \le p_i \le m_i - 1$, is shown for $n = p = 9$ in Table 1.

Table 1

k	1	2	3	4	5	6	7	8	9
$\{m_i\}$	1	15	91	286	495	462	210	36	1
$\{p_i\}$	1	8	28	56	70	56	28	8	1

Not all of these sets contribute to the result (3), because of the condition $1 \le p_i \le m_i - 1$.

The evaluation of formula (3) has been performed in REDUCE [7] for $n + p \le 14$, and in FORTRAN for higher values of $n + p$. Although REDUCE is very convenient because it gives the rational coefficients of $\zeta(m_1) \ldots \zeta(m_k)$ directly, FORTRAN was preferred for higher $n + p$ for reasons of computer time. The results have been checked by evaluating the definition integral (1) by numerical integration, replacing the limits 0 and 1 by $\varepsilon = 10^{-8}$ and $1 - \varepsilon$, respectively, and using Nielsen's 32 decimal table of $\zeta(m)$, $m = 2(1)70$, for the evaluation of $s_{n,p}$.

3. A TABLE OF THE INTEGRAL

We list the expressions for $s_{n,p}$, $n = 2(1)9$, $p = 2(1)n$. The values for the integral in (2) itself,

$$r_{n,p} = \int_0^1 t^{-1} \log^{n-1} t \log^p(1-t)\, dt = (-1)^{n+p-1}(n-1)!p!s_{n,p} , \qquad (20)$$

would lead for higher n or p to rather large coefficients. The expressions for $p = 1$ can be easily obtained from (8). The reference work [8, No. 4.2912] lists only the case $n = p = 1$, whereas Lewin [6] gives (20) for $n = 3, 4$, and $p = 2$.

Using the well-known relation [8, No. 9.5421],

$$\zeta(2\mu) = \frac{2^{2\mu-1}\pi^{2\mu}|B_{2\mu}|}{(2\mu)!} , \qquad (21)$$

where $B_{2\mu}$ are the Bernoulli numbers, the expressions for $r_{n,p}$ simplify to some extent. We give these values for $n = 1(1)7$, $p = 1(1)n$.

$$s_{22} = -\tfrac{1}{2}\,\zeta^2(2) + \tfrac{3}{2}\,\zeta(4)$$

$$s_{32} = -\zeta(2)\zeta(3) + 2\zeta(5)$$

$$s_{33} = \tfrac{1}{6}\,\zeta^3(2) - \tfrac{3}{2}\,\zeta(2)\zeta(4) - \zeta^2(3) + \tfrac{10}{3}\,\zeta(6)$$

$$s_{42} = -\zeta(2)\zeta(4) - \tfrac{1}{2}\,\zeta^2(3) + \tfrac{5}{2}\,\zeta(6)$$

$$s_{43} = \tfrac{1}{2}\,\zeta^2(2)\zeta(3) - 2\zeta(2)\zeta(5) - \tfrac{5}{2}\,\zeta(3)\zeta(4) + 5\zeta(7)$$

$$s_{44} = -\tfrac{1}{24}\,\zeta^4(2) + \tfrac{3}{4}\,\zeta^2(2)\zeta(4) + \zeta(2)\zeta^2(3) - \tfrac{10}{3}\,\zeta(2)\zeta(6) - 4\zeta(3)\zeta(5) - \tfrac{17}{8}\,\zeta^2(4) + \tfrac{35}{4}\,\zeta(8)$$

$$s_{52} = -\zeta(2)\zeta(5) - \zeta(3)\zeta(4) + 3\zeta(7)$$

$$s_{53} = \tfrac{1}{2}\,\zeta^2(2)\zeta(4) + \tfrac{1}{2}\,\zeta(2)\zeta^2(3) - \tfrac{5}{2}\,\zeta(2)\zeta(6) - 3\zeta(3)\zeta(5) - \tfrac{3}{2}\,\zeta^2(4) + 7\zeta(8)$$

$$s_{54} = -\tfrac{1}{6}\,\zeta^3(2)\zeta(3) + \zeta^2(2)\zeta(5) + \tfrac{5}{2}\,\zeta(2)\zeta(3)\zeta(4) - 5\zeta(2)\zeta(7) + \tfrac{1}{2}\,\zeta^3(3) - \tfrac{35}{6}\,\zeta(3)\zeta(6) - 6\zeta(4)\zeta(5) + 14\zeta(9)$$

$$s_{55} = \tfrac{1}{120}\,\zeta^5(2) - \tfrac{1}{4}\,\zeta^3(2)\zeta(4) - \tfrac{1}{2}\,\zeta^2(2)\zeta^2(3) + \tfrac{5}{3}\,\zeta^2(2)\zeta(6) + 4\zeta(2)\zeta(3)\zeta(5) + \tfrac{17}{8}\,\zeta(2)\zeta^2(4) - \tfrac{35}{4}\,\zeta(2)\zeta(8) + \tfrac{5}{2}\,\zeta^2(3)\zeta(4) - 10\zeta(3)\zeta(7) - 10\zeta(4)\zeta(6) - 5\zeta^2(5) + \tfrac{126}{5}\,\zeta(10)$$

$$s_{62} = -\zeta(2)\zeta(6) - \zeta(3)\zeta(5) - \tfrac{1}{2}\,\zeta^2(4) + \tfrac{7}{2}\,\zeta(8)$$

$$s_{63} = \tfrac{1}{2}\,\zeta^2(2)\zeta(5) + \zeta(2)\zeta(3)\zeta(4) - 3\zeta(2)\zeta(7) + \tfrac{1}{6}\,\zeta^3(3) - \tfrac{7}{2}\,\zeta(3)\zeta(6) - \tfrac{7}{2}\,\zeta(4)\zeta(5) + \tfrac{28}{3}\,\zeta(9)$$

$$s_{64} = -\tfrac{1}{6}\,\zeta^3(2)\zeta(4) - \tfrac{1}{4}\,\zeta^2(2)\zeta^2(3) + \tfrac{5}{4}\,\zeta^2(2)\zeta(6) + 3\zeta(2)\zeta(3)\zeta(5) + \tfrac{3}{2}\,\zeta(2)\zeta^2(4) - 7\zeta(2)\zeta(8) + \tfrac{7}{4}\,\zeta^2(3)\zeta(4) - 8\zeta(3)\zeta(7) - \tfrac{97}{12}\,\zeta(4)\zeta(6) - 4\zeta^2(5) + 21\zeta(10)$$

$$s_{65} = \tfrac{1}{24}\,\zeta^4(2)\zeta(3) - \tfrac{1}{3}\,\zeta^3(2)\zeta(5) - \tfrac{5}{4}\,\zeta^2(2)\zeta(3)\zeta(4) + \tfrac{5}{2}\,\zeta^2(2)\zeta(7) - \tfrac{1}{2}\,\zeta(2)\zeta^3(3) + \tfrac{35}{6}\,\zeta(2)\zeta(3)\zeta(6) + 6\zeta(2)\zeta(4)\zeta(5) - 14\zeta(2)\zeta(9) + \tfrac{7}{2}\,\zeta^2(3)\zeta(5) + \tfrac{29}{8}\,\zeta(3)\zeta^2(4) - \tfrac{63}{4}\,\zeta(3)\zeta(8) - \tfrac{31}{2}\,\zeta(4)\zeta(7) - \tfrac{91}{6}\,\zeta(5)\zeta(6) + 42\zeta(11)$$

$$s_{66} = -\tfrac{1}{720}\,\zeta^6(2) + \tfrac{1}{16}\,\zeta^4(2)\zeta(4) + \tfrac{1}{6}\,\zeta^3(2)\zeta^2(3) - \tfrac{5}{9}\,\zeta^3(2)\zeta(6) - 2\zeta^2(2)\zeta(3)\zeta(5) - \tfrac{17}{16}\,\zeta^2(2)\zeta^2(4) + \tfrac{35}{8}\,\zeta^2(2)\zeta(8) - \tfrac{5}{2}\,\zeta(2)\zeta^2(3)\zeta(4) + 10\zeta(2)\zeta(3)\zeta(7) + 10\zeta(2)\zeta(4)\zeta(6) + 5\zeta(2)\zeta^2(5) - \tfrac{126}{5}\,\zeta(2)\zeta(10) - \tfrac{1}{4}\,\zeta^4(3) + \tfrac{35}{6}\,\zeta^2(3)\zeta(6) + 12\zeta(3)\zeta(4)\zeta(5) - 28\zeta(3)\zeta(9) + \tfrac{33}{16}\,\zeta^3(4) - \tfrac{217}{8}\,\zeta(4)\zeta(8) - 26\zeta(5)\zeta(7) - \tfrac{461}{36}\,\zeta^2(6) + 77\zeta(12)$$

$$s_{72} = -\zeta(2)\zeta(7) - \zeta(3)\zeta(6) - \zeta(4)\zeta(5) + 4\zeta(9)$$

$$s_{73} = \tfrac{1}{2}\,\zeta^2(2)\zeta(6) + \zeta(2)\zeta(3)\zeta(5) + \tfrac{1}{2}\,\zeta(2)\zeta^2(4) - \tfrac{7}{2}\,\zeta(2)\zeta(8) + \tfrac{1}{2}\,\zeta^2(3)\zeta(4) - 4\zeta(3)\zeta(7) - 4\zeta(4)\zeta(6) - 2\zeta^2(5) + 12\zeta(10)$$

$$s_{74} = -\tfrac{1}{6}\,\zeta^3(2)\zeta(5) - \tfrac{1}{2}\,\zeta^2(2)\zeta(3)\zeta(4) + \tfrac{3}{2}\,\zeta^2(2)\zeta(7) - \tfrac{1}{6}\,\zeta(2)\zeta^3(3) + \tfrac{7}{2}\,\zeta(2)\zeta(3)\zeta(6) + \tfrac{7}{2}\,\zeta(2)\zeta(4)\zeta(5) - \tfrac{28}{3}\,\zeta(2)\zeta(9) + 2\zeta^2(3)\zeta(5) + 2\zeta(3)\zeta^2(4) - \tfrac{21}{2}\,\zeta(3)\zeta(8) - \tfrac{21}{2}\,\zeta(4)\zeta(7) - \tfrac{31}{3}\,\zeta(5)\zeta(6) + 30\zeta(11)$$

$$s_{75} = \tfrac{1}{24}\,\zeta^4(2)\zeta(4) + \tfrac{1}{12}\,\zeta^3(2)\zeta^2(3) - \tfrac{5}{12}\,\zeta^3(2)\zeta(6) - \tfrac{3}{2}\,\zeta^2(2)\zeta(3)\zeta(5) - \tfrac{3}{4}\,\zeta^2(2)\zeta^2(4) + \tfrac{7}{2}\,\zeta^2(2)\zeta(8) - \tfrac{7}{4}\,\zeta(2)\zeta^2(3)\zeta(4) + 8\zeta(2)\zeta(3)\zeta(7) + \tfrac{97}{12}\,\zeta(2)\zeta(4)\zeta(6) + 4\zeta(2)\zeta^2(5) - 21\zeta(2)\zeta(10) - \tfrac{1}{6}\,\zeta^4(3) + \tfrac{14}{3}\,\zeta^2(3)\zeta(6) + \tfrac{19}{2}\,\zeta(3)\zeta(4)\zeta(5) - \tfrac{70}{3}\,\zeta(3)\zeta(9) + \tfrac{13}{8}\,\zeta^3(4) - \tfrac{91}{4}\,\zeta(4)\zeta(8) - 22\zeta(5)\zeta(7) - \tfrac{65}{6}\,\zeta^2(6) + 66\zeta(12)$$

$$\begin{aligned}
s_{76} = {} & -\tfrac{1}{120}\,\zeta^5(2)\zeta(3) + \tfrac{1}{12}\,\zeta^4(2)\zeta(5) + \tfrac{5}{12}\,\zeta^3(2)\zeta(3)\zeta(4) - \tfrac{5}{6}\,\zeta^3(2)\zeta(7) \\
& + \tfrac{1}{4}\,\zeta^2(2)\zeta^3(3) - \tfrac{35}{12}\,\zeta^2(2)\zeta(3)\zeta(6) - 3\zeta^2(2)\zeta(4)\zeta(5) + 7\zeta^2(2)\zeta(9) \\
& - \tfrac{7}{2}\,\zeta(2)\zeta^2(3)\zeta(5) - \tfrac{29}{8}\,\zeta(2)\zeta(3)\zeta^2(4) + \tfrac{63}{4}\,\zeta(2)\zeta(3)\zeta(8) \\
& + \tfrac{31}{2}\,\zeta(2)\zeta(4)\zeta(7) + \tfrac{91}{6}\,\zeta(2)\zeta(5)\zeta(6) - 42\zeta(2)\zeta(11) - \tfrac{17}{12}\,\zeta^3(3)\zeta(4) \\
& + 9\zeta^2(3)\zeta(7) + \tfrac{217}{12}\,\zeta(3)\zeta(4)\zeta(6) + 9\zeta(3)\zeta^2(5) - \tfrac{231}{5}\,\zeta(3)\zeta(10) \\
& + \tfrac{37}{4}\,\zeta^2(4)\zeta(5) - \tfrac{133}{3}\,\zeta(4)\zeta(9) - 42\zeta(5)\zeta(8) - \tfrac{122}{3}\,\zeta(6)\zeta(7) \\
& + 132\zeta(13)
\end{aligned}$$

$$\begin{aligned}
s_{77} = {} & \tfrac{1}{5040}\,\zeta^7(2) - \tfrac{1}{80}\,\zeta^5(2)\zeta(4) - \tfrac{1}{24}\,\zeta^4(2)\zeta^2(3) + \tfrac{5}{36}\,\zeta^4(2)\zeta(6) \\
& + \tfrac{2}{3}\,\zeta^3(2)\zeta(3)\zeta(5) + \tfrac{17}{48}\,\zeta^3(2)\zeta^2(4) - \tfrac{35}{24}\,\zeta^3(2)\zeta(8) \\
& + \tfrac{5}{4}\,\zeta^2(2)\zeta^2(3)\zeta(4) - 5\zeta^2(2)\zeta(3)\zeta(7) - 5\zeta^2(2)\zeta(4)\zeta(6) - \tfrac{5}{2}\,\zeta^2(2)\zeta^2(5) \\
& + \tfrac{63}{5}\,\zeta^2(2)\zeta(10) + \tfrac{1}{4}\,\zeta(2)\zeta^4(3) - \tfrac{35}{6}\,\zeta(2)\zeta^2(3)\zeta(6) \\
& - 12\zeta(2)\zeta(3)\zeta(4)\zeta(5) + 28\zeta(2)\zeta(3)\zeta(9) - \tfrac{33}{16}\,\zeta(2)\zeta^3(4) \\
& + \tfrac{217}{8}\,\zeta(2)\zeta(4)\zeta(8) + 26\zeta(2)\zeta(5)\zeta(7) + \tfrac{461}{36}\,\zeta(2)\zeta^2(6) - 77\zeta(2)\zeta(12) \\
& - \tfrac{7}{3}\,\zeta^3(3)\zeta(5) - \tfrac{29}{8}\,\zeta^2(3)\zeta^2(4) + \tfrac{63}{4}\,\zeta^2(3)\zeta(8) + 31\zeta(3)\zeta(4)\zeta(7) \\
& + \tfrac{91}{3}\,\zeta(3)\zeta(5)\zeta(6) - 84\zeta(3)\zeta(11) + \tfrac{187}{12}\,\zeta^2(4)\zeta(6) + \tfrac{31}{2}\,\zeta(4)\zeta^2(5) \\
& - \tfrac{399}{5}\,\zeta(4)\zeta(10) - \tfrac{224}{3}\,\zeta(5)\zeta(9) - \tfrac{427}{6}\,\zeta(6)\zeta(8) - 35\zeta^2(7) \\
& + \tfrac{1716}{7}\,\zeta(14)
\end{aligned}$$

$$s_{82} = -\zeta(2)\zeta(8) - \zeta(3)\zeta(7) - \zeta(4)\zeta(6) - \tfrac{1}{2}\,\zeta^2(5) + \tfrac{9}{2}\,\zeta(10)$$

$$\begin{aligned}
s_{83} = {} & \tfrac{1}{2}\,\zeta^2(2)\zeta(7) + \zeta(2)\zeta(3)\zeta(6) + \zeta(2)\zeta(4)\zeta(5) - 4\zeta(2)\zeta(9) \\
& + \tfrac{1}{2}\,\zeta^2(3)\zeta(5) + \tfrac{1}{2}\,\zeta(3)\zeta^2(4) - \tfrac{9}{2}\,\zeta(3)\zeta(8) - \tfrac{9}{2}\,\zeta(4)\zeta(7) \\
& - \tfrac{9}{2}\,\zeta(5)\zeta(6) + 15\zeta(11)
\end{aligned}$$

$$\begin{aligned}
s_{84} = {} & -\tfrac{1}{6}\,\zeta^3(2)\zeta(6) - \tfrac{1}{2}\,\zeta^2(2)\zeta(3)\zeta(5) - \tfrac{1}{4}\,\zeta^2(2)\zeta^2(4) + \tfrac{7}{4}\,\zeta^2(2)\zeta(8) \\
& - \tfrac{1}{2}\,\zeta(2)\zeta^2(3)\zeta(4) + 4\zeta(2)\zeta(3)\zeta(7) + 4\zeta(2)\zeta(4)\zeta(6) + 2\zeta(2)\zeta^2(5) \\
& - 12\zeta(2)\zeta(10) - \tfrac{1}{24}\,\zeta^4(3) + \tfrac{9}{4}\,\zeta^2(3)\zeta(6) + \tfrac{9}{2}\,\zeta(3)\zeta(4)\zeta(5) \\
& - \tfrac{40}{3}\,\zeta(3)\zeta(9) + \tfrac{3}{4}\,\zeta^3(4) - \tfrac{53}{4}\,\zeta(4)\zeta(8) - 13\zeta(5)\zeta(7) - \tfrac{155}{24}\,\zeta^2(6) \\
& + \tfrac{165}{4}\,\zeta(12)
\end{aligned}$$

$$s_{85} = \tfrac{1}{24}\zeta^4(2)\zeta(5) + \tfrac{1}{6}\zeta^3(2)\zeta(3)\zeta(4) - \tfrac{1}{2}\zeta^3(2)\zeta(7) + \tfrac{1}{12}\zeta^2(2)\zeta^3(3)$$
$$- \tfrac{7}{4}\zeta^2(2)\zeta(3)\zeta(6) - \tfrac{7}{4}\zeta^2(2)\zeta(4)\zeta(5) + \tfrac{14}{3}\zeta^2(2)\zeta(9) - 2\zeta(2)\zeta^2(3)\zeta(5)$$
$$- 2\zeta(2)\zeta(3)\zeta^2(4) + \tfrac{21}{2}\zeta(2)\zeta(3)\zeta(8) + \tfrac{21}{2}\zeta(2)\zeta(4)\zeta(7)$$
$$+ \tfrac{31}{3}\zeta(2)\zeta(5)\zeta(6) - 30\zeta(2)\zeta(11) - \tfrac{3}{4}\zeta^3(3)\zeta(4) + 6\zeta^2(3)\zeta(7)$$
$$+ \tfrac{145}{12}\zeta(3)\zeta(4)\zeta(6) + 6\zeta(3)\zeta^2(5) - 33\zeta(3)\zeta(10) + \tfrac{49}{8}\zeta^2(4)\zeta(5)$$
$$- 32\zeta(4)\zeta(9) - \tfrac{123}{4}\zeta(5)\zeta(8) - 30\zeta(6)\zeta(7) + 99\zeta(13)$$

$$s_{86} = -\tfrac{1}{120}\zeta^5(2)\zeta(4) - \tfrac{1}{48}\zeta^4(2)\zeta^2(3) + \tfrac{5}{48}\zeta^4(2)\zeta(6) + \tfrac{1}{2}\zeta^3(2)\zeta(3)\zeta(5)$$
$$+ \tfrac{1}{4}\zeta^3(2)\zeta^2(4) - \tfrac{7}{6}\zeta^3(2)\zeta(8) + \tfrac{7}{8}\zeta^2(2)\zeta^2(3)\zeta(4) - 4\zeta^2(2)\zeta(3)\zeta(7)$$
$$- \tfrac{97}{24}\zeta^2(2)\zeta(4)\zeta(6) - 2\zeta^2(2)\zeta^2(5) + \tfrac{21}{2}\zeta^2(2)\zeta(10) + \tfrac{1}{6}\zeta(2)\zeta^4(3)$$
$$- \tfrac{14}{3}\zeta(2)\zeta^2(3)\zeta(6) - \tfrac{19}{2}\zeta(2)\zeta(3)\zeta(4)\zeta(5) + \tfrac{70}{3}\zeta(2)\zeta(3)\zeta(9)$$
$$- \tfrac{13}{8}\zeta(2)\zeta^3(4) + \tfrac{91}{4}\zeta(2)\zeta(4)\zeta(8) + 22\zeta(2)\zeta(5)\zeta(7) + \tfrac{65}{6}\zeta(2)\zeta^2(6)$$
$$- 66\zeta(2)\zeta(12) - \tfrac{11}{6}\zeta^3(3)\zeta(5) - \tfrac{45}{16}\zeta^2(3)\zeta^2(4) + \tfrac{105}{8}\zeta^2(3)\zeta(8)$$
$$+ 26\zeta(3)\zeta(4)\zeta(7) + \tfrac{51}{2}\zeta(3)\zeta(5)\zeta(6) - 72\zeta(3)\zeta(11) + \tfrac{209}{16}\zeta^2(4)\zeta(6)$$
$$+ 13\zeta(4)\zeta^2(5) - \tfrac{687}{10}\zeta(4)\zeta(10) - \tfrac{194}{3}\zeta(5)\zeta(9) - \tfrac{1487}{24}\zeta(6)\zeta(8)$$
$$- \tfrac{61}{2}\zeta^2(7) + \tfrac{429}{2}\zeta(14)$$

$$s_{87} = \tfrac{1}{720}\zeta^6(2)\zeta(3) - \tfrac{1}{60}\zeta^5(2)\zeta(5) - \tfrac{5}{48}\zeta^4(2)\zeta(3)\zeta(4) + \tfrac{5}{24}\zeta^4(2)\zeta(7)$$
$$- \tfrac{1}{12}\zeta^3(2)\zeta^3(3) + \tfrac{35}{36}\zeta^3(2)\zeta(3)\zeta(6) + \zeta^3(2)\zeta(4)\zeta(5) - \tfrac{7}{3}\zeta^3(2)\zeta(9)$$
$$+ \tfrac{7}{4}\zeta^2(2)\zeta^2(3)\zeta(5) + \tfrac{29}{16}\zeta^2(2)\zeta(3)\zeta^2(4) - \tfrac{63}{8}\zeta^2(2)\zeta(3)\zeta(8)$$
$$- \tfrac{31}{4}\zeta^2(2)\zeta(4)\zeta(7) - \tfrac{91}{12}\zeta^2(2)\zeta(5)\zeta(6) + 21\zeta^2(2)\zeta(11)$$
$$+ \tfrac{17}{12}\zeta(2)\zeta^3(3)\zeta(4) - 9\zeta(2)\zeta^2(3)\zeta(7) - \tfrac{217}{12}\zeta(2)\zeta(3)\zeta(4)\zeta(6)$$
$$- 9\zeta(2)\zeta(3)\zeta^2(5) + \tfrac{231}{5}\zeta(2)\zeta(3)\zeta(10) - \tfrac{37}{4}\zeta(2)\zeta^2(4)\zeta(5)$$
$$+ \tfrac{133}{3}\zeta(2)\zeta(4)\zeta(9) + 42\zeta(2)\zeta(5)\zeta(8) + \tfrac{122}{3}\zeta(2)\zeta(6)\zeta(7)$$
$$- 132\zeta(2)\zeta(13) + \tfrac{1}{12}\zeta^5(3) - \tfrac{7}{2}\zeta^3(3)\zeta(6) - \tfrac{43}{4}\zeta^2(3)\zeta(4)\zeta(5)$$
$$+ \tfrac{77}{3}\zeta^2(3)\zeta(9) - \tfrac{59}{16}\zeta(3)\zeta^3(4) + \tfrac{399}{8}\zeta(3)\zeta(4)\zeta(8) + 48\zeta(3)\zeta(5)\zeta(7)$$
$$+ \tfrac{851}{36}\zeta(3)\zeta^2(6) - 143\zeta(3)\zeta(12) + \tfrac{197}{8}\zeta^2(4)\zeta(7) + \tfrac{193}{4}\zeta(4)\zeta(5)\zeta(6)$$
$$- 135\zeta(4)\zeta(11) + 8\zeta^3(5) - \tfrac{627}{5}\zeta(5)\zeta(10) - \tfrac{355}{3}\zeta(6)\zeta(9)$$
$$- \tfrac{459}{4}\zeta(7)\zeta(8) + 429\zeta(15)$$

$$s_{88} = -\tfrac{1}{40320}\zeta^8(2) + \tfrac{1}{480}\zeta^6(2)\zeta(4) + \tfrac{1}{120}\zeta^5(2)\zeta^2(3) - \tfrac{1}{36}\zeta^5(2)\zeta(6)$$
$$- \tfrac{1}{6}\zeta^4(2)\zeta(3)\zeta(5) - \tfrac{17}{192}\zeta^4(2)\zeta^2(4) + \tfrac{35}{96}\zeta^4(2)\zeta(8)$$
$$- \tfrac{5}{12}\zeta^3(2)\zeta^2(3)\zeta(4) + \tfrac{5}{3}\zeta^3(2)\zeta(3)\zeta(7) + \tfrac{5}{3}\zeta^3(2)\zeta(4)\zeta(6)$$
$$+ \tfrac{5}{6}\zeta^3(2)\zeta^2(5) - \tfrac{21}{5}\zeta^3(2)\zeta(10) - \tfrac{1}{8}\zeta^2(2)\zeta^4(3) + \tfrac{35}{12}\zeta^2(2)\zeta^2(3)\zeta(6)$$
$$+ 6\zeta^2(2)\zeta(3)\zeta(4)\zeta(5) - 14\zeta^2(2)\zeta(3)\zeta(9) + \tfrac{33}{32}\zeta^2(2)\zeta^3(4)$$
$$- \tfrac{217}{16}\zeta^2(2)\zeta(4)\zeta(8) - 13\zeta^2(2)\zeta(5)\zeta(9) - \tfrac{461}{72}\zeta^2(2)\zeta^2(6)$$
$$+ \tfrac{77}{2}\zeta^2(2)\zeta(12) + \tfrac{7}{3}\zeta(2)\zeta^3(3)\zeta(5) + \tfrac{29}{8}\zeta(2)\zeta^2(3)\zeta^2(4)$$
$$- \tfrac{63}{4}\zeta(2)\zeta^2(3)\zeta(8) - 31\zeta(2)\zeta(3)\zeta(4)\zeta(7) - \tfrac{91}{3}\zeta(2)\zeta(3)\zeta(5)\zeta(6)$$
$$+ 84\zeta(2)\zeta(3)\zeta(11) - \tfrac{187}{12}\zeta(2)\zeta^2(4)\zeta(6) - \tfrac{31}{2}\zeta(2)\zeta(4)\zeta^2(5)$$
$$+ \tfrac{399}{5}\zeta(2)\zeta(4)\zeta(10) + \tfrac{224}{3}\zeta(2)\zeta(5)\zeta(9) + \tfrac{427}{6}\zeta(2)\zeta(6)\zeta(8)$$
$$+ 35\zeta(2)\zeta^2(7) - \tfrac{1716}{7}\zeta(2)\zeta(14) + \tfrac{17}{24}\zeta^4(3)\zeta(4) - 6\zeta^3(3)\zeta(7)$$
$$- \tfrac{217}{12}\zeta^2(3)\zeta(4)\zeta(6) - 9\zeta^2(3)\zeta^2(5) + \tfrac{231}{5}\zeta^2(3)\tau(10)$$
$$- \tfrac{37}{2}\zeta(3)\zeta^2(4)\zeta(5) + \tfrac{266}{3}\zeta(3)\zeta(4)\zeta(9) + 84\zeta(3)\zeta(5)\zeta(8)$$
$$+ \tfrac{244}{3}\zeta(3)\zeta(6)\zeta(7) - 264\zeta(3)\zeta(13) - \tfrac{203}{128}\zeta^4(4) + \tfrac{1379}{32}\zeta^2(4)\zeta(8)$$
$$+ 83\zeta(4)\zeta(5)\zeta(7) + \tfrac{327}{8}\zeta(4)\zeta^2(6) - \tfrac{495}{2}\zeta(4)\zeta(12) + \tfrac{122}{3}\zeta^2(5)\zeta(6)$$
$$- 228\zeta(5)\zeta(11) - 213\zeta(6)\zeta(10) - 204\zeta(7)\zeta(9) - \tfrac{3217}{32}\zeta^2(8)$$
$$+ \tfrac{6435}{8}\zeta(16)$$

$$s_{92} = -\zeta(2)\zeta(9) - \zeta(3)\zeta(8) - \zeta(4)\zeta(7) - \zeta(5)\zeta(6) + 5\zeta(11)$$

$$s_{93} = \tfrac{1}{2}\zeta^2(2)\zeta(8) + \zeta(2)\zeta(3)\zeta(7) + \zeta(2)\zeta(4)\zeta(6) + \tfrac{1}{2}\zeta(2)\zeta^2(5)$$
$$- \tfrac{9}{2}\zeta(2)\zeta(10) + \tfrac{1}{2}\zeta^2(3)\zeta(6) + \zeta(3)\zeta(4)\zeta(5) - 5\zeta(3)\zeta(9) + \tfrac{1}{6}\zeta^3(4)$$
$$- 5\zeta(4)\zeta(8) - 5\zeta(5)\zeta(7) - \tfrac{5}{2}\zeta^2(6) + \tfrac{55}{3}\zeta(12)$$

$$s_{94} = -\tfrac{1}{6}\zeta^3(2)\zeta(7) - \tfrac{1}{2}\zeta^2(2)\zeta(3)\zeta(6) - \tfrac{1}{2}\zeta^2(2)\zeta(4)\zeta(5) + 2\zeta^2(2)\zeta(9)$$
$$- \tfrac{1}{2}\zeta(2)\zeta^2(3)\zeta(5) - \tfrac{1}{2}\zeta(2)\zeta(3)\zeta^2(4) + \tfrac{9}{2}\zeta(2)\zeta(3)\zeta(8)$$
$$+ \tfrac{9}{2}\zeta(2)\zeta(4)\zeta(7) + \tfrac{9}{2}\zeta(2)\zeta(5)\zeta(6) - 15\zeta(2)\zeta(11) - \tfrac{1}{6}\zeta^3(3)\zeta(4)$$
$$+ \tfrac{5}{2}\zeta^2(3)\zeta(7) + 5\zeta(3)\zeta(4)\zeta(6) + \tfrac{5}{2}\zeta(3)\zeta^2(5) - \tfrac{33}{2}\zeta(3)\zeta(10)$$
$$+ \tfrac{5}{2}\zeta^2(4)\zeta(5) - \tfrac{49}{3}\zeta(4)\zeta(9) - 16\zeta(5)\zeta(8) - \tfrac{95}{6}\zeta(6)\zeta(7) + 55\zeta(13)$$

$$
\begin{aligned}
s_{95} = {} & \tfrac{1}{24}\zeta^4(2)\zeta(6) + \tfrac{1}{6}\zeta^3(2)\zeta(3)\zeta(5) + \tfrac{1}{12}\zeta^3(2)\zeta^2(4) - \tfrac{7}{12}\zeta^3(2)\zeta(8) \\
& + \tfrac{1}{4}\zeta^2(2)\zeta^2(3)\zeta(4) - 2\zeta^2(2)\zeta(3)\zeta(7) - 2\zeta^2(2)\zeta(4)\zeta(6) - \zeta^2(2)\zeta^2(5) \\
& + 6\zeta^2(2)\zeta(10) + \tfrac{1}{24}\zeta(2)\zeta^4(3) - \tfrac{9}{4}\zeta(2)\zeta^2(3)\zeta(6) - \tfrac{9}{2}\zeta(2)\zeta(3)\zeta(4)\zeta(5) \\
& + \tfrac{40}{3}\zeta(2)\zeta(3)\zeta(9) - \tfrac{3}{4}\zeta(2)\zeta^3(4) + \tfrac{53}{4}\zeta(2)\zeta(4)\zeta(8) \\
& + 13\zeta(2)\zeta(5)\zeta(7) + \tfrac{155}{24}\zeta(2)\zeta^2(6) - \tfrac{165}{4}\zeta(2)\zeta(12) - \tfrac{5}{6}\zeta^3(3)\zeta(5) \\
& - \tfrac{5}{4}\zeta^2(3)\zeta^2(4) + \tfrac{15}{2}\zeta^2(3)\zeta(8) + 15\zeta(3)\zeta(4)\zeta(7) + \tfrac{89}{6}\zeta(3)\zeta(5)\zeta(6) \\
& - 45\zeta(3)\zeta(11) + \tfrac{181}{24}\zeta^2(4)\zeta(6) + \tfrac{15}{2}\zeta(4)\zeta^2(5) - \tfrac{87}{2}\zeta(4)\zeta(10) \\
& - \tfrac{125}{3}\zeta(5)\zeta(9) - \tfrac{485}{12}\zeta(6)\zeta(8) - 20\zeta^2(7) + 143\zeta(14)
\end{aligned}
$$

$$
\begin{aligned}
s_{96} = {} & -\tfrac{1}{120}\zeta^5(2)\zeta(5) - \tfrac{1}{24}\zeta^4(2)\zeta(3)\zeta(4) + \tfrac{1}{8}\zeta^4(2)\zeta(7) - \\
& - \tfrac{1}{36}\zeta^3(2)\zeta^3(3) + \tfrac{7}{12}\zeta^3(2)\zeta(3)\zeta(6) + \tfrac{7}{12}\zeta^3(2)\zeta(4)\zeta(5) \\
& - \tfrac{14}{9}\zeta^3(2)\zeta(9) + \zeta^2(2)\zeta^2(3)\zeta(5) + \zeta^2(2)\zeta(3)\zeta^2(4) - \tfrac{21}{4}\zeta^2(2)\zeta(3)\zeta(8) \\
& - \tfrac{21}{4}\zeta^2(2)\zeta(4)\zeta(7) - \tfrac{31}{6}\zeta^2(2)\zeta(5)\zeta(6) + 15\zeta^2(2)\zeta(11) \\
& + \tfrac{3}{4}\zeta(2)\zeta^3(3)\zeta(4) - 6\zeta(2)\zeta^2(3)\zeta(7) - \tfrac{145}{12}\zeta(2)\zeta(3)\zeta(4)\zeta(6) \\
& - 6\zeta(2)\zeta(3)\zeta^2(5) + 33\zeta(2)\zeta(3)\zeta(10) - \tfrac{49}{8}\zeta(2)\zeta^2(4)\zeta(5) \\
& + 32\zeta(2)\zeta(4)\zeta(9) + \tfrac{123}{4}\zeta(2)\zeta(5)\zeta(8) + 30\zeta(2)\zeta(6)\zeta(7) - 99\zeta(2)\zeta(13) \\
& + \tfrac{1}{24}\zeta^5(3) - \tfrac{83}{36}\zeta^3(3)\zeta(6) - 7\zeta^2(3)\zeta(4)\zeta(5) + \tfrac{55}{3}\zeta^2(3)\zeta(9) \\
& - \tfrac{19}{8}\zeta(3)\zeta^3(4) + 36\zeta(3)\zeta(4)\zeta(8) + 35\zeta(3)\zeta(5)\zeta(7) + \tfrac{415}{24}\zeta(3)\zeta^2(6) \\
& - \tfrac{429}{4}\zeta(3)\zeta(12) + \tfrac{143}{8}\zeta^2(4)\zeta(8) + \tfrac{211}{6}\zeta(4)\zeta(5)\zeta(6) - 102\zeta(4)\zeta(11) \\
& + \tfrac{35}{6}\zeta^3(5) - \tfrac{957}{10}\zeta(5)\zeta(10) - \tfrac{920}{9}\zeta(6)\zeta(9) - \tfrac{355}{4}\zeta(7)\zeta(8) \\
& + \tfrac{1001}{3}\zeta(15)
\end{aligned}
$$

$$
\begin{aligned}
s_{97} = {} & \tfrac{1}{720}\zeta^6(2)\zeta(4) + \tfrac{1}{240}\zeta^5(2)\zeta^2(3) - \tfrac{1}{48}\zeta^5(2)\zeta(6) - \tfrac{1}{8}\zeta^4(2)\zeta(3)\zeta(5) \\
& - \tfrac{1}{16}\zeta^4(2)\zeta^2(4) + \tfrac{7}{24}\zeta^4(2)\zeta(8) - \tfrac{7}{24}\zeta^3(2)\zeta^2(3)\zeta(4) \\
& + \tfrac{4}{3}\zeta^3(2)\zeta(3)\zeta(7) + \tfrac{97}{72}\zeta^3(2)\zeta(4)\zeta(6) + \tfrac{2}{3}\zeta^3(2)\zeta^2(5) \\
& - \tfrac{7}{2}\zeta^3(2)\zeta(10) - \tfrac{1}{12}\zeta^2(2)\zeta^4(3) + \tfrac{7}{3}\zeta^2(2)\zeta^2(3)\zeta(6) \\
& + \tfrac{19}{4}\zeta^2(2)\zeta(3)\zeta(4)\zeta(5) - \tfrac{35}{3}\zeta^2(2)\zeta(3)\zeta(9) + \tfrac{13}{16}\zeta^2(2)\zeta^3(4) \\
& - \tfrac{91}{8}\zeta^2(2)\zeta(4)\zeta(8) - 11\zeta^2(2)\zeta(5)\zeta(7) - \tfrac{65}{12}\zeta^2(2)\zeta^2(6) \\
& + 33\zeta^2(2)\zeta(12) + \tfrac{11}{6}\zeta(2)\zeta^3(3)\zeta(5) + \tfrac{45}{16}\zeta(2)\zeta^2(3)\zeta^2(4) \\
& - \tfrac{105}{8}\zeta(2)\zeta^2(3)\zeta(8) - 26\zeta(2)\zeta(3)\zeta(4)\zeta(7) - \tfrac{51}{2}\zeta(2)\zeta(3)\zeta(5)\zeta(6) \\
& + 72\zeta(2)\zeta(3)\zeta(11) - \tfrac{209}{16}\zeta(2)\zeta^2(4)\zeta(6) - 13\zeta(2)\zeta(4)\zeta^2(5) \\
& + \tfrac{687}{10}\zeta(2)\zeta(4)\zeta(10) + \tfrac{194}{3}\zeta(2)\zeta(5)\zeta(9) + \tfrac{1487}{24}\zeta(2)\zeta(6)\zeta(8)
\end{aligned}
$$

$$
\begin{aligned}
&= + \tfrac{61}{2}\,\zeta(2)\zeta^2(7) - \tfrac{429}{2}\,\zeta(2)\zeta(14) + \tfrac{13}{24}\,\zeta^4(3)\zeta(4) - 5\zeta^3(3)\zeta(7) \\
&\quad - \tfrac{181}{12}\,\zeta^2(3)\zeta(4)\zeta(6) - \tfrac{15}{2}\,\zeta^2(3)\zeta^2(5) + \tfrac{198}{5}\,\zeta^2(3)\zeta(10) \\
&\quad - \tfrac{123}{8}\,\zeta(3)\zeta^2(4)\zeta(5) + \tfrac{229}{3}\,\zeta(3)\zeta(4)\zeta(9) + \tfrac{291}{4}\,\zeta(3)\zeta(5)\zeta(8) \\
&\quad + \tfrac{212}{3}\,\zeta(3)\zeta(6)\zeta(7) - 231\zeta(3)\zeta(13) - \tfrac{21}{16}\,\zeta^4(4) - \tfrac{149}{4}\,\zeta^2(4)\zeta(8) \\
&\quad + 72\zeta(4)\zeta(5)\zeta(7) + \tfrac{2557}{72}\,\zeta(4)\zeta^2(6) - \tfrac{869}{4}\,\zeta(4)\zeta(12) \\
&\quad + \tfrac{106}{3}\,\zeta^2(5)\zeta(6) - 201\zeta(5)\zeta(11) - \tfrac{377}{2}\,\zeta(6)\zeta(10) - 181\zeta(7)\zeta(9) \\
&\quad - \tfrac{357}{4}\,\zeta^2(8) + 715\zeta(16)
\end{aligned}
$$

$$
\begin{aligned}
s_{98} &= -\tfrac{1}{5040}\,\zeta^7(2)\zeta(3) + \tfrac{1}{360}\,\zeta^6(2)\zeta(5) + \tfrac{1}{48}\,\zeta^5(2)\zeta(3)\zeta(4) \\
&\quad - \tfrac{1}{24}\,\zeta^5(2)\zeta(7) + \tfrac{1}{48}\,\zeta^4(2)\zeta^3(3) - \tfrac{35}{144}\,\zeta^4(2)\zeta(3)\zeta(6) \\
&\quad - \tfrac{1}{4}\,\zeta^4(2)\zeta(4)\zeta(5) + \tfrac{7}{12}\,\zeta^4(2)\zeta(9) - \tfrac{7}{12}\,\zeta^3(2)\zeta^2(3)\zeta(5) \\
&\quad - \tfrac{29}{48}\,\zeta^3(2)\zeta(3)\zeta^2(4) + \tfrac{21}{8}\,\zeta^3(2)\zeta(3)\zeta(8) + \tfrac{31}{12}\,\zeta^3(2)\zeta(4)\zeta(7) \\
&\quad + \tfrac{91}{36}\,\zeta^3(2)\zeta(5)\zeta(6) - 7\zeta^3(2)\zeta(11) - \tfrac{17}{24}\,\zeta^2(2)\zeta^3(3)\zeta(4) \\
&\quad + \tfrac{9}{2}\,\zeta^2(2)\zeta^2(3)\zeta(7) + \tfrac{217}{24}\,\zeta^2(2)\zeta(3)\zeta(4)\zeta(6) + \tfrac{9}{2}\,\zeta^2(2)\zeta(3)\zeta^2(5) \\
&\quad - \tfrac{231}{10}\,\zeta^2(2)\zeta(3)\zeta(10) + \tfrac{37}{8}\,\zeta^2(2)\zeta^2(4)\zeta(5) + \tfrac{133}{6}\,\zeta^2(2)\zeta(4)\zeta(9) \\
&\quad - 21\zeta^2(2)\zeta(5)\zeta(8) - \tfrac{61}{3}\,\zeta^2(2)\zeta(6)\zeta(7) + 66\zeta^2(2)\zeta(13) - \tfrac{1}{12}\,\zeta(2)\zeta^5(3) \\
&\quad + \tfrac{7}{2}\,\zeta(2)\zeta^3(3)\zeta(6) + \tfrac{43}{4}\,\zeta(2)\zeta^2(3)\zeta(4)\zeta(5) - \tfrac{77}{3}\,\zeta(2)\zeta^2(3)\zeta(9) \\
&\quad + \tfrac{59}{16}\,\zeta(2)\zeta(3)\zeta^3(4) - \tfrac{399}{8}\,\zeta(2)\zeta(3)\zeta(4)\zeta(8) - 48\zeta(2)\zeta(3)\zeta(5)\zeta(7) \\
&\quad - \tfrac{851}{36}\,\zeta(2)\zeta(3)\zeta^2(6) + 143\zeta(2)\zeta(3)\zeta(12) - \tfrac{197}{8}\,\zeta(2)\zeta^2(4)\zeta(7) \\
&\quad - \tfrac{193}{4}\,\zeta(2)\zeta(4)\zeta(5)\zeta(6) + 135\zeta(2)\zeta(4)\zeta(11) - 8\zeta(2)\zeta^3(5) \\
&\quad + \tfrac{627}{5}\,\zeta(2)\zeta(5)\zeta(10) + \tfrac{355}{3}\,\zeta(2)\zeta(6)\zeta(9) + \tfrac{459}{4}\,\zeta(2)\zeta(7)\zeta(8) \\
&\quad - 429\zeta(2)\zeta(15) + \tfrac{25}{24}\,\zeta^4(3)\zeta(5) + \tfrac{103}{48}\,\zeta^3(3)\zeta^2(4) - \tfrac{77}{8}\,\zeta^3(3)\zeta(8) \\
&\quad - \tfrac{57}{2}\,\zeta^2(3)\zeta(4)\zeta(7) - \tfrac{335}{12}\,\zeta^2(3)\zeta(5)\zeta(6) + 78\zeta^2(3)\zeta(11) \\
&\quad - \tfrac{1375}{48}\,\zeta(3)\zeta^2(4)\zeta(6) - \tfrac{57}{2}\,\zeta(3)\zeta(4)\zeta^2(5) + \tfrac{297}{2}\,\zeta(3)\zeta(4)\zeta(10) \\
&\quad + \tfrac{418}{3}\,\zeta(3)\zeta(5)\zeta(9) + \tfrac{1065}{8}\,\zeta(3)\zeta(6)\zeta(8) + \tfrac{131}{2}\,\zeta(3)\zeta^2(7) \\
&\quad - \tfrac{6435}{14}\,\zeta(3)\zeta(14) - \tfrac{39}{4}\,\zeta^3(4)\zeta(5) + \tfrac{857}{12}\,\zeta^2(4)\zeta(9) \\
&\quad + \tfrac{543}{4}\,\zeta(4)\zeta(5)\zeta(8) + \tfrac{395}{3}\,\zeta(4)\zeta(6)\zeta(7) - 429\zeta(4)\zeta(13) \\
&\quad + \tfrac{131}{2}\,\zeta^2(5)\zeta(7) + \tfrac{4649}{72}\,\zeta(5)\zeta^2(6) - \tfrac{1573}{4}\,\zeta(5)\zeta(12) \\
&\quad - 365\zeta(6)\zeta(11) - \tfrac{693}{2}\,\zeta(7)\zeta(10) - \tfrac{675}{2}\,\zeta(8)\zeta(9) + 1430\zeta(17)
\end{aligned}
$$

$$
\begin{aligned}
s_{99} = {} & \tfrac{1}{362880}\,\zeta^9(2) - \tfrac{1}{3360}\,\zeta^7(2)\zeta(4) - \tfrac{1}{720}\,\zeta^6(2)\zeta^2(3) + \tfrac{1}{216}\,\zeta^6(2)\zeta(6) \\
& + \tfrac{1}{30}\,\zeta^5(2)\zeta(3)\zeta(5) + \tfrac{17}{960}\,\zeta^5(2)\zeta^2(4) - \tfrac{7}{96}\,\zeta^5(2)\zeta(8) \\
& + \tfrac{5}{48}\,\zeta^4(2)\zeta^2(3)\zeta(4) - \tfrac{5}{12}\,\zeta^4(2)\zeta(3)\zeta(7) - \tfrac{5}{12}\,\zeta^4(2)\zeta(4)\zeta(6) \\
& - \tfrac{5}{24}\,\zeta^4(2)\zeta^2(5) + \tfrac{21}{20}\,\zeta^4(2)\zeta(10) + \tfrac{1}{24}\,\zeta^3(2)\zeta^4(3) \\
& - \tfrac{35}{36}\,\zeta^3(2)\zeta^2(3)\zeta(6) - 2\zeta^3(2)\zeta(3)\zeta(4)\zeta(5) + \tfrac{14}{3}\,\zeta^3(2)\zeta(3)\zeta(9) \\
& - \tfrac{11}{32}\,\zeta^3(2)\zeta^3(4) + \tfrac{217}{48}\,\zeta^3(2)\zeta(4)\zeta(8) + \tfrac{13}{3}\,\zeta^3(2)\zeta(5)\zeta(7) \\
& + \tfrac{461}{216}\,\zeta^3(2)\zeta^2(6) - \tfrac{77}{6}\,\zeta^3(2)\zeta(12) - \tfrac{7}{6}\,\zeta^2(2)\zeta^3(3)\zeta(5) \\
& - \tfrac{29}{16}\,\zeta^2(2)\zeta^2(3)\zeta^2(4) + \tfrac{63}{8}\,\zeta^2(2)\zeta^2(3)\zeta(8) + \tfrac{31}{2}\,\zeta^2(2)\zeta(3)\zeta(4)\zeta(7) \\
& + \tfrac{91}{6}\,\zeta^2(2)\zeta(3)\zeta(5)\zeta(6) - 42\zeta^2(2)\zeta(3)\zeta(11) + \tfrac{187}{24}\,\zeta^2(2)\zeta^2(4)\zeta(6) \\
& - \tfrac{31}{4}\,\zeta^2(2)\zeta(4)\zeta^2(5) - \tfrac{399}{10}\,\zeta^2(2)\zeta(4)\zeta(10) - \tfrac{112}{3}\,\zeta^2(2)\zeta(5)\zeta(9) \\
& - \tfrac{427}{12}\,\zeta^2(2)\zeta(6)\zeta(8) - \tfrac{35}{2}\,\zeta^2(2)\zeta^2(7) + \tfrac{858}{7}\,\zeta^2(2)\zeta(14) \\
& - \tfrac{17}{24}\,\zeta(2)\zeta^4(3)\zeta(4) + 6\zeta(2)\zeta^3(3)\zeta(7) + \tfrac{217}{12}\,\zeta(2)\zeta^2(3)\zeta(4)\zeta(6) \\
& + 9\zeta(2)\zeta^2(3)\zeta^2(5) - \tfrac{231}{5}\,\zeta(2)\zeta^2(3)\zeta(10) + \tfrac{37}{2}\,\zeta(2)\zeta(3)\zeta^2(4)\zeta(5) \\
& - \tfrac{266}{3}\,\zeta(2)\zeta(3)\zeta(4)\zeta(9) - 84\zeta(2)\zeta(3)\zeta(5)\zeta(8) - \tfrac{244}{3}\,\zeta(2)\zeta(3)\zeta(6)\zeta(7) \\
& + 264\zeta(2)\zeta(3)\zeta(13) + \tfrac{203}{128}\,\zeta(2)\zeta^4(4) - \tfrac{1379}{32}\,\zeta(2)\zeta^2(4)\zeta(8) \\
& - 83\zeta(2)\zeta(4)\zeta(5)\zeta(7) - \tfrac{327}{8}\,\zeta(2)\zeta(4)\zeta^2(6) + \tfrac{495}{2}\,\zeta(2)\zeta(4)\zeta(12) \\
& - \tfrac{122}{3}\,\zeta(2)\zeta^2(5)\zeta(6) + 228\zeta(2)\zeta(5)\zeta(11) + 213\zeta(2)\zeta(6)\zeta(10) \\
& + 204\zeta(2)\zeta(7)\zeta(9) + \tfrac{3217}{32}\,\zeta(2)\zeta^2(8) - \tfrac{6435}{8}\,\zeta(2)\zeta(16) - \tfrac{1}{36}\,\zeta^6(3) \\
& - \tfrac{7}{4}\,\zeta^4(3)\zeta(6) + \tfrac{43}{6}\,\zeta^3(3)\zeta(4)\zeta(5) - \tfrac{154}{9}\,\zeta^3(3)\zeta(9) \\
& + \tfrac{59}{16}\,\zeta^2(3)\zeta^3(4) - \tfrac{399}{8}\,\zeta^2(3)\zeta(4)\zeta(8) - 48\zeta^2(3)\zeta(5)\zeta(7) \\
& - \tfrac{851}{36}\,\zeta^2(3)\zeta^2(6) + 143\zeta^2(3)\zeta(12) - \tfrac{197}{4}\,\zeta(3)\zeta^2(4)\zeta(7) \\
& - \tfrac{193}{2}\,\zeta(3)\zeta(4)\zeta(5)\zeta(6) + 270\zeta(3)\zeta(4)\zeta(11) - 16\zeta(3)\zeta^3(5) \\
& + \tfrac{1254}{5}\,\zeta(3)\zeta(5)\zeta(10) + \tfrac{710}{3}\,\zeta(3)\zeta(6)\zeta(9) + \tfrac{459}{2}\,\zeta(3)\zeta(7)\zeta(8) \\
& - 858\zeta(3)\zeta(15) - \tfrac{33}{2}\,\zeta^3(4)\zeta(6) - \tfrac{197}{8}\,\zeta^2(4)\zeta^2(5) + \tfrac{2571}{20}\,\zeta^2(4)\zeta(10) \\
& + \tfrac{724}{3}\,\zeta(4)\zeta(5)\zeta(9) + \tfrac{692}{3}\,\zeta(4)\zeta(6)\zeta(8) + \tfrac{227}{2}\,\zeta(4)\zeta^2(7) \\
& - \tfrac{5577}{7}\,\zeta(4)\zeta(14) + \tfrac{459}{4}\,\zeta^2(5)\zeta(8) + \tfrac{668}{3}\,\zeta(5)\zeta(6)\zeta(7) \\
& - 726\zeta(5)\zeta(13) + \tfrac{11855}{324}\,\zeta^3(6) - \tfrac{4015}{6}\,\zeta(6)\zeta(12) - 630\zeta(7)\zeta(11) \\
& - \tfrac{1215}{2}\,\zeta(8)\zeta(10) - \tfrac{2701}{9}\,\zeta^2(9) + \tfrac{24310}{9}\,\zeta(18)
\end{aligned}
$$

$r_{11} = -\frac{1}{6}\pi^2$

$r_{21} = \zeta(3)$

$r_{22} = -\frac{1}{180}\pi^4$

$r_{31} = -\frac{1}{45}\pi^4$

$r_{32} = -\frac{2}{3}\pi^2\zeta(3) + 8\zeta(5)$

$r_{33} = -\frac{23}{1260}\pi^6 + 12\zeta^2(3)$

$r_{41} = 6\zeta(5)$

$r_{42} = -\frac{1}{105}\pi^6 + 6\zeta^2(3)$

$r_{43} = -\frac{1}{2}\pi^4\zeta(3) - 12\pi^2\zeta(5) + 180\zeta(7)$

$r_{44} = -\frac{499}{12600}\pi^8 - 24\pi^2\zeta^2(3) + 576\zeta(3)\zeta(5)$

$r_{51} = -\frac{8}{315}\pi^6$

$r_{52} = -\frac{8}{15}\pi^4\zeta(3) - 8\pi^2\zeta(5) + 144\zeta(7)$

$r_{53} = -\frac{61}{1575}\pi^8 - 12\pi^2\zeta^2(3) + 432\zeta(3)\zeta(5)$

$r_{54} = -\frac{4}{3}\pi^6\zeta(3) - \frac{112}{5}\pi^4\zeta(5) - 480\pi^2\zeta(7) + 288\zeta^3(3) + 8064\zeta(9)$

$r_{55} = -\frac{149}{660}\pi^{10} - 40\pi^4\zeta^2(3) - 1920\pi^2\zeta(3)\zeta(5) + 28800\zeta(3)\zeta(7) + 14400\zeta^2(5)$

$r_{61} = 120\zeta(7)$

$r_{62} = -\frac{2}{63}\pi^8 + 240\zeta(3)\zeta(5)$

$r_{63} = -\frac{4}{3}\pi^6\zeta(3) - 18\pi^4\zeta(5) - 360\pi^2\zeta(7) + 120\zeta^3(3) + 6720\zeta(9)$

$r_{64} = -\frac{278}{1485}\pi^{10} - 36\pi^4\zeta^2(3) - 1440\pi^2\zeta(3)\zeta(5) + 23040\zeta(3)\zeta(7) + 11520\zeta^2(5)$

$r_{65} = -\frac{47}{6}\pi^8\zeta(3) - \frac{280}{3}\pi^6\zeta(5) - 1480\pi^4\zeta(7) - 1200\pi^2\zeta^3(3) - 33600\pi^2\zeta(9)$
$\quad + 50400\zeta^2(3)\zeta(5) + 604800\zeta(11)$

$r_{66} = -\frac{4714153}{2522520}\pi^{12} - 200\pi^6\zeta^2(3) - 6720\pi^4\zeta(3)\zeta(5) - 144000\pi^2\zeta(3)\zeta(7)$
$\quad - 72000\pi^2\zeta^2(5) + 21600\zeta^4(3) + 2419200\zeta(3)\zeta(9) + 2246400\zeta(5)\zeta(7)$

$r_{71} = -\frac{8}{105}\pi^8$

$r_{72} = -\frac{32}{21}\pi^6\zeta(3) - 16\pi^4\zeta(5) - 240\pi^2\zeta(7) + 5760\zeta(9)$

$$r_{73} = -{}^{74}/_{385}\,\pi^{10} - 24\pi^4\zeta^2(3) - 720\pi^2\zeta(3)\zeta(5) + 17280\zeta(3)\zeta(7) + 8640\zeta^2(5)$$

$$r_{74} = -{}^{104}/_{15}\,\pi^8\zeta(3) - {}^{632}/_{7}\,\pi^6\zeta(5) - 1296\pi^4\zeta(7) - 480\pi^2\zeta^3(3) - 26880\pi^2\zeta(9) + 34560\zeta^2(3)\zeta(5) + 518400\zeta(11)$$

$$r_{75} = -{}^{1048079}/_{630630}\,\pi^{12} - 180\pi^6\zeta^2(3) - 5520\pi^4\zeta(3)\zeta(5) - 115200\pi^2\zeta(3)\zeta(7) - 57600\pi^2\zeta^2(5) + 14400\zeta^4(3) + 2016000\zeta(3)\zeta(9) + 1900800\zeta(5)\zeta(7)$$

$$r_{76} = -{}^{223}/_{3}\,\pi^{10}\zeta(3) - 772\pi^8\zeta(5) - {}^{66000}/_{7}\,\pi^6\zeta(7) - 4560\pi^4\zeta^3(3) - 154560\pi^4\zeta(9) - 302400\pi^2\zeta^2(3)\zeta(5) - 3628800\pi^2\zeta(11) + 4665600\zeta^2(3)\zeta(7) + 4665600\zeta(3)\zeta^2(5) + 68428800\zeta(13)$$

$$r_{77} = -{}^{5101555}/_{216216}\,\pi^{14} - 1974\pi^8\zeta^2(3) - 47040\pi^6\zeta(3)\zeta(5) - 745920\pi^4\zeta(3)\zeta(7) - 372960\pi^4\zeta^2(5) - 151200\pi^2\zeta^4(3) - 16934400\pi^2\zeta(3)\zeta(9) - 15724800\pi^2\zeta(5)\zeta(7) + 8467200\zeta^3(3)\zeta(5) + 304819200\zeta(3)\zeta(11) + 270950400\zeta(5)\zeta(9) + 127008000\zeta^2(7)$$

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