



DISSERTATION

Studie des seltenen Zerfalls

$$K_L^0 \rightarrow \pi^0 \pi^\pm e^\mp \nu_e \text{ (Ke4)}$$

am CERN Experiment NA48

(in englischer Sprache)

ausgeführt zum Zwecke der Erlangung des akademischen
Grades eines Doktors der technischen Wissenschaften
unter der Leitung von

tit.ao.Univ.Prof. Dr. Manfred Markytan
E 136

Institut für Theoretische Physik

eingereicht an der Technischen Universität Wien
an der Fakultät für technische Naturwissenschaften und
Informatik

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Genf, am 13. September 2001

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Deutsche Kurzfassung

Die Kaonphysik ist eng verknüpft mit einigen der überraschendsten und interessantesten Entdeckungen der Elementarteilchenphysik im letzten halben Jahrhundert; seit man die fundamentale Bedeutung der Symmetrien und der damit verbundenen Erhaltungssätze erkannt hatte, fand man auch immer wieder Verletzungen dieser Symmetrien. Zunächst war es die Verletzung der Parität (1956), die man durch genaue Untersuchungen des radioaktiven Betazerfalls nachweisen konnte [2]. Immerhin schien trotzdem die kombinierte Symmetrie CP (Parität und Ladungskonjugation) weiter gültig zu sein, bis man ein paar Jahre später (1964) in einem Aufsehen erregenden Experiment den CP-verletzenden Zerfall eines langlebigen neutralen Kaons in zwei (statt normalerweise durch die Symmetrie diktierte drei) Pionen beobachtete [1].

Nachdem die Existenz der CP-Verletzung unzweifelhaft feststand gab es eine ganze Reihe von Erklärungsmodellen. Ein wesentlicher Unterschied zwischen diesen Modellen war ihre Aussage über die Existenz der sogenannten direkten CP-Verletzung, bei der die Symmetrie explizit beim Zerfallsprozess gebrochen wird. Während das superschwache Modell eindeutig keine direkte CP-Verletzung zulässt, ist eine solche im Rahmen des Standardmodells möglich solange es mindestens drei Generationen von Teilchen gibt.

Um die Frage nach der Existenz direkter CP-Verletzung experimentell zu klären gab es bereits vor gut 10 Jahren eine erste Runde von Experimenten - NA31 am CERN und E731 am Fermilab - die jedoch aufgrund des äußerst anspruchsvollen Experiments keine eindeutige Antwort geben konnten¹. Die Erfahrungen mit diesen Experimenten konnten jedoch bei der Planung einer zweiten Runde von Experimenten eingebracht werden, wobei KTeV am Fermilab und NA48 am CERN jedoch unterschiedlichen Philosophien folgten. Beide Experimente hatten die Messung der direkten CP-Verletzung als ihr Hauptziel, parallel dazu wurden aber auch eine ganze Reihe von seltenen Kaonzerfällen untersucht.

Die vorliegende Arbeit beruht auf den Daten, die im Rahmen des NA48 Experiments im Sommer des Jahres 1999 genommen wurden. Ihr Ziel war der Nachweis und die Analyse eines seltenen semileptonischen Kaonzerfalls,

¹es ist auch anzumerken, dass zur damaligen Zeit die theoretischen Vorhersagen für die direkte CP-Verletzung um eine Größenordnung höher lagen, was sich jedoch nicht bestätigte

der unter der Bezeichnung "Ke4" bekannt ist. Darunter versteht man den Zerfall $K_L \rightarrow \pi^0 \pi^\pm e^\mp \nu_e$, der sich vom häufigsten K_L Zerfall Ke3 nur durch ein zusätzliches neutrales Pion unterscheidet, aber alleine durch den stark eingeschränkten Phasenraum etwa 8000 mal seltener ist.

Während der geladene Ke4 Zerfall (gemeint ist der entsprechende Zerfall des geladenen Kaons) experimentell bereits mit mehreren hunderttausend Ereignissen beobachtet wurde [22], waren es beim neutralen Ke4 Zerfall bisher weltweit nur etwas mehr als 700 Zerfälle [32]. Dabei wartet der neutrale Ke4 Zerfall im Vergleich zu seinem geladenen Gegenpart mit einer interessanten Eigenschaft auf: wie auch in dieser Arbeit näher erläutert wird verschwindet aus Symmetriegründen der führende Term des F-Formfaktors, sodass sich die Pionen in einem praktisch reinem p-Zustand befinden. Dadurch wird nicht nur die Untersuchung ansonst überdeckter Beiträge in den Formfaktoren, sondern auch die Messung des L_3 -Kopplungsparameters der chiralen Störungstheorie ohne störende andere Beiträge möglich.

Diese Arbeit unterteilt sich grob in drei Teile. Im ersten, theoretischen Teil wird nach einer kurzen Diskussion der Phänomenologie des neutralen Kaonsystems die Theorie des Ke4-Zerfalls ausführlich behandelt. Hier werden die Formfaktoren zunächst modell-unabhängig eingeführt und diskutiert, danach im Rahmen der chiralen Störungstheorie berechnet. Es werden auch Methoden ausgearbeitet, mit denen eine experimentelle Untersuchung der Formfaktoren erfolgen kann.

Im zweiten Teil wird das NA48 Experiment mit seinen Detektoren vorgestellt und erläutert, auf welche Weise der Ke4-Zerfall getriggert werden kann. Auch die Messung der Effizienz dieses Triggers wird hier beschrieben.

Der dritte Teil präsentiert die eigentliche Analyse: die Rekonstruktion des Ke4-Zerfalls, die Selektion der Ereignisse, die Bestimmung der Formfaktoren und schließlich die Messung des Verzweigungsverhältnisses (BR). Weitere Kapitel befassen sich mit dem Untergrund durch andere Zerfälle sowie der Akzeptanzkorrektur.

Im Anhang finden sich neben einer Formel- und Konstantensammlung längere Rechnungen allgemeinerer Art, sowie ausführliche Diskussionen einiger im Laufe dieser Arbeit auftretender Probleme wie etwa die Kinematik eines 4-Teilchen-Zerfalls und die Behandlung der zwei physikalischen Lösungen für den Impuls des undetektierten Neutrinos.

DISSERTATION

A Study of the Rare Decay $K_L^0 \rightarrow \pi^0 \pi^\pm e^\mp \nu_e$ (Ke4) at the CERN CP-Violation Experiment NA48

A Thesis

Presented to the Faculty of Science and Informatics
Vienna University of Technology

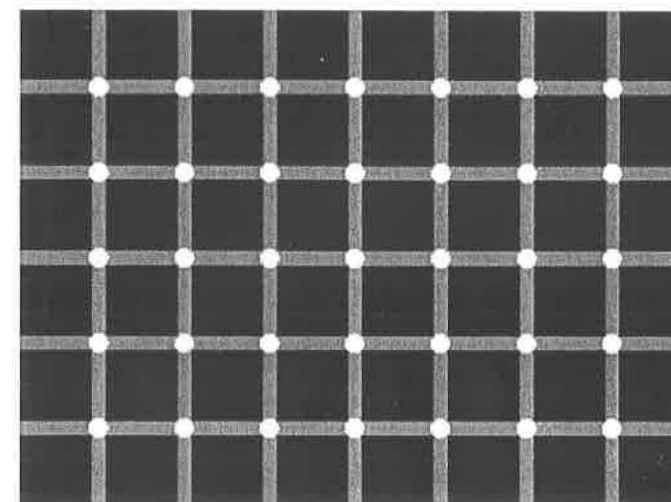
Under the Supervision of
Prof. Dr. Manfred Markytan
Institute for Theoretical Physics

In Partial Fulfillment
Of the Requirements for the Degree
Doctor of Technical Sciences

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September 2001



black dots

If the basic structure of energy and all of its properties could be accurately discerned including its waveform and frequency which as I stated may play a key role in gravity by causing the space (energy medium to warp) then it may be possible to produce antigravity effects by taking this waveform and shifting it out of phase by 180 degrees. Perhaps this is already being done.

[C.N. Vind, posting in alt.alien.visitors]

Le bon sens est la chose du monde la mieux partagée: car chacun pense en être si bien pourvu, que ceux même qui sont les plus difficiles à contenter en toute autre chose, n'ont point coutume d'en désirer plus qu'ils en ont.

[René Descartes, Discours de la Méthode]

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Preface

Kaon physics are closely related to some of the most surprising and interesting discoveries of high energy physics in the last half century; since physicists realised the fundamental importance of symmetries and the corresponding conservation laws, they also found violations of these symmetries. The first violation known was that of parity P (1956), seen in subtle studies of the radio-active beta decay [2]. However, the combined symmetry CP (parity and charge conjugation) seemed to hold still, until only a few years later (1964) the surprising result of a kaon experiment gave evidence to CP -violation as well [1].

As soon as the existence of CP -violation was out of doubt, there were a couple of different theoretical models. A major difference between those was their prediction with respect of so-called direct CP -violation, where CP is explicitly broken during the particle decay (whereas indirect CP -violation arises from a state-mixing in the original kaon). The super-weak model [3] excludes direct CP -violation, while the standard models requires at least three generations of particles in order to allow for direct CP -violation.

To settle the open question of direct CP -violation, there was already one first round of experiments about a decade ago - NA31 at CERN and E731 at Fermilab. Due to the high difficulty of the experiments, though, there was no clear evidence for direct CP -violation from those experiments². However, the experiences achieved were used in the planning and design of a second round of experiments, where NA48 at CERN followed a somewhat different philosophy than KTeV at Fermilab. Both experiments have their primary goal in measuring direct CP -violation, but in addition also a long list of rare kaon decays are studied.

This thesis is based on data which was taken with the NA48 experiment at CERN in summer 1999. Its goal was to find evidence for and analyse the rare semi-leptonic neutral kaon decay known as Ke_4 . This is short for the decay $K_L \rightarrow \pi^0 \pi^\pm e^\mp \nu_e$ which differs from the most common K_L decay Ke_3 only by an additional neutral pion, but is already by limited phase-space roughly 8000 times more rare.

²is has to be noted that theoretical predictions for direct CP -violation have been one order of magnitude large at that time - this prediction was falsified by the two experiments

While the charged K_4 decay (the corresponding decay of a charged kaon) is experimentally well known with 400k of events [22], for neutral K_4 only around 700 decays have been observed [32]. Nevertheless the neutral K_4 decay in comparison with its charged counterpart has an interesting property: as explained later in this work, the leading term of the F form factor vanishes for symmetry reasons. The consequence is that the pions are in a (practically) pure p-wave state, allowing not only to study otherwise covered form factors, but also the undisturbed measurement of the L_3 coupling parameter of chiral perturbation theory.

This thesis is divided into three distinct parts: the first, purely theoretical part gives a short discussion of neutral kaon phenomenology and then covers the theory of K_4 decay in detail. The form factors are studied in a model-independent way as far as possible, and then calculated within the framework of chiral perturbation theory. Additionally, also methods are presented to experimentally access and study the form factors to check the theoretical results.

The second part explains the NA48 experiment, its detector, and its trigger systems. The special trigger used for K_4 is discussed and its efficiency measured.

Finally the third part contains the reconstruction and selection of K_4 decays, acceptance and background studies, the analysis of form factors and finally the determination of the K_4 branching ratio.

The appendix is the place for lengthy calculations of general nature, as well as detailed discussions of some problems like the kinematics of a 4-body decay and the treatment of the two physical solutions for the (undetectable) neutrino. It also contains a list of formulae and constants.

Part I

Theory



saxophone player

Chapter 1

Phenomenology of the Neutral Kaon System

This chapter deals with general properties of the neutral kaon system - the $K_0 - \bar{K}_0$ -transitions and their consequence: the weak eigenstates K_L and K_S with significantly different life times. The connection with CP-symmetry is discussed, and then the effects of (small) CP-violation are studied. Finally, the distinction between indirect and direct CP-violation is made.

1.1 The Weak Eigenstates of the Neutral Kaon

The eigenstates of the neutral kaon in respect of strong and electromagnetic interaction are given by

$$|K_0\rangle = |d\bar{s}\rangle \quad |\bar{K}_0\rangle = |\bar{d}s\rangle \quad (1.1)$$

In absence of weak interaction strangeness is conserved, and since there are no lighter particles with $S \neq 0$, the above particles would be stable and distinctively observable. Weak interaction however can carry the s-quark over to a u-quark governed by the element V_{us} of the CKM-matrix (see fig. 1.1).

Similar transitions exist for c and t quarks. Since K_0 and $\bar{K}_0$ are neutral, the combination of quark-transitions like fig. 1.1 leads to a second order weak transition between K_0 and $\bar{K}_0$ states through Feynman graphs like that in fig. 1.2. Note that inside the loop may also appear c and t quarks - a fact that is crucial for the occurrence of CP violation discussed later in this chapter.

For studying such transitions we consider an effective Hamiltonian H which includes the weak interaction and acts on the strong eigenstates $|K_0\rangle$ and $|\bar{K}_0\rangle$. These states form an orthogonal basis of a 2-dimensional Hilbert

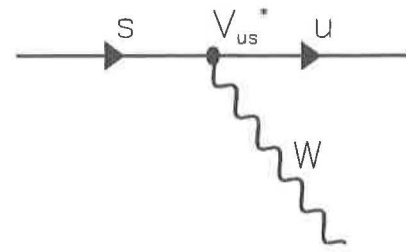
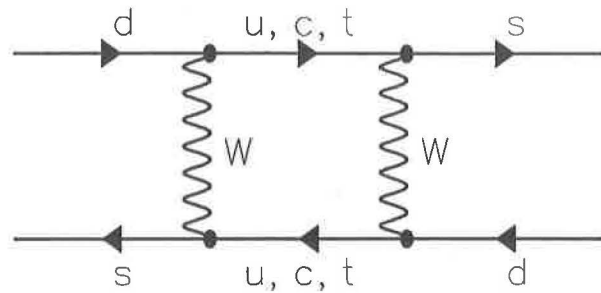


Figure 1.1: weak transition of a s-quark to a u-quark

Figure 1.2: example Feynman graph for the transition $K_0 \leftrightarrow \bar{K}_0$ via weak interaction ($\Delta S = 2$); the vertices carry the corresponding elements of the CKM-matrix

space; since we are only interested in transitions between these eigenstates, we restrict ourselves to this subspace of the whole Hilbert space, where H can be written as a 2×2 -Matrix¹:

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \quad |K_0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |\bar{K}_0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (1.2)$$

Of course it is generally not guaranteed that states in this subspace stay there for $t \rightarrow \infty$ - physically speaking, the kaons might decay to other particles like pions, which are not part of this subspace. As a consequence, the matrix H generally is *not* hermitian. However, it can always be split in a hermitian and anti-hermitian part:

$$H = M - \frac{i}{2}\Gamma \quad (1.3)$$

The matrices M and Γ are hermitian by definition:

¹for a more detailed discussion and proper definition of H see e.g. [4] or [5]

$$\begin{aligned} M_{11} &= M_{11}^* & M_{22} &= M_{22}^* & M_{12} &= M_{21}^* \\ \Gamma_{11} &= \Gamma_{11}^* & \Gamma_{22} &= \Gamma_{22}^* & \Gamma_{12} &= \Gamma_{21}^* \end{aligned} \quad (1.4)$$

Solving the Schrödinger equation $i\frac{d}{dt}|\Psi\rangle = H|\Psi\rangle$ one finds the solution

$$|\Psi_a(t)\rangle = e^{-i(M_a - \frac{i}{2}\Gamma_a)t} |\Psi_a\rangle \quad (1.5)$$

with the (real) eigenvalues M_a, Γ_a and the eigenstates $|\Psi_a\rangle$. If one now calculates the probability P_a to find the initial ($t = 0$) state $|\Psi_a\rangle$ at some time t , one finds

$$P_a = |\langle\Psi_a(0)|\Psi_a(t)\rangle|^2 = e^{-\Gamma_a t} \quad (1.6)$$

which gives rise to the interpretation of Γ as *decay width*, while M is the *mass* of the eigenstate (H shall be given in the rest frame of the kaon).

To discuss the properties of the eigenvalues and -states of H , it is not necessary to compute the matrix elements H_{ij} (which would not be straightforward, because nonperturbative strong interactions had to be taken into account): fortunately, some fundamental symmetries alone are sufficient to get some interesting properties of the solutions.

First, CPT-Symmetry² gives [4]

$$M_{11} = M_{22} \quad \Gamma_{11} = \Gamma_{22} \quad (1.7)$$

Since the CPT-theorem guarantees CPT-Symmetry under very general assumptions on the theoretical side, and also holds very well experimentally, (1.7) can be regarded as exact. CP-symmetry, though, can be broken - in the standard model, for instance, CP-violation enters via a phase in the CKM-matrix³. However, CP-violation turns out to be small experimentally, so it is worthwhile to study the limit of exact CP-conservation; in this case, one finds

$$M_{12} = M_{21} \quad \Gamma_{12} = \Gamma_{21} \quad (1.8)$$

if one defines the relative phases of $|K_0\rangle$ and $|\bar{K}_0\rangle$ accordingly⁴. Together with (1.4) this means that M_{12} and Γ_{12} are real.

For the following, exact CP-conservation is assumed; the consequences of a (small) CP-violation are studied in the next section. Exact CP-Symmetry now leads to the matrix

²symmetry under the combined transformations Charge conjugation, Parity and Time reversal

³it is interesting to remark that this requires at least 3 generations of particles; without the b- and t-quark, there could be no CP-violation within the standard model

⁴ $CP|K_0\rangle = |\bar{K}_0\rangle$ and $CP|\bar{K}_0\rangle = |K_0\rangle$; this phase convention is not necessary, but simplifies the calculations below; the general calculation can be found e.g. in [35]

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{11} \end{pmatrix} \quad H_{ij} = M_{ij} - \frac{i}{2}\Gamma_{ij} \quad (1.9)$$

with the eigenvalues

$$H_{1,2} = H_{11} \pm H_{12} \quad (1.10)$$

and the normalized eigenstates

$$|K_{1,2}\rangle = \frac{1}{\sqrt{2}}(|K_0\rangle \pm |\bar{K}_0\rangle) \quad (1.11)$$

These states are not only eigenstates to H , but also of CP :

$$CP|K_{1,2}\rangle = \pm|K_{1,2}\rangle \quad (1.12)$$

The masses of the two kaon states are given by⁵

$$M_{1,2} = M \mp \Delta \quad M = M_{11}, \Delta = -M_{12} \quad (1.13)$$

As the element M_{12} is related to the energy of the weak transition $|K_0\rangle \leftrightarrow |\bar{K}_0\rangle$ while M_{11} contains the binding energy of the strong interaction between the constituent quarks, it can be expected that $\Delta \ll M$. Indeed, experimentally one finds [34]

$$M = 497.672 \pm 0.031 \text{ MeV} \quad \Delta = (1.745 \pm 0.004) \times 10^{-12} \text{ MeV} \quad (1.14)$$

The situation is different for $\Gamma_{1,2}$: any decay involves a transition $s \rightarrow u$, so $|\Gamma_{12}|$ need not be much smaller than $|\Gamma_{11}|$. Actually, in 2nd order perturbation theory it follows [5]⁶ that

$$\Gamma_{11} \approx \Gamma_{12} \quad (1.15)$$

This leads to

$$\Gamma_1 \approx 2\Gamma_{11} \quad \Gamma_2 \approx 0 \quad (1.16)$$

which means that the CP-odd state $|K_2\rangle$ is almost stable in comparison with the CP-even state $|K_1\rangle$, at least has a significantly longer mean lifetime. The physical explanation for this is that for the primary decay mode $K \rightarrow \pi\pi$, the $\pi\pi$ is CP-even, and therefore forbidden for the CP-odd kaon state $|K_2\rangle$ under our current assumption of exact CP-conservation.

Experimentally, one finds [34]

⁵considering the experimental fact that K_2 is heavier than K_1 and therefore $M_{12} < 0$

⁶mind the different phase convention in [5]

$$\tau_1 = \Gamma_1^{-1} = (0.8935 \pm 0.0008) \times 10^{-10} \text{ s} \quad \tau_2 = \Gamma_2^{-1} = (5.17 \pm 0.04) \times 10^{-8} \text{ s} \quad (1.17)$$

To conclude: under exact CP-symmetry one has two weak kaon eigenstates mixed from the strong eigenstates like

$$|K_{1,2}\rangle = \frac{1}{\sqrt{2}}(|K_0\rangle \pm |\bar{K}_0\rangle) \quad (1.18)$$

with almost the same masses, but quite different lifetimes.

1.2 CP-Violation in the K_0 -System

Without CP-symmetry, we do not have (1.8), but still have (1.4) and (1.7). The main consequence of this is that – with our phase convention – M_{12} and Γ_{12} are no longer real, but get a phase. The CP-conserving matrix (1.9) has to be generalized to

$$H = \begin{pmatrix} H_{11} & H_{12}(1+2\epsilon) \\ H_{12}(1-2\epsilon) & H_{11} \end{pmatrix} \quad (1.19)$$

where H_{12} contains only the real parts of M_{12} and Γ_{12} , while ϵ collects both phases like this⁷:

$$\epsilon := \frac{\frac{1}{2}Im\Gamma_{12} + iImM_{12}}{2ReM_{12} - \frac{i}{2}Re\Gamma_{12}} \quad (1.20)$$

Assuming a *small* CP-violation, ϵ is also small, which motivates the form (1.19). Neglecting all terms $O(\epsilon^2)$, this CP-violating matrix gives the same eigenvalues⁸

$$H_{S,L} = H_{11} \pm H_{12} \quad (1.21)$$

and therefore also the same masses and lifetimes. However, the eigenstates change – at $O(\epsilon)$ – to

$$|K_{S,L}\rangle = \frac{1}{\sqrt{2}}((1+\epsilon)|K_0\rangle \pm (1-\epsilon)|\bar{K}_0\rangle) \quad (1.22)$$

which of course can also be expressed in $|K_1\rangle$ and $|K_2\rangle$:

⁷this definition of ϵ , especially with the factor $\frac{1}{2}$ will turn out to be exactly the standard ϵ -parameter of CP-violation

⁸to distinguish from the exact CP-conserving solutions, we now use the indices S, L instead of 1, 2 for the two eigenstates as the CP-conserving one, where S stands for *short-lived* and L for *long-lived*

$$|K_S\rangle = |K_1\rangle + \epsilon|K_2\rangle \quad (1.23)$$

$$|K_L\rangle = |K_2\rangle + \epsilon|K_1\rangle \quad (1.24)$$

This can be interpreted as such: in the presence of *small* CP-violation, the kaon eigenstates are almost CP-eigenstates, but with a small admixture of the opposite CP-eigenstate of the order ϵ . Masses and lifetimes are not affected in first order.

1.3 Indirect and Direct CP-Violation

Looking at the decay modes of $K_{L,S}$, one has to distinguish two different kinds of CP-violation: the first one, called *indirect* CP-violation, happens when the small admixture of opposite CP decays in a CP-conserving way. This can also be interpreted as a combination of a so-called *box-graph* like fig. 1.2 with a subsequent CP-conserving decay graph.

The second possibility is a *direct* CP-violating decay, governed by Feynman graphs that cannot be derived from a combination of a box graph and a CP-conserving decay. Examples are so-called *penguin-graphs* like in fig. 1.3.

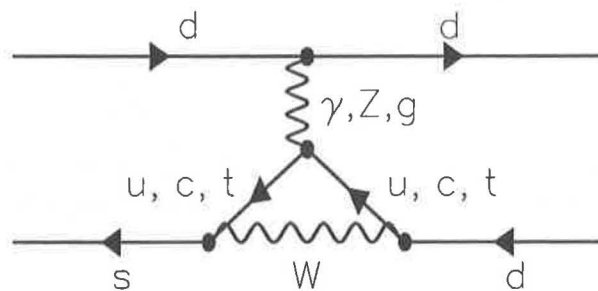


Figure 1.3: a penguin graph, responsible for direct CP-violation

The main difference between indirect and direct CP-violation is that if one looks at the ratios of the probabilities of a CP-violating decay and the corresponding CP-conserving one, the decay graph cancels in the case of indirect CP-violation, so the ratio does not depend on the specific decay process, and can be expressed through ϵ , which is therefore known as the parameter of indirect CP-violation.

In the case of direct CP-violation, the decay graphs do not cancel, and therefore the result generally depends on the decay mode. The effect can

be parameterized in some parameter ϵ' that can be related to differences in the branching ratios of the decays $K \rightarrow \pi^0\pi^0$ and $K \rightarrow \pi^+\pi^-$ [7]:

$$Re(\frac{\epsilon'}{\epsilon}) \approx \frac{1}{6}(1 - R) \quad R = \frac{\frac{\Gamma(K_L \rightarrow \pi^0\pi^0)}{\Gamma(K_S \rightarrow \pi^0\pi^0)}}{\frac{\Gamma(K_L \rightarrow \pi^+\pi^-)}{\Gamma(K_S \rightarrow \pi^+\pi^-)}} \quad (1.25)$$

As the analysis of the K_{e4} decay does not involve the study of direct CP-violation, for further details on direct CP-violation and the parameter ϵ' we refer to e.g. [6] or [33].

Chapter 2

The K_{e4} Decay

While the previous chapter dealt with general properties of the neutral kaon system, we now concentrate on the main topic of this thesis, the K_{e4} decay. After definition of this decay, the kinematics and the matrix element are studied. Some theoretical predictions are discussed, and finally it is shown how the theoretical parameters of interest can be extracted from experimental data.

2.1 Introduction

A special class of neutral kaon decay modes are the *semi-leptonic modes*, characterized by the occurrence of a lepton-neutrino pair ($l\bar{\nu}_l, \bar{l}\nu_l$) that originates from a W-interaction as sketched in fig. 2.1.

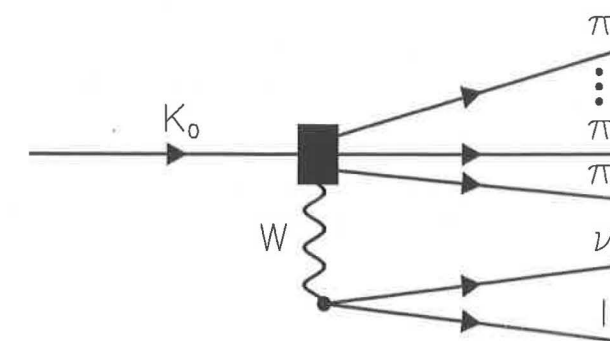


Figure 2.1: sketch of a generic semi-leptonic kaon decay

One classifies the leptonic and semi-leptonic decays of mesons by the

number of particles in the final state and by the type of the lepton. One example, which turns out to be the most probable K_L decay ($BR = 38.78 \pm 0.28\%$) [34], is $K_{e3} : K_L \rightarrow \pi^\mp e^\pm \nu/\bar{\nu}$. For the sake of readability and simplicity, this is henceforth simply written as $K_L \rightarrow \pi^+ e^- \nu$, but meant to a) stand also for the charge conjugated mode and b) hold lepton number conservation (that is, ν has to be replaced by $\bar{\nu}$ if necessary).

While K_{e3} is a frequent decay mode, the mode with one more pion in the final state, called K_{e4} , is rare[32]:

$$K_{e4} : K_L \rightarrow \pi^0 \pi^+ e^- \nu \quad BR = (5.18 \pm 0.29) \times 10^{-5} \quad (2.1)$$

This K_{e4} decay has some theoretically interesting properties which will be discussed later in this chapter.

2.2 Kinematics

As a 4-body-decay, the kinematics of Ke4 is non-trivial and requires some discussion and definition of proper phase space variables.

In principle, the phase space is 4×4 -dimensional. Since the masses of the final state particles are fixed, this reduces to 4×3 . 4-momentum conservation reduces by 4 degrees of freedom to an 8-dimensional space. Finally, as the physics do not depend on any special orientation in the phase space (isotropic symmetry), one can fix the three Euler angles and ends up with 5 physical degrees of freedom, so 5 variables are necessary to describe the physics of the Ke4 decay. Following the work of Cabibbo and Maksymowicz [8], one uses the following, henceforth called *Cabibbo variables*:

1. $m_{\pi\pi}$: invariant mass of the $\pi\pi$ -system, ranges from $m_{\pi^0} + m_{\pi^+}$ to $m_K - m_e$ (275 - 497 MeV).
2. $m_{e\nu}$: invariant mass of the $e\nu$ -system, range from m_e to $m_K - m_{\pi^0} - m_{\pi^+}$ (0.5 - 222 MeV), but correlated with $m_{\pi\pi}$ by energy conservation: $m_{e\nu} \leq M_K - m_{\pi\pi}$.
3. $\theta_{\pi\pi}$: angle of the charged pion momentum in the dipion rest frame with respect of the pion momentum in the kaon rest frame (range 0 to π). See fig. 2.2.
4. $\theta_{e\nu}$: angle of the electron momentum in the dilepton rest frame with respect of the dilepton momentum in the kaon rest frame (range 0 to π). See fig. 2.3.
5. ϕ : angle of the planes defined by dipion and dilepton momenta in the kaon rest frame, with respect to the direction of the dipion momentum (range $-\pi$ to π), see fig. 2.4. Explicitly, the definition of ϕ is:

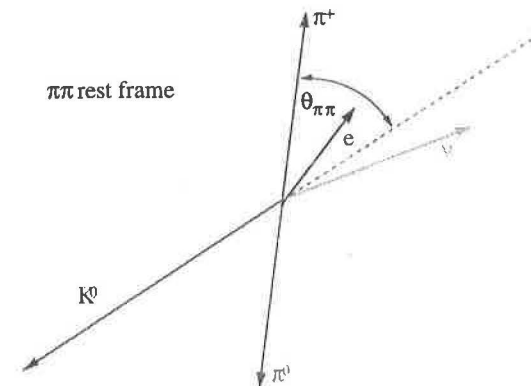


Figure 2.2: definition of Cabibbo variable $\theta_{\pi\pi}$ ($\pi\pi$ -rest frame)

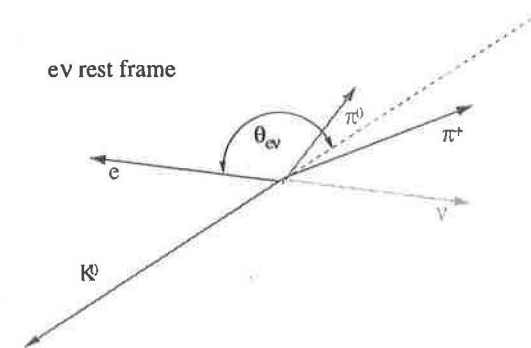


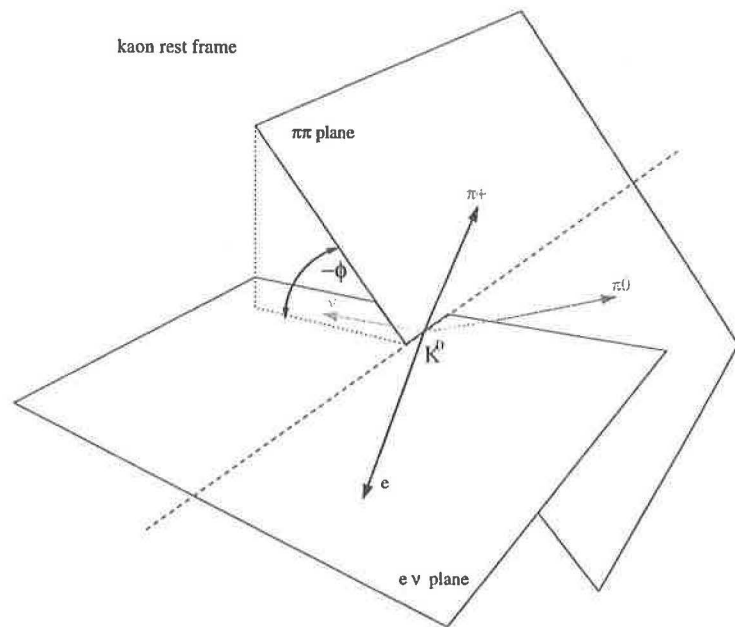
Figure 2.3: definition of Cabibbo variable $\theta_{e\nu}$ ($e\nu$ -rest frame)

$$\begin{aligned} r_{\pi\pi} &:= \frac{p_{\pi^+} + p_{\pi^0}}{|p_{\pi^+} + p_{\pi^0}|} \\ n_{\pi\pi} &:= \frac{p_{\pi^+} \times p_{\pi^0}}{|p_{\pi^+} \times p_{\pi^0}|} \quad n_{e\nu} := \frac{p_e \times p_\nu}{|p_e \times p_\nu|} \\ \cos(\phi) &:= n_{\pi\pi} \cdot n_{e\nu} \quad \sin(\phi) := (n_{\pi\pi} \times n_{e\nu}) \cdot r_{\pi\pi} \end{aligned} \quad (2.2)$$

Please note the ranges $[0, \pi]$ for $\theta_{\pi\pi}$ and $\theta_{e\nu}$, $[-\pi, \pi]$ for ϕ – this choice covers the whole phase space.

The Cabibbo variables can now be related to the 4-momenta of the particles. But instead of using the single particle momenta, it is more convenient to define variables that are symmetric (antisymmetric) under particle exchange:

$$P_{\pi\pi} := p_{\pi^+} + p_{\pi^0} \quad Q_{\pi\pi} := p_{\pi^+} - p_{\pi^0} \quad P_{e\nu} := p_e + p_\nu \quad Q_{e\nu} := p_e - p_\nu \quad (2.3)$$

Figure 2.4: definition of Cabibbo variable ϕ (kaon rest frame)

The corresponding scalar products can be expressed using the Cabibbo variables (see appendix, page 163). If one neglects the electron mass ($m_e \approx 0$)¹ the result is:

$$P_{\pi\pi}^2 = m_{\pi\pi}^2 \quad (2.4)$$

$$P_{e\nu}^2 = m_{e\nu}^2 \quad (2.5)$$

$$Q_{\pi\pi}^2 = \epsilon^2 \frac{m^4}{m_{\pi\pi}^2} - m_{\pi\pi}^2 \sigma_\pi^2 \quad (2.6)$$

$$Q_{e\nu}^2 = -m_{e\nu}^2$$

$$P_{\pi\pi} Q_{\pi\pi} = \epsilon m^2 \quad (2.7)$$

$$P_{e\nu} Q_{e\nu} = O(m_e^2) \approx 0$$

$$P_{\pi\pi} P_{e\nu} = \bar{p}$$

$$P_{\pi\pi} Q_{e\nu} = \sigma \cos \theta_{e\nu} + O(m_e^2)$$

$$P_{e\nu} Q_{\pi\pi} = \sigma \sigma_\pi \cos \theta_{\pi\pi} + \epsilon \frac{m^2}{m_{\pi\pi}^2} \bar{p}$$

$$Q_{\pi\pi} Q_{e\nu} = \epsilon \frac{m^2}{m_{\pi\pi}^2} \sigma \cos \theta_{e\nu} + \sigma_\pi \bar{p} \cos \theta_{\pi\pi} \cos \theta_{e\nu} + \sigma_\pi m_{\pi\pi} m_{e\nu} \sin \theta_{\pi\pi} \sin \theta_{e\nu} \cos \phi + O(m_e^2)$$

¹the neglected terms are typically $O(\frac{m_e^2}{m^2}) = O(10^{-6})$; only for $m_{e\nu} \approx 0$ these terms contribute, but can be safely neglected for $m_{e\nu} > 0$

$$\epsilon_{\alpha\beta\gamma\delta} P_{\pi\pi}^\alpha Q_{\pi\pi}^\beta P_{e\nu}^\gamma Q_{e\nu}^\delta = -\sigma \sigma_\pi m_{\pi\pi} m_{e\nu} \sin \theta_{\pi\pi} \sin \theta_{e\nu} \sin \phi + O(m_e^2)$$

with

$$\bar{p} := \frac{1}{2}(m^2 - m_{\pi\pi}^2 - m_{e\nu}^2) \quad (2.8)$$

$$\sigma := \sqrt{\bar{p}^2 - m_{\pi\pi}^2 m_{e\nu}^2} \quad (2.9)$$

$$\sigma_\pi := \sqrt{1 - c_\pi \frac{m^2}{m_{\pi\pi}^2} + \epsilon^2 \frac{m^4}{m_{\pi\pi}^4}} \quad (2.10)$$

and the constants [34]

$$\epsilon := \frac{m_{\pi^+}^2 - m_{\pi^0}^2}{m^2} = (5.09192 \pm 0.00064) \times 10^{-3} \quad (2.11)$$

$$c_\pi := \frac{2(m_{\pi^+}^2 + m_{\pi^0}^2)}{m^2} = 0.304416 \pm 0.000038 \quad (2.12)$$

$$(2.13)$$

Note that ϵ is a small parameter that characterizes the effect of the mass difference between charged and neutral pions; it can also be regarded as parameter of isospin violation.

2.3 Matrix Element

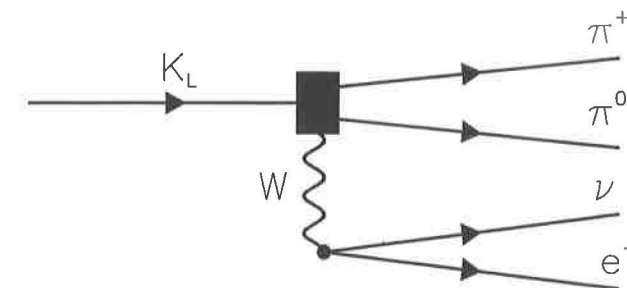


Figure 2.5: sketch of Ke4 decay graph

2.3.1 Introduction of the Form Factors

The matrix element of the neutral Ke4 decay² can be written as [35]

²in fact this is also the form for the charged Ke4 decay; actually most of the following also applies for the charged mode, but still we will specialize on the neutral Ke4

$$T = \frac{G_F}{\sqrt{2}} V_{us}^* L_\mu (V^\mu - A^\mu) \quad (2.14)$$

L_μ is the well known (V-A)-current for the lepton pair

$$L_\mu := \bar{u}(p_e) \gamma_\mu (1 - \gamma_5) v(p_\nu) \quad (2.15)$$

while V_μ and A_μ are effective (pseudo-) vector currents³ describing the coupling of the strongly interacting hadrons. Since these vector currents have to be 4-vectors that can depend only on the dipion momenta $P_{\pi\pi}, Q_{\pi\pi}$ and the total momentum of the kaon $p = P_{\pi\pi} + P_{e\nu}$, the most general form is given by

$$A^\mu = -i \frac{1}{m} [\bar{F}(P_{\pi\pi}, Q_{\pi\pi}, P_{e\nu}) \cdot P_{\pi\pi}^\mu + \bar{G}(P_{\pi\pi}, Q_{\pi\pi}, P_{e\nu}) \cdot Q_{\pi\pi}^\mu + \bar{R}(P_{\pi\pi}, Q_{\pi\pi}, P_{e\nu}) \cdot P_{e\nu}^\mu] \quad (2.16)$$

$$V_\mu = -\frac{1}{m^3} \bar{H}(P_{\pi\pi}, Q_{\pi\pi}, P_{e\nu}) \cdot \epsilon_{\mu\alpha\beta\gamma} P_{\pi\pi}^\alpha Q_{\pi\pi}^\beta P_{e\nu}^\gamma \quad (2.17)$$

$\bar{F}, \bar{G}, \bar{R}, \bar{H}$ - the Ke4 form factors - are complex Lorentz scalars that only can depend on the variables $P_{\pi\pi}, Q_{\pi\pi}, P_{e\nu}$. These form factors either can be measured (one of the aims of this work) or be predicted using methods like χ PT (chiral perturbation theory) [10]. It is convenient to write the complex form factors as norm and phase like this:

$$\bar{F} = F e^{i\phi_F} \quad (2.18)$$

the form factors will be discussed in some more detail later.

2.3.2 The Partial Decay Rate

The matrix element is related to the physical observable partial decay rate like ⁴:

$$d\Gamma = \frac{1}{2m} \sum_{spins} |T|^2 dLIPS(p; p_{\pi^+}, p_{\pi^0}, p_e, p_\nu) \quad (2.19)$$

The phase space integration is studied in the appendix (page 170) and gives

$$d\Gamma = \frac{G_F^2 |V_{us}|^2 N(m_{\pi\pi}, m_{e\nu})}{J_5(m_{\pi\pi}, m_{e\nu}, \cos \theta_{\pi\pi}, \cos \theta_{e\nu}, \phi) dm_{\pi\pi} dm_{e\nu} d(\cos \theta_{\pi\pi}) d(\cos \theta_{e\nu}) d\phi}$$

³due to the odd parity of the three particles K_L, π^+, π^0 the matrix element of the vector V_μ transforms like an axial vector, while the matrix element of the axial A_μ transforms like a vector

⁴as the spins of the leptons are not measured, the sum over all spins is performed

with the definition of $J_5(m_{\pi\pi}, m_{e\nu}, \theta_{\pi\pi}, \theta_{e\nu}, \phi)$ like in [35]:

$$\sum_{spins} |T|^2 = \frac{2G_F^2}{m^2} |V_{us}|^2 J_5(m_{\pi\pi}, m_{e\nu}, \theta_{\pi\pi}, \theta_{e\nu}, \phi) \quad (2.20)$$

Evaluating $|T|^2$ gives

$$\sum_{spins} |T|^2 = \frac{G_F^2}{2} |V_{us}|^2 \underbrace{\sum_{spins} L_\mu^\dagger L_\nu (V^{*\mu} V^\nu - V^{*\mu} A^\nu - A^{*\mu} V^\nu + A^{*\mu} A^\nu)}_{=: 4T_{\mu\nu}} \quad (2.21)$$

The leptonic tensor $T_{\mu\nu}$ gives (see Appendix, page 173):

$$T^{\mu\nu} = P_{e\nu}^\mu P_{e\nu}^\nu - Q_{e\nu}^\mu Q_{e\nu}^\nu - 2\bar{p}_{e\nu} g^{\mu\nu} - i\epsilon^{\mu\nu\alpha\beta} P_{e\nu\alpha} Q_{e\nu\beta} \quad (2.22)$$

For further calculations it is convenient to split T in symmetric and anti-symmetric parts with respect to $\mu\nu$:

$$\begin{aligned} \{T^{\mu\nu}\} &= P_{e\nu}^\mu P_{e\nu}^\nu - Q_{e\nu}^\mu Q_{e\nu}^\nu - 2\bar{p}_{e\nu} g^{\mu\nu} \\ [T^{\mu\nu}] &= -i\epsilon^{\mu\nu\alpha\beta} P_{e\nu\alpha} Q_{e\nu\beta} \end{aligned} \quad (2.23)$$

The same can be done for the hadronic parts of the matrix element. Using the short notations

$$P^\mu := P_{\pi\pi}^\mu \quad (2.24)$$

$$Q^\mu := Q_{\pi\pi}^\mu \quad (2.25)$$

$$L^\mu := P_{e\nu}^\mu \quad (2.26)$$

$$T^\mu := Q_{e\nu}^\mu \quad (2.27)$$

$$N_\mu := \epsilon_{\mu\alpha\beta\gamma} P^\alpha Q^\beta L^\gamma \quad (2.28)$$

$$\langle PQLT \rangle := \epsilon_{\alpha\beta\gamma\delta} P^\alpha Q^\beta L^\gamma T^\delta \quad (2.29)$$

one has

$$\{m^6 V_\mu^* V_\nu\} = H^2 N_\mu N_\nu$$

$$[m^6 V_\mu^* V_\nu] = 0$$

$$\begin{aligned} \{m^2 A_\mu^* A_\nu\} &= F^2 P_\mu P_\nu + G^2 Q_\mu Q_\nu + R^2 L_\mu L_\nu + \\ &+ FG \cos(\phi_G - \phi_F) (P_\mu Q_\nu + P_\nu Q_\mu) + \\ &+ FR \cos(\phi_R - \phi_F) (P_\mu L_\nu + P_\nu L_\mu) + \\ &+ GR \cos(\phi_R - \phi_G) (Q_\mu L_\nu + Q_\nu L_\mu) \end{aligned}$$

$$\begin{aligned}
[m^2 A_\mu^* A_\nu] &= iFG \sin(\phi_G - \phi_F)(P_\mu Q_\nu - P_\nu Q_\mu) + \\
&\quad + iFR \sin(\phi_R - \phi_F)(P_\mu L_\nu - P_\nu L_\mu) + \\
&\quad + iGR \sin(\phi_R - \phi_G)(Q_\mu L_\nu - Q_\nu L_\mu) \\
\{m^4(V_\mu^* A_\nu + A_\mu^* V_\nu)\} &= FH \sin(\phi_H - \phi_F)(N_\mu P_\nu + N_\nu P_\mu) + \\
&\quad + GH \sin(\phi_H - \phi_G)(N_\mu Q_\nu + N_\nu Q_\mu) + \\
&\quad + RH \sin(\phi_H - \phi_R)(N_\mu L_\nu + N_\nu L_\mu) \\
[m^4(V_\mu^* A_\nu + A_\mu^* V_\nu)] &= iFH \cos(\phi_H - \phi_F)(N_\mu P_\nu - N_\nu P_\mu) + \\
&\quad + iGH \cos(\phi_H - \phi_G)(N_\mu Q_\nu - N_\nu Q_\mu) + \\
&\quad + iRH \cos(\phi_H - \phi_R)(N_\mu L_\nu - N_\nu L_\mu)
\end{aligned}$$

Since only combinations of symmetric (anti-symmetric) terms contribute to the previously defined J_5 one has

$$\begin{aligned}
J_5 &= F^2 \cdot [(PL)^2 - (PT)^2 - 2\bar{p}_{e\nu} P^2] + \\
&\quad G^2 \cdot [(QL)^2 - (QT)^2 - 2\bar{p}_{e\nu} Q^2] + \\
&\quad R^2 \cdot [(L)^4 - (LT)^2 - 2\bar{p}_{e\nu} L^2] + \\
&\quad \frac{1}{m^4} H^2 \cdot [2\bar{p}_{e\nu} [P^2 Q^2 L^2 + 2(PQ)(PL)(QL) - P^2(QL)^2 - Q^2(PL)^2 - L^2(PQ)^2] \\
&\quad - < PQLT >^2] + \\
&\quad 2FG \cos(\phi_G - \phi_F) \cdot [(PL)(QL) - (PT)(QT) - 2\bar{p}_{e\nu}(PQ)] + \\
&\quad 2FG \sin(\phi_G - \phi_F) \cdot < PQLT > + \\
&\quad 2FR \cos(\phi_R - \phi_F) \cdot [(PL)L^2 - (PT)(LT) - 2\bar{p}_{e\nu}(PL)] + \\
&\quad \frac{2}{m^2} FH \cos(\phi_H - \phi_F) \cdot [(PL)(PQ)(LT) - P^2(QL)(LT) + P^2 L^2(QT) - (PL)^2(QT) + \\
&\quad + (QL)(PL)(PT) - (PQ)L^2(PT)] + \\
&\quad \frac{2}{m^2} FH \sin(\phi_H - \phi_F) \cdot < PQLT > (PT) + \\
&\quad 2GR \cos(\phi_R - \phi_G) \cdot [(QL)L^2 - (QT)(LT) - 2\bar{p}_{e\nu}(QL)] + \\
&\quad \frac{2}{m^2} GH \cos(\phi_H - \phi_G) \cdot [(PL)Q^2(LT) - (PQ)(QL)(LT) + (PQ)L^2(QT) - \\
&\quad - (PL)(QL)(QT) + (QL)^2(PT) - Q^2 L^2(PT)] + \\
&\quad \frac{2}{m^2} GH \sin(\phi_H - \phi_G) \cdot < PQLT > (QT) + \\
&\quad \frac{2}{m^2} RH \sin(\phi_H - \phi_R) \cdot < PQLT > (LT)
\end{aligned}$$

Using the Cabibbo variables (and neglecting the electron mass) this becomes

$$J_5 = F^2 \cdot \sigma^2 \sin^2 \theta_{e\nu} + \quad (2.3)$$

$$\begin{aligned}
&G^2 \cdot \left[m_{\pi\pi}^2 m_{e\nu}^2 \sigma_\pi^2 \sin^2 \theta_{\pi\pi} (1 - \sin^2 \theta_{e\nu} \cos^2 \phi) + \epsilon^2 \frac{m^4}{m_{\pi\pi}^4} \sigma^2 \sin^2 \theta_{e\nu} + \right. \\
&\quad \left. + \bar{p} \sigma_\pi^2 \cos \theta_{\pi\pi} (\bar{p} \cos \theta_{\pi\pi} \sin^2 \theta_{e\nu} - m_{\pi\pi} m_{e\nu} \sin \theta_{\pi\pi} \sin 2\theta_{e\nu} \cos \phi) + \right. \\
&\quad \left. + \epsilon \frac{m^2}{m_{\pi\pi}^2} \sigma \sigma_\pi (2\bar{p} \cos \theta_{\pi\pi} \sin^2 \theta_{e\nu} - m_{\pi\pi} m_{e\nu} \sin \theta_{\pi\pi} \sin 2\theta_{e\nu} \cos \phi) \right] + \\
&H^2 \cdot \frac{m_{\pi\pi}^2 m_{e\nu}^2 \sigma_\pi^2}{m^4} \sin^2 \theta_{\pi\pi} [1 - \sin^2 \theta_{e\nu} \sin^2 \phi] + \\
&FG \cos(\phi_G - \phi_F) \cdot [2\epsilon \frac{m^2}{m_{\pi\pi}^2} \sigma^2 \sin^2 \theta_{e\nu} + 2\sigma \sigma_\pi \bar{p} \cos \theta_{\pi\pi} \sin^2 \theta_{e\nu} - \\
&\quad - m_{\pi\pi} m_{e\nu} \sigma \sigma_\pi \sin \theta_{\pi\pi} \sin 2\theta_{e\nu} \cos \phi] - \\
&- FG \sin(\phi_G - \phi_F) \cdot 2m_{\pi\pi} m_{e\nu} \sigma \sigma_\pi \sin \theta_{\pi\pi} \sin \theta_{e\nu} \sin \phi - \\
&- FH \cos(\phi_H - \phi_F) \cdot \frac{2m_{\pi\pi} m_{e\nu} \sigma^2 \sigma_\pi}{m^2} \sin \theta_{\pi\pi} \sin \theta_{e\nu} \cos \phi - \\
&- FH \sin(\phi_H - \phi_F) \cdot \frac{m_{\pi\pi} m_{e\nu} \sigma^2 \sigma_\pi}{m^2} \sin \theta_{\pi\pi} \sin 2\theta_{e\nu} \sin \phi - \\
&- GH \cos(\phi_H - \phi_G) \cdot \left[\frac{m_{\pi\pi} m_{e\nu} \sigma \sigma_\pi}{m^2} [\bar{p} \sin 2\theta_{\pi\pi} \sin \theta_{e\nu} \cos \phi - 2m_{\pi\pi} m_{e\nu} \sin^2 \theta_{\pi\pi} \cos \theta_{e\nu}] + \right. \\
&\quad \left. + 2\epsilon \frac{m_{e\nu}}{m_{\pi\pi}} \sigma^2 \sigma_\pi \sin \theta_{\pi\pi} \sin \theta_{e\nu} \cos \phi \right] - \\
&- GH \sin(\phi_H - \phi_G) \cdot \left[\frac{m_{\pi\pi} m_{e\nu} \sigma \sigma_\pi}{m^2} [(\epsilon \frac{m^2}{m_{\pi\pi}^2} \sigma + \sigma_\pi \bar{p} \cos \theta_{\pi\pi}) \sin \theta_{\pi\pi} \sin 2\theta_{e\nu} \sin \phi + \right. \\
&\quad \left. + m_{\pi\pi} m_{e\nu} \sigma_\pi \sin^2 \theta_{\pi\pi} \sin^2 \theta_{e\nu} \sin 2\phi] \right]
\end{aligned}$$

note that the terms proportional to R^2 , FR , GR , RH vanish for $m_e \approx 0$, so the influence of R on the matrix element is negligible.

2.3.3 Integrating over ϕ

The structure of J_5 is rather complicated, and there is not enough experimental data to verify it in this form; however, since the form factors cannot depend on the Cabibbo variables $\theta_{e\nu}$ and ϕ ⁵, it is possible to perform the integrations over these two variables.

As a first step, integration over ϕ leads to

$$J_4 := \int_{-\pi}^{\pi} d\phi J_5 = F^2 \cdot 2\pi \sigma^2 \sin^2 \theta_{e\nu} + \quad (2.31)$$

$$G^2 \cdot \pi \left[m_{\pi\pi}^2 m_{e\nu}^2 \sigma_\pi^2 \sin^2 \theta_{\pi\pi} (2 - \sin^2 \theta_{e\nu}) + 2\epsilon^2 \frac{m^4}{m_{\pi\pi}^4} \sigma^2 \sin^2 \theta_{e\nu} + \right.$$

⁵they can only depend on the momentum of the exchanged virtual W -boson ($= L_\mu$), and not on details of its decay to the lepton pair - this is already contained in the structure of the leptonic tensor $T_{\mu\nu}$

$$\begin{aligned}
& + 2(2\epsilon \frac{m^2}{m_{\pi\pi}^2} \sigma + \bar{p}\sigma_\pi \cos \theta_{\pi\pi}) \bar{p}\sigma_\pi \cos \theta_{\pi\pi} \sin^2 \theta_{e\nu} \Big] + \\
& H^2 \cdot \pi \frac{m_{\pi\pi}^2 m_{e\nu}^2 \sigma^2 \sigma_\pi^2}{m^4} \sin^2 \theta_{\pi\pi} [2 - \sin^2 \theta_{e\nu}] + \\
& FG \cos(\phi_G - \phi_F) \cdot 4\pi\sigma \left(\epsilon \frac{m^2}{m_{\pi\pi}^2} \sigma + \sigma_\pi \bar{p} \cos \theta_{\pi\pi} \right) \sin^2 \theta_{e\nu} + \\
& GH \cos(\phi_H - \phi_G) \cdot \frac{2\pi\sigma\sigma_\pi}{m^2} [2m_{\pi\pi}^2 m_{e\nu}^2 \sigma_\pi \sin^2 \theta_{\pi\pi} + \epsilon \frac{m^2}{m_{\pi\pi}^2} \sigma \bar{p} \cos \theta_{\pi\pi}] \cos \theta_{e\nu}
\end{aligned}$$

Note that the dependence on $\theta_{e\nu}$ is of the form

$$J_4 = c_1 + c_2 \cos \theta_{e\nu} + c_3 \cos^2 \theta_{e\nu} \quad (2.32)$$

This prediction is a consequence of the (V-A)-coupling of the lepton pair and independent of the form factors (which only determine the c_i) and can be compared with experimental results.

2.3.4 Integrating over $\theta_{e\nu}$

The next step is to perform also the $\theta_{e\nu}$ integration; using

$$\int_{-1}^1 \cos \theta d(\cos \theta) = 0 \quad \int_{-1}^1 \sin^2 \theta d(\cos \theta) = \frac{4}{3} \quad \int_{-1}^1 \cos^2 \theta d(\cos \theta) = \frac{2}{3} \quad (2.33)$$

one has

$$\begin{aligned}
J_3 & := \int d(\cos \theta_{e\nu}) J_4 = F^2 \cdot \frac{8\pi}{3} \sigma^2 + \\
& G^2 \cdot \frac{8\pi}{3} \left[\epsilon^2 \frac{m^4}{m_{\pi\pi}^4} \sigma^2 + m_{\pi\pi}^2 m_{e\nu}^2 \sigma_\pi^2 \sin^2 \theta_{\pi\pi} + (2\epsilon \frac{m^2}{m_{\pi\pi}^2} \sigma + \bar{p}\sigma_\pi \cos \theta_{\pi\pi}) \bar{p}\sigma_\pi \cos \theta_{\pi\pi} \right] + \\
& H^2 \cdot \frac{8\pi}{3} \frac{m_{\pi\pi}^2 m_{e\nu}^2 \sigma^2 \sigma_\pi^2}{m^4} \sin^2 \theta_{\pi\pi} + \\
& FG \cos(\phi_G - \phi_F) \cdot \frac{16\pi}{3} \sigma \left[\epsilon \frac{m^2}{m_{\pi\pi}^2} \sigma + \sigma_\pi \bar{p} \cos \theta_{\pi\pi} \right]
\end{aligned}$$

with the definition of ⁶

$$F_1 := \sigma \cdot \bar{F} + \left(\epsilon \frac{m^2}{m_{\pi\pi}^2} \sigma + \bar{p}\sigma_\pi \cos \theta_{\pi\pi} \right) \cdot \bar{G} \quad (2.34)$$

⁶note that this definition differs slightly from the general definition given in [35] because it includes also a term proportional to ϵ ; of course this term could be neglected, but it is kept to keep track of the error induced by this isospin violation

$$F_2 := m_{\pi\pi} m_{e\nu} \sigma_\pi \sin \theta_{\pi\pi} \cdot \bar{G} \quad (2.35)$$

$$F_3 := \frac{m_{\pi\pi} m_{e\nu} \sigma \sigma_\pi \sin \theta_{\pi\pi}}{m^2} \cdot \bar{H} \quad (2.36)$$

$$(2.37)$$

this can also be written as

$$J_3 = \frac{8\pi}{3} [|F_1|^2 + |F_2|^2 + |F_3|^2] \quad (2.38)$$

2.3.5 Partial Wave Expansion

Since the form factors can depend on $\theta_{\pi\pi}$, it is convenient to write partial wave expansions

$$\begin{aligned}
F &= \sum_l P_l(\cos \theta_{\pi\pi}) f_l \\
G &= \sum_l P'_l(\cos \theta_{\pi\pi}) g_l \\
H &= \sum_l P'_l(\cos \theta_{\pi\pi}) h_l
\end{aligned}$$

note that G and H have been expanded in derivatives of Legendre polynomials to relate the coefficients g_l and h_l to the corresponding form factors of elastic $\pi\pi$ -scattering [9]. Equivalently, also the F_i can be expanded as

$$F_1 = \sigma \sum_l P_l(\cos \theta_{\pi\pi}) \tilde{f}_l \quad (2.39)$$

$$F_2 = m_{\pi\pi} m_{e\nu} \sigma_\pi \sum_l \sin \theta_{\pi\pi} P'_l(\cos \theta_{\pi\pi}) \tilde{g}_l \quad (2.40)$$

$$F_3 = \frac{m_{\pi\pi} m_{e\nu} \sigma \sigma_\pi}{m^2} \sum_l \sin \theta_{\pi\pi} P'_l(\cos \theta_{\pi\pi}) \tilde{h}_l \quad (2.41)$$

Obviously there is

$$\tilde{g}_l = g_l \quad \tilde{h}_l = h_l \quad \text{but} \quad \tilde{f}_l \neq f_l \quad (2.42)$$

2.3.6 Watson-Fermi Theorem and $\pi\pi$ Phases

The Watson-Fermi theorem [20] states that the phases of $\tilde{f}_l, g_l, h_l$ coincide with the phases of elastic $\pi\pi$ scattering δ_l^I in the following way:

$$\begin{aligned}
\arg(\tilde{f}_{2l}) &= \arg(g_{2l}) = \arg(h_{2l}) = \delta_{2l}^0 \\
\arg(\tilde{f}_{2l+1}) &= \arg(g_{2l+1}) = \arg(h_{2l+1}) = \delta_{2l+1}^1
\end{aligned} \quad (2.43)$$

Note that the phase is the same for all three form factors for a given l .

2.3.7 Implications of Isospin- and Bose Symmetry

Applying isospin- and Bose-symmetry, it turns out⁷ that only the anti-symmetric part with respect to exchange of the two pions contributes to the matrix element, and subsequently the form factor F has to be anti-symmetrized, while G and H are symmetrized, all weighted with $\sqrt{2}$:

$$\begin{aligned} F &\rightarrow \sqrt{2} \frac{1}{2} [F(\pi^+, \pi^0) - F(\pi^0, \pi^+)] \\ G &\rightarrow \sqrt{2} \frac{1}{2} [G(\pi^+, \pi^0) + G(\pi^0, \pi^+)] \\ H &\rightarrow \sqrt{2} \frac{1}{2} [H(\pi^+, \pi^0) + H(\pi^0, \pi^+)] \end{aligned} \quad (2.44)$$

Exchange of the pions means $\theta_{\pi\pi} \rightarrow \pi - \theta_{\pi\pi}$, $\phi \rightarrow \pi - \phi$; the parity of P_l is $(-1)^l$, and of P'_l is $(-1)^{l-1}$, so obviously all coefficients f_l, g_l, h_l with even l vanish, especially the leading order f_0 , also called f_s for s-wave:

$$f_0 =: f_s = 0 \quad \rightarrow \quad \tilde{f}_s = \epsilon \frac{m^2}{m_{\pi\pi}^2} g_1 =: \epsilon \frac{m^2}{m_{\pi\pi}^2} g_p \quad (2.45)$$

This is a theoretical prediction of isospin symmetry that can be tested experimentally (see discussion later in this chapter).

2.3.8 The Partial Decay Rate for Leading Order Partial Waves

The leading order form factors are now given by

$$\begin{aligned} F_1 &= \sqrt{2} \sigma \cos \theta_{\pi\pi} f_p(m_{\pi\pi}, m_{e\nu}) + \sqrt{2} \left(\epsilon \frac{m^2}{m_{\pi\pi}^2} \sigma + \bar{p} \sigma_\pi \cos \theta_{\pi\pi} \right) g_p(m_{\pi\pi}, m_{e\nu}) \\ F_2 &= \sqrt{2} m_{\pi\pi} m_{e\nu} \sigma_\pi \sin \theta_{\pi\pi} g_p(m_{\pi\pi}, m_{e\nu}) \\ F_3 &= \sqrt{2} \frac{m_{\pi\pi} m_{e\nu} \sigma \sigma_\pi}{m^2} \sin \theta_{\pi\pi} h_p(m_{\pi\pi}, m_{e\nu}) \end{aligned} \quad (2.46)$$

$$(2.47)$$

Since all form factors have the same phase $e^{i\delta_1^1}$, it is not observable in leading order and will be dropped. The dependence of J_3 on $\theta_{\pi\pi}$ is

$$J_3 = c_1 + c_2 \cos \theta_{\pi\pi} + c_3 \cos^2 \theta_{\pi\pi} \quad (2.48)$$

where c_2 is of $O(\epsilon)$.

The physical observable is the partial decay rate

$$d\Gamma = G_F^2 |V_{us}|^2 N(m_{\pi\pi}, m_{e\nu}) J_3(m_{\pi\pi}, m_{e\nu}, \cos \theta_{\pi\pi}) dm_{\pi\pi} dm_{e\nu} d(\cos \theta_{\pi\pi}) \quad (2.49)$$

⁷see discussion in the appendix, page 181

with

$$N(m_{\pi\pi}, m_{e\nu}) := \frac{m_{\pi\pi} m_{e\nu} \sigma \sigma_\pi}{2^{11} \pi^6 m^5} \quad (2.50)$$

If one uses only the leading form factors from above to integrate over $\cos \theta_{\pi\pi}$, one has with (2.33)

$$J_2 = \frac{32\pi}{9} \left[|\sigma f_p + \sigma_\pi \bar{p} g_p|^2 + (2m_{\pi\pi}^2 m_{e\nu}^2 \sigma_\pi^2 + 3\epsilon^2 \frac{m^4}{m_{\pi\pi}^4} \sigma^2) |g_p|^2 + 2 \frac{m_{\pi\pi}^2 m_{e\nu}^2 \sigma^2 \sigma_\pi^2}{m^4} |h_p|^2 \right] \quad (2.51)$$

in

$$d\Gamma = G_F^2 |V_{us}|^2 N(m_{\pi\pi}, m_{e\nu}) J_2(m_{\pi\pi}, m_{e\nu}) dm_{\pi\pi} dm_{e\nu} \quad (2.52)$$

It is now useful to transform this expression to dimensionless variables⁸

$$x := \frac{m_{\pi\pi}}{m} \quad y := \frac{m_{e\nu}^2}{m^2} \quad (2.53)$$

with the additional definition of

$$\sigma_{xy} := \frac{2}{m^2} \sigma = \sqrt{(1-x^2-y)^2 - 4x^2y} \quad (2.54)$$

$$\sigma_x := x \sigma_\pi = \sqrt{x^2 - c_\pi + \frac{\epsilon^2}{x^2}} \quad (2.55)$$

one arrives at

$$\begin{aligned} d\Gamma &= \Gamma^0 \cdot \left[\sigma_x \sigma_{xy}^3 |f_p|^2 + 2 \frac{\sigma_x^2}{x} (1-x^2-y) \sigma_{xy}^2 |f_p g_p| \cos \phi_{fg} + \right. \\ &\quad \left. + \left(\frac{\sigma_x^3}{x^2} \sigma_{xy}^3 + 12 \sigma_x^3 y \sigma_{xy} + 3 \epsilon^2 \frac{\sigma_x}{x^4} \sigma_{xy}^3 \right) |g_p|^2 + 2 \sigma_x^3 y \sigma_{xy}^3 |h_p|^2 \right] dx dy \end{aligned} \quad (2.56)$$

with [34]

$$\Gamma^0 := \frac{G_F^2 |V_{us}|^2 m^5}{2^{10} 3^2 \pi^5} = (7.10 \pm 0.15) \times 10^{-20} \text{ GeV} \quad (2.57)$$

The error of Γ^0 is dominated by the uncertainty on V_{us} .

⁸note that y is defined quadratic to simplify expressions below

2.3.9 The $m_{\pi\pi}$ Distribution

To get the $m_{\pi\pi}$ distribution one has to integrate over $m_{e\nu}$. Since in principle all form factors can depend on $m_{e\nu}$, they are expanded in y in order to check the relevance of this possible dependence for the result:

$$\begin{aligned} |f_p|^2 &= |f_p^0|^2 \left(1 + \sum_{k=1}^{\infty} \tilde{\lambda}_{f,k} y^k\right) \\ |f_p g_p| \cos \phi_{fg} &= |f_p^0 g_p^0| \cos \phi_{fg}^0 \left(1 + \sum_{k=1}^{\infty} \tilde{\lambda}_{fg,k} y^k\right) \\ |g_p|^2 &= |g_p^0|^2 \left(1 + \sum_{k=1}^{\infty} \tilde{\lambda}_{g,k} y^k\right) \\ |h_p|^2 &= |h_p^0|^2 \left(1 + \sum_{k=1}^{\infty} \tilde{\lambda}_{h,k} y^k\right) \end{aligned} \quad (2.58)$$

The result of the y -integration will be of the general form

$$\frac{d\Gamma}{dx} = \Gamma^0 \cdot \left[|f_p^0|^2 I_f(x) + 2|f_p^0 g_p^0| \cos \phi_{fg}^0 I_{fg}(x) + |g_p^0|^2 I_g(x) + |h_p^0|^2 I_h(x) \right] \quad (2.59)$$

where the $I_w, w = f, fg, g, h$ are of the form

$$I_w(x) = I_w^{(0)} + \sum \tilde{\lambda}_{w,k} I_w^{(k)} \quad (2.60)$$

Introducing the general integral⁹

$$I_{a,b}(x) := \int_0^{(1-x)^2} dy y^a \sigma_{xy}^b \quad (2.61)$$

one finds

$$\begin{aligned} I_f^{(k)}(x) &= \sigma_x I_{k,3}(x) \\ I_{fg}^{(k)}(x) &= \frac{\sigma_x^2}{x} [(1-x^2) I_{k,2}(x) - I_{k+1,2}(x)] \\ I_g^{(k)}(x) &= \sigma_x \left[\frac{\sigma_x^2}{x^2} I_{k,3}(x) + 12\sigma_x^2 I_{k+1,1}(x) + 3\epsilon^2 \frac{1}{x^4} I_{k,3}(x) \right] \\ I_h^{(k)}(x) &= 2\sigma_x^3 I_{k+1,3}(x) \end{aligned} \quad (2.62)$$

⁹the integral range follows from the correlation between $m_{\pi\pi}$ and $m_{e\nu}$ requiring $m_{\pi\pi} + m_{e\nu} \leq m$

The properties of the integrals $I_{a,b}(x)$ are studied in the appendix (page 174). One important relation is an upper bound on $I_{k,b}$:

$$I_{k,b}^{mean} \leq \frac{8 \cdot 3^b}{(k+1)(2k+b+3)2^{2(k+b)}} I_{0,1}^{mean} \quad (2.63)$$

Possible effects of $m_{e\nu}$ dependence of the form factors are obviously suppressed unless $\tilde{\lambda}_{w,k} \gg 1$. Therefore, any $m_{e\nu}$ dependence will henceforth be neglected. At leading order ($k=0$) the following functions are needed:

$$\begin{aligned} I_{1,1} &= \frac{1}{6}(1-x^2)[1+10x^2+x^4] + 2x^2(1+x^2)\ln x \\ I_{0,2} &= \frac{1}{3}(1-x)^4[1+4x+x^2] \\ I_{1,2} &= \frac{1}{12}(1-x)^6[1+6x+x^2] \\ I_{0,3} &= \frac{1}{4}(1-x^4)[1-8x^2+x^4] - 6x^4\ln x \\ I_{1,3} &= \frac{1}{20}(1-x^2)[1-14x^2-94x^4-14x^6+x^8] - 6x^4(1+x^2)\ln x \end{aligned}$$

which finally gives

$$\begin{aligned} I_f(x) &= \sigma_x(x) \left[\frac{1}{4}(1-x^4)(1-8x^2+x^4) - 6x^4\ln x \right] \\ I_{fg}(x) &= \frac{\sigma_x^2(x)(1-x)^5}{12x} [3+15x+25x^2+5x^3] \\ I_g(x) &= \frac{1}{x^4} [x^2\sigma_x^2(x) + 3\epsilon^2] \cdot I_f(x) + 2\sigma_x^3(x) [(1-x^2)(1+10x^2+x^4) + 12x^2(1+x^2)\ln x] \\ I_h(x) &= \sigma_x^3(x) \left[\frac{1}{10}(1-x^2)(1-14x^2-94x^4-14x^6+x^8) - 12x^4(1+x^2)\ln x \right] \end{aligned}$$

The term proportional to ϵ^2 in I_g contributes only for x -values very close to the lower bound ($x-x_0 = O(10^{-4})$), so can be neglected at this point.

The functions I_w are shown in fig. 2.6(a) to scale - it is eye-striking that the h -contribution is negligible (of course unless the form factor h is much larger than f or g).

To compare the different shapes, fig. 2.6(b) shows the normalized functions. The shapes differ mainly by their maximum, which is lowest for f and highest for g .

It is interesting to remark that if there were a contribution from f_s , it would contribute to I_f , but not to the interference I_{fg} ¹⁰. As the shape of I_f is quite distinct from the others, it will be thus possible to check for any occurrence of f_s in the data with reasonable accuracy.

¹⁰this contribution would be proportional to $\cos \theta_{\pi\pi}$ and cancels by integrating over this variable

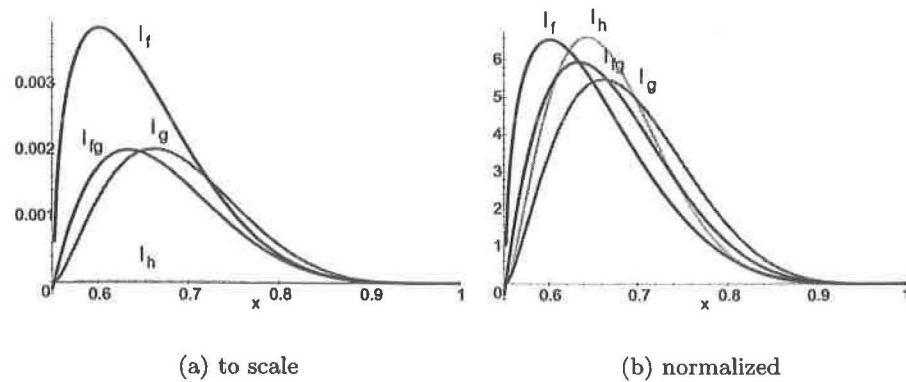


Figure 2.6: shapes of $m_{\pi\pi}$ distributions for form factors f, fg, g and h ($x = \frac{m_{\pi\pi}}{m}$)

2.3.10 The Total Decay Rate

As a last step, the total decay rate can be computed like this

$$\Gamma = \int_{x_0}^1 dx \frac{d\Gamma}{dx} = \Gamma^0 [c_f |f_p|^2 + 2c_{fg} |f_p g_p| \cos \phi_{fg} + c_g |g_p|^2 + c_h |h_p|^2]$$

$$x_0 := \frac{m_{\pi^+} + m_{\pi^0}}{m} = 0.551662 \pm 0.000034 \quad (2.64)$$

where the form factors have been expanded in powers of $(x - x_0)^2$

$$|f_p|^2 = |f_p^0|^2 \left(1 + \sum_{k=1}^{\infty} \lambda_{f,k} (x - x_0)^{2k}\right)$$

$$|f_p g_p| \cos \phi_{fg} = |f_p^0 g_p^0| \cos \phi_{fg}^0 \left(1 + \sum_{k=1}^{\infty} \lambda_{fg,k} (x - x_0)^{2k}\right)$$

$$|g_p|^2 = |g_p^0|^2 \left(1 + \sum_{k=1}^{\infty} \lambda_{g,k} (x - x_0)^{2k}\right)$$

$$|h_p|^2 = |h_p^0|^2 \left(1 + \sum_{k=1}^{\infty} \lambda_{h,k} (x - x_0)^{2k}\right)$$

and the constants $c_w, w = f, fg, g, h$ are thus given by

$$c_w := \int_{x_0}^1 dx I_w(x) + \sum_{k=1}^{\infty} \lambda_{w,k} \int_{x_0}^1 dx (x - x_0)^{2k} I_w(x) \quad (2.65)$$

Because of the estimation

$$\int_{x_0}^1 dx (x - x_0)^{2k} I_w(x) \leq (1 - x_0)^{2k} \int_{x_0}^1 dx I_w(x) < \frac{1}{2^{2k}} \int_{x_0}^1 dx I_w(x) \quad (2.66)$$

only the terms for $k \leq 1$ are kept. Using the values from [34] one finds numerically

$$\begin{aligned} c_f &= (5.8831 \pm 0.0026 + (0.08290 \pm 0.004) \lambda_f) \times 10^{-4} \\ c_{fg} &= (3.3739 \pm 0.0017 + (0.06237 \pm 0.004) \lambda_{fg}) \times 10^{-4} \\ c_g &= (3.6880 \pm 0.0020 + (0.08338 \pm 0.005) \lambda_g) \times 10^{-4} \\ c_h &= (3.9491 \pm 0.0025 + (0.06843 \pm 0.005) \lambda_h) \times 10^{-6} \end{aligned}$$

The errors are dominated by the error of the neutral kaon mass m . Note that the influence of the λ_w and of c_h is smaller than the uncertainty on the overall constant Γ^0 ; therefore, as far as the total decay rate is concerned, everything but c_f, c_{fg} and c_g may be neglected (of course unless the other form factors are very large).

If one normalizes to the K_L life time [34]

$$\tau_{K_L} = (5.17 \pm 0.04) \times 10^{-8} \text{s} = (7.86 \pm 0.06) \times 10^{16} \text{GeV}^{-1} \quad (2.67)$$

one gets the branching ratio

$$BR(Ke4) = \tau_{K_L} \Gamma = (3.283 \pm 0.074) \times 10^{-6} \cdot [|f_p|^2 + 2r_{fg} |f_p g_p| \cos \phi_{fg} + r_g |g_p|^2] \quad (2.68)$$

with

$$r_{fg} = (57.349 \pm 0.038)\% \quad r_g = (62.6889 \pm 0.00044)\% \quad (2.69)$$

Obviously the branching ratio depends significantly on all three contributions: the f_p and g_p form factors and their interference.

2.4 Extraction of the Form Factors

In the study above, certain assumptions about the form factors have been made in order to simplify the calculations. In this section we will show how these assumptions can be tested experimentally by extracting the form factors from measured distributions.

To extract the form factors, there are several possible ways; the best way would of course be to study the measured partial decay rate in all three

relevant Cabibbo variables¹¹. This, however, requires large statistics¹², thus the approach has to be somewhat different.

The way of choice is to study only distributions in one variable, but use different weights when integrating over the other variables. Using this method it is possible to separate the form factors F_i as defined in (2.39), as will be shown below:

First, one goes back to the definition of J_4 in (2.31) and defines a weighted J_4^ϕ like this:

$$J_4^\phi := \int d\phi \cos 2\phi J_5 \quad (2.70)$$

Using

$$\int \cos 2\phi d\phi = 0 \quad \int \cos 2\phi \cos \phi d\phi = 0 \quad \int \cos 2\phi \cos^2 \phi d\phi = \frac{\pi}{2} \quad (2.71)$$

one gets from (2.30)

$$J_4^\phi = -G^2 \cdot \frac{\pi}{2} \sigma_\pi^2 m_{\pi\pi}^2 m_{e\nu}^2 \sin^2 \theta_{\pi\pi} \sin^2 \theta_{e\nu} + H^2 \cdot \frac{\pi}{2} \frac{m_{\pi\pi}^2 m_{e\nu}^2 \sigma_\pi^2}{m^4} \sin^2 \theta_{\pi\pi} \sin^2 \theta_{e\nu} \quad (2.72)$$

Integrating over $\theta_{e\nu}$ then gives

$$J_3^\phi = -\frac{2\pi}{3} [|F_2|^2 - |F_3|^2] \quad (2.73)$$

Equivalently, we define a weighted J_3^θ as

$$J_3^\theta := \int d(\cos \theta_{e\nu}) (3 \cos^2 \theta_{e\nu} - 1) J_4 \quad (2.74)$$

Because of

$$\begin{aligned} \int d(\cos \theta_{e\nu}) (3 \cos^2 \theta_{e\nu} - 1) &= 0 \\ \int d(\cos \theta_{e\nu}) (3 \cos^2 \theta_{e\nu} - 1) \cos \theta_{e\nu} &= 0 \\ \int d(\cos \theta_{e\nu}) (3 \cos^2 \theta_{e\nu} - 1) \cos^2 \theta_{e\nu} &= \frac{8}{15} \end{aligned} \quad (2.75)$$

this results in

¹¹this is $m_{\pi\pi}, m_{e\nu}$ and $\cos \theta_{\pi\pi}$

¹²just over the thumb: if the resolution in each Cabibbo variable would be just 10 bins, then there would be already 10^3 bins in total. As statistics for this Ke4 study is roughly of the same order, one could expect only a couple of events per bin

$$J_3^\theta = -\frac{8\pi}{15} [2|F_1|^2 - |F_2|^2 - |F_3|^2] \quad (2.76)$$

It is now possible to solve for the form factors F_i :

$$|F_1|^2 = \frac{1}{8\pi} [J_3 - 5J_3^\theta] \quad (2.77)$$

$$|F_2|^2 = \frac{1}{16\pi} [2J_3 - 12J_3^\phi + 5J_3^\theta] \quad (2.78)$$

$$|F_3|^2 = \frac{1}{16\pi} [2J_3 + 12J_3^\phi + 5J_3^\theta] \quad (2.79)$$

It is remarkable that up to this point no special assumptions about the form factors have been made. However, as statistics are insufficient to study the distribution in all three variables, one has to average over two variables to get a distribution in the third one - doing so requires certain assumptions about the dependence of the form factors on the averaged variables; of course it is still possible to perform a weighted average if appropriate.

2.4.1 Distributions in $\cos \theta_{\pi\pi}$

To study the form factors as a function of $\cos \theta_{\pi\pi}$ one looks at the partial wave expansion

$$\begin{aligned} \sum_l P_l(\cos \theta_{\pi\pi}) f_l &= f_s + f_p \cos \theta_{\pi\pi} + \dots \\ \sum_l \sin \theta_{\pi\pi} P'_l(\cos \theta_{\pi\pi}) g_l &= g_p \sin \theta_{\pi\pi} + 3g_d \sin \theta_{\pi\pi} \cos \theta_{\pi\pi} + \dots \\ \sum_l \sin \theta_{\pi\pi} P'_l(\cos \theta_{\pi\pi}) h_l &= h_p \sin \theta_{\pi\pi} + 3h_d \sin \theta_{\pi\pi} \cos \theta_{\pi\pi} + \dots \end{aligned}$$

Integrating over the invariant masses gives - besides an irrelevant normalization constant

$$\begin{aligned} \frac{d\Gamma_1}{d\cos \theta} &= \langle |f_s|^2 \rangle + 2\langle |f_s||f_p| \rangle \cos \phi_{sp} \cos \theta_{\pi\pi} + \langle |f_p|^2 \rangle \cos^2 \theta_{\pi\pi} \\ \frac{d\Gamma_2}{d\cos \theta} &= \langle |g_p|^2 \rangle \sin^2 \theta_{\pi\pi} + 6\langle |g_p||g_d| \rangle \cos \phi_{gd} \sin^2 \theta_{\pi\pi} \cos \theta_{\pi\pi} + 9\langle |g_d|^2 \rangle \sin^2 \theta_{\pi\pi} \cos^2 \theta_{\pi\pi} \\ \frac{d\Gamma_3}{d\cos \theta} &= \langle |h_p|^2 \rangle \sin^2 \theta_{\pi\pi} + 6\langle |h_p||h_d| \rangle \cos \phi_{gd} \sin^2 \theta_{\pi\pi} \cos \theta_{\pi\pi} + 9\langle |h_d|^2 \rangle \sin^2 \theta_{\pi\pi} \cos^2 \theta_{\pi\pi} \end{aligned}$$

where the Γ_i are defined in the sense of (2.85) and $\langle f(m_{\pi\pi}, m_{e\nu}) \rangle$ means the mass-average of the corresponding function. Since the weight is always positive, one has in any case

$$\langle |f|^2 \rangle < c \Leftrightarrow |f_{max}|^2 < \frac{c}{w_{max}} \quad (2.80)$$

where w_{max} is the largest weight. As a special case one has of course also

$$\langle |f|^2 \rangle = 0 \Leftrightarrow |f|^2 = 0 \quad (2.81)$$

this way it is possible to estimate the significance of higher order partial waves, or especially to check whether f_s really vanishes as required by isospin symmetry.

2.4.2 Distributions in $m_{\pi\pi}$

Integrating over $\cos \theta_{\pi\pi}$ generally gives a weighted mean of the form factors; similar to above, this mean can still be used to study the significance of contributions. It is also possible to choose a special weight to specifically select a partial wave. Here we assume that only the leading partial waves contribute significantly:

$$\sigma^2 |\tilde{f}_p|^2 = \frac{3}{64\pi} [J_2 - 5J_2^\theta] \quad (2.82)$$

$$m_{\pi\pi}^2 m_{e\nu}^2 \sigma_\pi^2 |g_p|^2 = \frac{3}{64\pi} [2J_2 - 12J_2^\phi + 5J_2^\theta] \quad (2.83)$$

$$m_{\pi\pi}^2 m_{e\nu}^2 \sigma_\pi^2 \frac{\sigma^2}{m^4} |h_p|^2 = \frac{3}{64\pi} [2J_2 + 12J_2^\phi + 5J_2^\theta] \quad (2.84)$$

Defining the distributions in $m_{\pi\pi}$ as¹³

$$\begin{aligned} \frac{d\Gamma_1}{dx} &:= \frac{d\Gamma}{dx} - 5 \frac{d\Gamma^\theta}{dx} \\ \frac{d\Gamma_2}{dx} &:= 2 \frac{d\Gamma}{dx} - 12 \frac{d\Gamma^\phi}{dx} + 5 \frac{d\Gamma^\theta}{dx} \\ \frac{d\Gamma_3}{dx} &:= 2 \frac{d\Gamma}{dx} + 12 \frac{d\Gamma^\phi}{dx} + 5 \frac{d\Gamma^\theta}{dx} \end{aligned} \quad (2.85)$$

one has up to an irrelevant proportional factor

$$\begin{aligned} \frac{d\Gamma_1}{dx} &= \sigma_x \int dy y \sigma_{xy}^3 |\tilde{f}_p|^2 \\ \frac{d\Gamma_2}{dx} &= \sigma_x^3 \int dy y \sigma_{xy} |g_p|^2 \end{aligned}$$

¹³the indices θ and ϕ indicate the corresponding weighting

$$\frac{d\Gamma_3}{dx} = \sigma_x^3 \int dy y \sigma_{xy}^3 |h_p|^2 \quad (2.86)$$

At this point some assumption has to be made concerning the dependence of the form factors on $m_{e\nu}$ in order to be able to perform the integral. With an expansion like in (2.58), the result for the integral can be written as

$$\begin{aligned} \frac{d\Gamma_1}{dx} &= |\tilde{f}_p|^2 \sigma_x (I_{0,3} + \sum_{k=1}^{\infty} \tilde{\lambda}_{f,k} I_{k,3}) \\ \frac{d\Gamma_2}{dx} &= |g_p|^2 \sigma_x^3 (I_{1,1} + \sum_{k=1}^{\infty} \tilde{\lambda}_{f,k} I_{k+1,1}) \\ \frac{d\Gamma_3}{dx} &= |h_p|^2 \sigma_x^3 (I_{1,3} + \sum_{k=1}^{\infty} \tilde{\lambda}_{f,k} I_{k+1,3}) \end{aligned} \quad (2.87)$$

Unless $\tilde{\lambda}_k \gg 1$, the higher order terms can be neglected to first order; then, since the $I_{a,b}$ are known, the $m_{\pi\pi}$ dependence of the form factors can be studied.

2.4.3 Distributions in $m_{e\nu}$

The distribution in $m_{e\nu}$ could be treated analogously to $m_{\pi\pi}$; however, this will not give any more information, and the analysis of the $m_{e\nu}$ distribution would also suffer from the ambiguity in the neutrino reconstruction (see discussion in appendix, page 193). Therefore, no further discussion is given here.

2.5 Extraction of Form Factor Phases

As already discussed above, the Watson-Fermi theorem [20] states that all form factors of same orbital momentum l with respect to the pions have the same phase; Bose- and isospin symmetry implies that there is no s-wave contribution to F , therefore the leading order is purely p-wave. This results in the same phase for all form factors.

Experimentally, this can be checked looking at an asymmetry in ϕ defined as¹⁴

$$A_\phi(m_{\pi\pi}, m_{e\nu}) := \frac{\int d(\cos \theta_{\pi\pi}) \int d(\cos \theta_{e\nu}) \int d\phi \Gamma_5 \operatorname{sgn}(\phi)}{\int d(\cos \theta_{\pi\pi}) \int d(\cos \theta_{e\nu}) \int d\phi \Gamma_5 \cos \phi \cos \theta_{e\nu}} \quad (2.88)$$

¹⁴we use $\operatorname{sgn}(\phi)$ as weight rather than $\sin \phi$ to make full use of the experimental statistics

with

$$\Gamma_5 := \frac{d\Gamma}{dm_{\pi\pi} dm_{e\nu} d(\cos\theta_{\pi\pi}) d(\cos\theta_{e\nu}) d\phi} \quad (2.89)$$

Using (2.30) one finds

$$A_\phi(m_{\pi\pi}, m_{e\nu}) = \frac{16}{\pi} \tan(\phi_G - \phi_F) \quad (2.90)$$

assuming that the phases do not depend on $\cos\theta_{\pi\pi}$. Experimentally, due to limited statistics, one also integrates over one of the invariant masses to study the (average) dependence of the phase difference on the other.

If the theoretical expectation is right, the experimental results should be compatible with 0.

2.6 Theoretical Predictions for the Form Factors

Everything up to now was studied with minimal assumptions about the properties of the form factors. This section now discusses theoretical predictions.

Theoretical predictions for form factors require a model to calculate the strong interactions in the Ke4 decay. One way is the chiral perturbation theory (χPT). A detailed discussion of χPT is beyond the scope of this work; we refer to [10],[11],[12]. Here we only list the results relevant for neutral Ke4, based on [35].

The current experimental uncertainty on the Ke4 branching ratio is $\approx 5\%$ [34], and it will not be significantly reduced by the results of this work. Thus, in our theoretical discussion, we will neglect contributions smaller than 1%.

2.6.1 Form factors at Tree Level

At leading order one finds, applying the symmetrization rules (2.44)

$$\begin{aligned} F_0 &= 0 \\ G_0 &= \frac{m}{\sqrt{2}F_\pi} = 3.808 \pm 0.011 \\ H_0 &= 0 \end{aligned}$$

with the pion decay constant [34]

$$F_\pi = 92.419 \pm 0.26 \text{ MeV} \quad (2.91)$$

It should be noted that F vanishes at leading order due to the antisymmetrization - this is only the case for the neutral Ke4 decay, charged Ke4 have $F \neq 0$.

2.6.2 1-loop Corrections

Since F vanishes at leading order, it is interesting to find the first non-vanishing contribution. It turns out that also most of the 1-loop corrections cancel due to the antisymmetrization, the only remaining is

$$F_{(1)} = 4G_0^3 L_3^r \frac{\sigma_x}{x} \sigma_{xy} \cos\theta_{\pi\pi} + U_F \quad (2.92)$$

U_F are unitarity corrections discussed in the appendix (page 183). L_3^r is one of the nine coupling constants L_i^r that appear in second order χPT to cope with divergencies. Theoretically one gets from chiral dimensional analysis [10] the approximate upper bound

$$L_i^r \leq \frac{3}{2^6 \pi^2} \approx 5 \cdot 10^{-3} \quad (2.93)$$

since $4G_0^3 \frac{3}{2^6 \pi^2} \approx 1$, it is convenient to define a set of constants c_i of $O(1)$ like this (values taken from [11]):

$$\begin{aligned} c_1 &:= 2G_0^3 c_\pi L_5^r \approx (4.7 \pm 1.7)\% \\ c_2 &:= 2G_0^3 (2L_3^r + 2L_9^r - c_\pi L_5^r) \approx 0.70 \pm 0.37 \\ c_3 &:= 4G_0^3 (L_3^r + L_9^r) \approx 0.75 \pm 0.28 \\ c_4 &:= 4G_0^3 L_3^r \approx -0.77 \pm 0.28 \\ c_5 &:= 4G_0^3 L_9^r \approx 1.52 \pm 0.16 \end{aligned} \quad (2.94)$$

G_1 depends on L_3^r, L_5^r and L_9^r :

$$G_{(1)} = -2G_0^3 [2(1+x^2-y)L_3^r - c_\pi L_5^r - 2yL_9^r] + U_G = -c_4(1+x^2-y) + c_1 + c_5 y + U_G \quad (2.95)$$

U_G again contains unitarity corrections, plus a constant depending on the renormalization scale (C_G in [35]). Note that $\text{Im}U_G \neq 0$, it carries the phase required by unitarity. As shown in the appendix (page 183), the phase comes only from a $\pi\pi$ -loop and is therefore related to the $\pi\pi$ scattering phase shifts.

As also shown in the appendix, the unitarity contributions can be approximated as¹⁵

$$\begin{aligned} U_F &\approx -u_F \cos\theta_{\pi\pi} & u_F &= 0.005 \\ U_G &\approx u_G \frac{\sigma_x^2}{x} + c_G + i \frac{G_0^3}{48\pi} \frac{\sigma_x^3}{x} & u_G &= 0.34 \quad c_G = -0.018 \end{aligned}$$

¹⁵actually the approximation for U_F is an upper bound, but this contribution turns out to be negligible anyway

Since the exact corrections have a complicated structure, we will (for the following discussion) use these approximations; for the final BR calculation, we will compare with the exact expressions for a cross check.

For H the first not vanishing contribution is due to the chiral anomaly [13] and predicted as

$$H_{(1)} = -\frac{G_0^3}{2\pi^2} = -2.797 \pm 0.024 \quad (2.96)$$

2.6.3 Predictions for f_p, g_p and h_p

For the leading order partial wave expansion coefficients one now has

$$f_p(x, y) = c_4 \frac{\sigma_x}{x} \sigma_{xy} - u_F \approx (0.20 \pm 0.06)c_4 \leq 0.32c_4 \quad (2.97)$$

$$g_p(x, y) = G_0 - c_4(1 + x^2 - y) + c_1 + c_5y + u_G \frac{\sigma_x^2}{x} + c_G + i \frac{G_0^3}{48\pi} \frac{\sigma_x^3}{x}$$

$$h_p(x, y) = -\frac{G_0^3}{2\pi^2} = -2.80 \pm 0.02$$

As the 1-loop corrections are roughly of the same order as the tree level, it can be assumed that also higher order corrections have considerable contributions. There are theoretical methods for such corrections [14], and $O(p^6)$ contributions have been extensively studied e.g. in [19]. However, for this work we will not go beyond $O(p^4)$, although we are aware that higher order corrections may significantly change the results.

2.6.4 Prediction for the $m_{\pi\pi}$ Distribution

Inserting the form factor predictions in (2.56) and integrating y one finds that most of the unitarity correction dependent terms contribute with less than 1% and can therefore be neglected. The only non-negligible contribution comes from the interference of U_G with the L_i^r dependent terms in g_p . We also neglect the contribution from h_p since it was already shown above that this contribution is kinematically highly suppressed (see fig. 2.7(a)). The result then is

$$\frac{d\Gamma}{dx} = \Gamma^0 [I_f(x) + 2I_{fg}(x) + I_g(x)] \quad (2.98)$$

where

$$I_f(x) = c_4^2 \frac{\sigma_x^3}{x^2} I_{0,5} = 5c_4^2 \frac{\sigma_x^3}{x^2} [(1+x^2)I_{1,3} - I_{2,3}]$$

$$I_{fg}(x) = c_4 \frac{\sigma_x^3}{x^2} \cdot [(1-x^2)f_{14}(x)I_{0,3} + f_{25}(x)I_{1,3} + c_3I_{2,3}]$$

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$$I_g(x) = \frac{\sigma_x^3}{x^2} \cdot [12x^2 K_{1,1} + K_{0,3}]$$

with the constants c_i as defined above and the definitions

$$f_{ij}(x) := G_0 + c_i - c_j(1+x^2)$$

$$K_{a,b} := (f_{14}(x) + 2u_G \frac{\sigma_x^2}{x})f_{14}(x)I_{a,b} + 2c_3(f_{14}(x) + u_G \frac{\sigma_x^2}{x})I_{a+1,b} + c_3^2 I_{a+2,b}$$

where the last two terms of $K_{a,b}$ can safely be neglected.

The functions $I_w(x)$, $w = f, fg, g$ are shown in fig. 2.7(a) - obviously the I_f contribution is negligible, but there is a small, yet non-negligible, contribution from interference I_{fg} . The $m_{\pi\pi}$ distribution is dominated by the function I_g which has a maximum at $m_{\pi\pi} \approx 330\text{MeV}$.

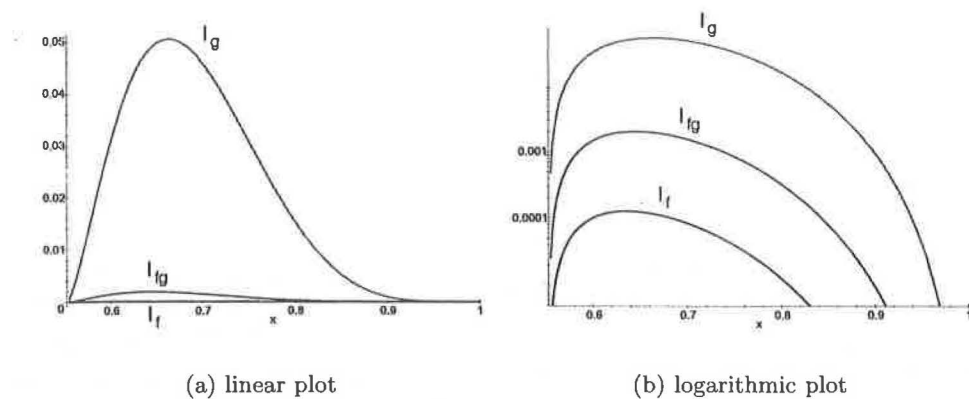


Figure 2.7: shapes of $m_{\pi\pi}$ distributions for form factors f, fg, g and h (to scale; $x = \frac{m_{\pi\pi}}{m}$)

2.6.5 Prediction for the Ke4 Branching Ratio

Integrating over x gives

$$\begin{aligned} c_f &= \int_{x_0}^1 dx I_f(x) = (2.1 \pm 1.3) \times 10^{-5} \\ c_{fg} &= \int_{x_0}^1 dx I_{fg}(x) = (-3.3 \pm 1.4) \times 10^{-4} \\ c_g &= \int_{x_0}^1 dx I_g(x) = (9.5 \pm 1.6) \times 10^{-3} \end{aligned} \quad (2.99)$$

where the errors of course are dominated by the uncertainty on the L_i^r parameters. This gives a predicted branching ratio for the $Ke4$ decay of

$$BR(Ke4)_{\chi PT} = \tau_{K_L} \Gamma = (4.9 \pm 0.8) \times 10^{-5} \quad (2.100)$$

An exact calculation of the BR, including all form factors F, G and H and the exact unitarity correction gives the same result¹⁶.

This is $\approx 5\%$ smaller than the experimental value of [34]

$$BR(Ke4)_{exp} = (5.18 \pm 0.29) \times 10^{-5} \quad (2.101)$$

and a prediction of VMSS (vector meson symmetry scheme) [15]

$$BR(Ke4)_{VMSS} = 5.2 \times 10^{-5} \quad (2.102)$$

However, a closer look at [32] reveals that in that analysis the (negative) contribution of the f_g -interference has been neglected. Without this contribution we get

$$BR(Ke4)'_{\chi PT} = (5.2 \pm 0.9) \times 10^{-5} \quad (2.103)$$

which is within rounding identical with the experimental value as measured by [32]. However, we believe that it is correct to take the contribution from the interference into account. In this context it is important to note that one of the most accurate measurements of the L_3^r parameter is also based on [32]. The uncertainty on L_3^r is large enough to accommodate for a 5% deviation of experimental result and theoretical prediction. We therefore suggest to reverse the problem and use the measured BR to get an improved value for L_3^r that accounts for the interference term:

2.6.6 Determination of L_3^r using the $Ke4$ Branching Ratio

Looking at the dependence of the predicted BR on the L_i^r parameters (fig. 2.6.6) it is interesting to remark that the prediction mainly depends on L_3^r , and only by a negligible part on L_5^r and L_9^r . Even more interesting is that obviously the BR allows a better determination of the L_3^r parameter than the error given in [11]:

Fig. 2.9(a) shows a scatter plot of the predicted BR versus the L_3^r parameter; to generate this plot, the values for L_5^r, L_9^r have been varied obeying a gaussian distribution with the mean values and standard deviations given in [11], while L_3^r was flatly distributed in $\pm 10^{-2}$. As expected from the results above, the BR mainly depends on L_3 and is only little smeared by the uncertainties on L_5 and L_9 .

¹⁶there is a small difference not seen because of rounding; the biggest effect comes from the neglected shift c_G of the unitarity correction U_G – since the shift is negative, it reduces the BR value by a small amount.

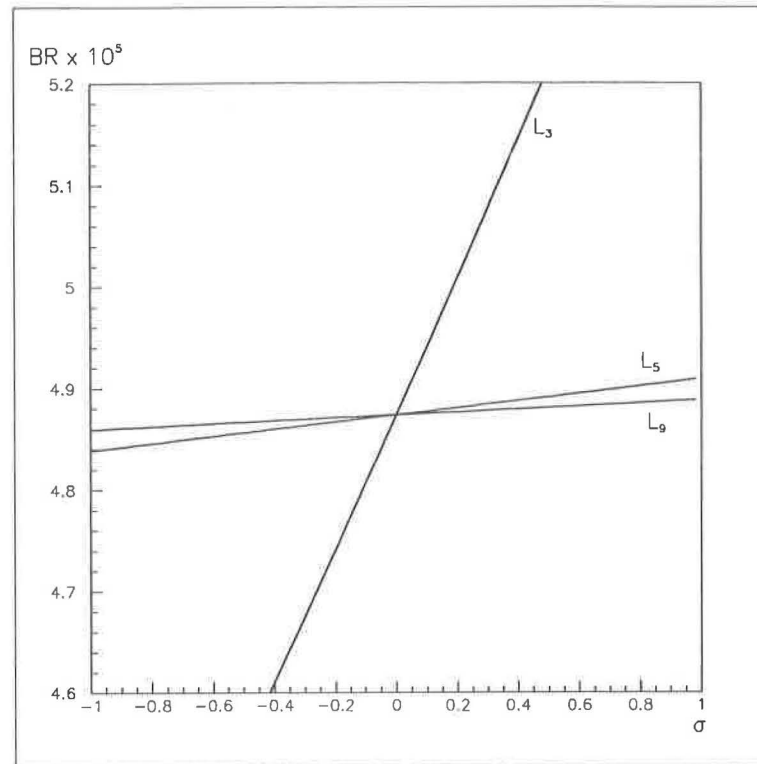


Figure 2.8: dependence of the predicted $Ke4$ branching ratio on the χPT parameters L_3^r, L_5^r and L_9^r ; shown is the variation for each L_i^r in the range $\pm 1\sigma$

We now get a measurement of L_3^r by weighting¹⁷ the data with the ratio of the desired to the observed distribution

$$w = \frac{e^{-\frac{(BR - BR_{PDG})^2}{2\sigma_{PDG}^2}}}{w(BR)} \quad (2.104)$$

The observed distribution for a flat L_3 , $w(BR)$, empirically was found to be $w(BR) \approx \frac{c}{\sqrt{BR}}$.

Applying above weighting one gets the distribution shown in fig. 2.9(b) – thus, the improved measurement of L_3^r is

$$L_3^r = (-4.0 \pm 0.5) \times 10^{-3} \quad (2.105)$$

This differs of course from the result given in [32] (although based on

¹⁷this procedure was chosen because it is non-trivial to invert the equations to get L_3 as a function of the predicted BR; another advantage of this method is that it automatically gives the statistical error on L_3 due to the uncertainty of the measured BR as well as of L_5 and L_9

the same experimental data), but is within errors well compatible with other measurements of L_3 , as e.g. listed in [16], although the mean value seems somewhat lower. As mentioned already above, the inclusion of $O(p^6)$ terms can significantly change the result [19].

Later in this thesis we will give a measurement of L_3^r based on our own experimental data.

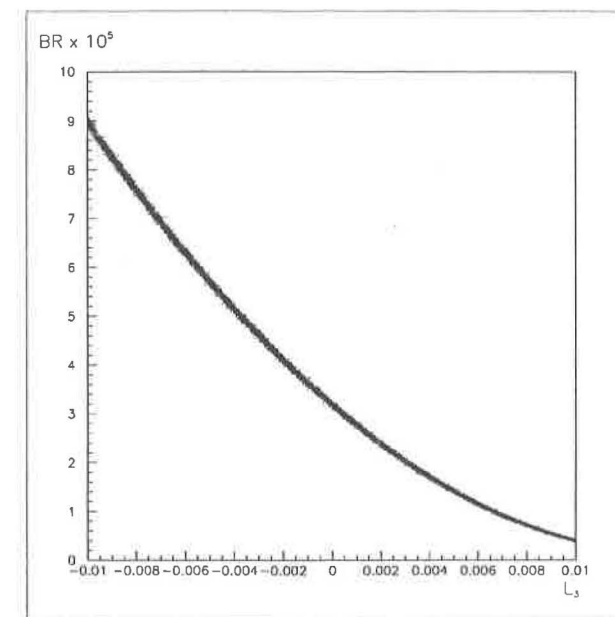
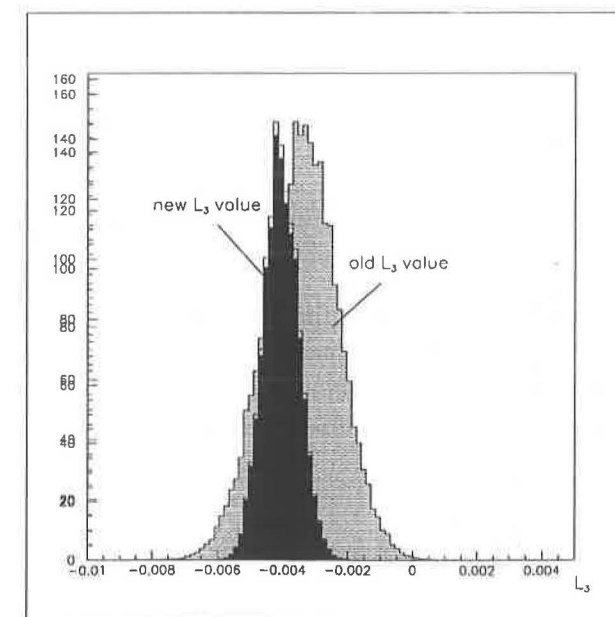
(a) scatter plot of predicted BR vs. L_3^r (b) weighted distribution of L_3^r

Figure 2.9: due to a strong dependence of the neutral Ke4 branching ratio (BR) on the L_3^r parameter, it is possible to improve the L_3^r measurement (see text)

2.6.7 The Slope λ_g in χPT

In the literature ([35], [32], [16], ...) the form factors are usually expanded at threshold $m_{\pi\pi} = m_{\pi^+} + m_{\pi^0} =: 2m_\pi$ as a function of

$$q^2 := \frac{m_{\pi\pi}^2 - 4m_\pi^2}{4m_\pi^2} \quad (2.106)$$

like

$$\begin{aligned} F(m_{\pi\pi}, m_{e\nu}, \cos \theta_{\pi\pi}) &= F(2m_\pi, m_{e\nu}, \cos \theta_{\pi\pi}) \cdot (1 + \lambda_f q^2) \\ G(m_{\pi\pi}, m_{e\nu}, \cos \theta_{\pi\pi}) &= G(2m_\pi, m_{e\nu}, \cos \theta_{\pi\pi}) \cdot (1 + \lambda_g q^2) \end{aligned}$$

The reason for this expansion was mainly experimental observation. Using the χPT predictions from above, one finds

$$\begin{aligned} f_p(2m_\pi, y) &=: f_p^0 = -u_F \approx 0 \\ g_p(2m_\pi, y) &=: g_p^0 = G_0 - c_4(1 + c_\pi - y) + c_1 + c_5 y + c_G \approx 5.044 \pm 0.071 \end{aligned} \quad (2.107)$$

if one now tries to calculate λ_f , one finds that it is not defined because f_p goes like $\sqrt{q^2}$ for $q^2 \rightarrow 0$. For λ_g one finds

$$\begin{aligned} \lambda_g &= \frac{1}{g_p^0} \frac{dg_p}{dq^2} \Big|_{q^2=0} = \frac{1}{g_p^0} \frac{2m_\pi^2}{m^2 x} \frac{dg_p}{dx} = \\ &= \frac{1}{g_p^0} \frac{2m_\pi}{m} u_G - \frac{G_0}{g_p^0} \frac{8m_\pi^2}{F_\pi^2} L_3 = \frac{G_0}{g_p^0} [0.049 - \frac{8m_\pi^2}{F_\pi^2} L_3] \approx 0.0896 \pm 0.0067 \end{aligned} \quad (2.108)$$

This is only equivalent to the result of [16] (used in [32]) if one assumes $g_p^0 \approx G_0$, which is not really the case since the 1-loop corrections to G_0 cannot be neglected. Additionally, the first term for λ_g corresponding to the unitarity corrections also gets L_3 dependent (because of the normalization to g_p^0).

2.7 Summary and Conclusion

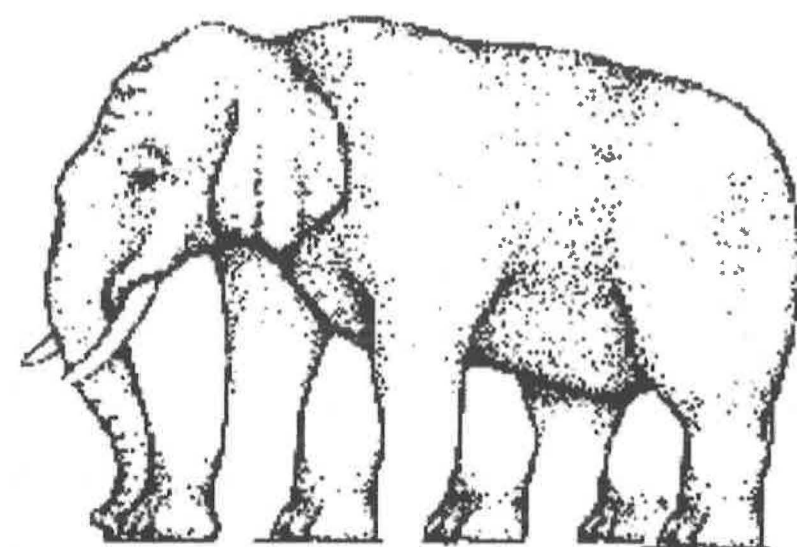
It has been shown that the neutral Ke4 decay has some theoretically very interesting properties: Because of Isospin- and Bose-symmetry, the pions are primarily in a p-wave state, which suppresses the influence of the F form factor on the decay width. The R form factor is unobservable because of $m_e \approx 0$ and the H form factor is negligible for kinematical reasons. Thus, the neutral Ke4 decay is dominated only by the G form factor, although it was shown that also an interference term with F has a non-negligible effect.

For an experimental study of the form factors, a method based on weighted means over two Cabibbo variables ($\theta_{e\nu}$ and ϕ) has been presented that allows to decouple the form factors to study each one separately, independent of specific theoretical assumptions. This allows to cross-check theoretical predictions as well as certain assumptions that have to be made.

Chiral perturbation theory (χPT) was used to give a theoretical prediction for the form factors. This results in a theoretical branching ratio that is very sensitive to the coupling parameter L_3 of χPT (this is a unique feature of the neutral Ke4 decay, the charged Ke4 decay depends on a mixture of several coupling parameters L_i). It is therefore possible to measure cleanly L_3 with considerably higher accuracy than with other decays.

Part II

Experiment



4 or 5?

Chapter 3

The NA48 Experiment

This chapter describes the NA48 experiment at CERN that delivered the data for this Ke4 study. After some general information of NA48's purpose and general concept, the detector is described; no attempt is made to give a full technical description of the detector, since there is much literature about that [23]. Instead, we give an overview of the experiment and its detectors as far as relevant for this analysis.

The NA48 experiment is located at the European Laboratory for Particle Physics, CERN near Geneva, Switzerland. Its primary goal is the high precision measurement of the direct CP-violation parameter ϵ' . It uses a proton beam provided by CERN's SPS (Super Proton Synchrotron).

NA48 is a collaboration of Austria, England, France, Germany, Italy and Poland. First work on NA48 started in 1991, first data was taken in 1996. Full data taking took place in 1997-2001¹. Ke4 data was taken in 1999 and 2001, this thesis is based on 1999 data only.

3.1 General Concept of NA48

The NA48 experiment is the consequence of the experiences gained with its predecessor NA31, which was CERN's first attempt in measuring direct CP violation. The main difficulty of this measurement, whose desired accuracy requires a high statistics experiment, is to control the systematics. So one problem with NA31 was to correct for differences between the K_S and K_L beams produced at different times (only one at a time).

NA48's approach is to cancel as many systematic effects - at least at first order - as possible. So K_S and K_L beams are produced simultaneously from

¹due to a fire accident at SPS, the '97 run lost two months; the run 2000 was hampered by an accident destroying the drift-chambers that happened in late 1999 - as the chambers could not be replaced in time, this run was done without spectrometer

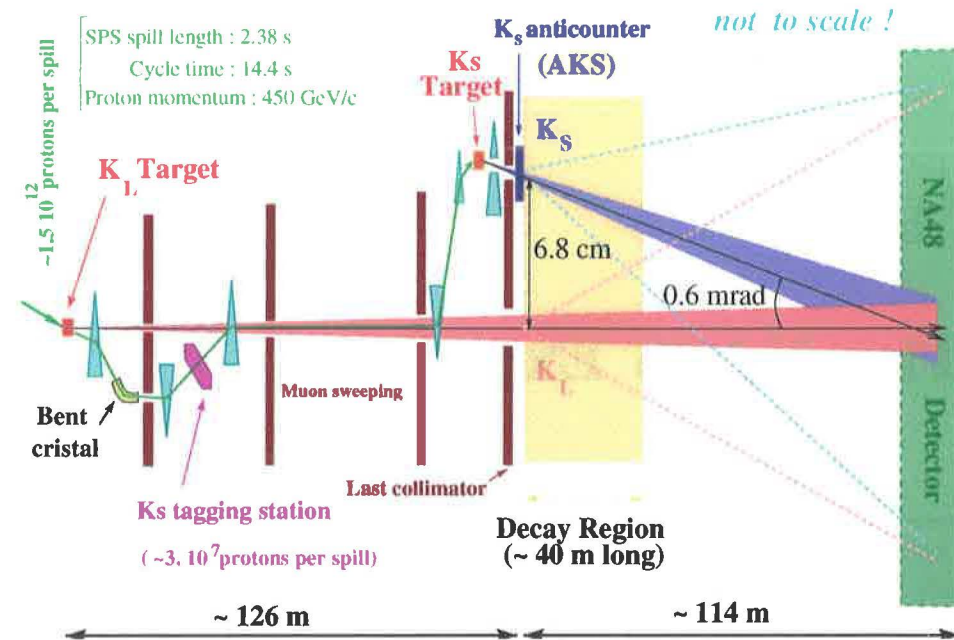


Figure 3.1: The dual beams concept of NA48

the same proton beam, thus cancelling variations in intensity. The setup can be seen in figure 3.1: there are two targets, the so-called K_L target that is far enough from the fiducial decay region to ensure that the K_S -component has vanished, and a second K_S target immediately before the fiducial decay region.

The proton beam from the SPS ($1.5 \cdot 10^{12}$ protons per burst at 450 GeV) first hits the K_L target at an angle of 2.4 mrad. The non-interacting protons are deflected to a bent crystal which attenuates the beam finally hitting the K_S target such that the rate of (CP-conserving) $K_S \rightarrow 2\pi$ decays is comparable to (CP-violating) $K_L \rightarrow 2\pi$ decays.

To identify the origin of a decaying kaon, a tagger system [30] is used that consists of an array of scintillators shortly after the bent crystal and tags all protons going to the K_S target. This allows, after correcting for approx. 10% of K_L that accidentally coincide with a tagger hit (the so-called *dilution*), to distinguish K_S from K_L decays.

The high rate of kaon decays in NA48 requires a detector and trigger with no or very small dead time and high time-, space- and energy-resolution. Both the detector and trigger are described in the following sections.

3.2 The NA48 Detector

3.2.1 Overview

The composition of the NA48 detector reflects the need for a detector for neutral $K \rightarrow \pi^0\pi^0$ events on one hand, and charged $K \rightarrow \pi^+\pi^-$ events on the other. The former is detected in an electro-magnetic calorimeter filled with liquid Krypton (LKR), the latter with a spectrometer consisting of a magnet and four drift-chambers (DCH). Supplementary detectors are a hadron calorimeter (HAC) located just behind the LKR, and a muon detector used as veto (MUV). There are also charged and neutral hodoscopes (HOD, NHOD).

The spectrometer is separated from the decay region (vacuum $< 10^{-4}$ mbar) by a Kevlar window and is filled with Helium. To avoid beam-gas scattering in the kaon beam, there is a beam pipe going through the spectrometer.

The most important parts of these subsystems (from the point of view of K_{e4} analysis) are described in some more detail in the following subsections.

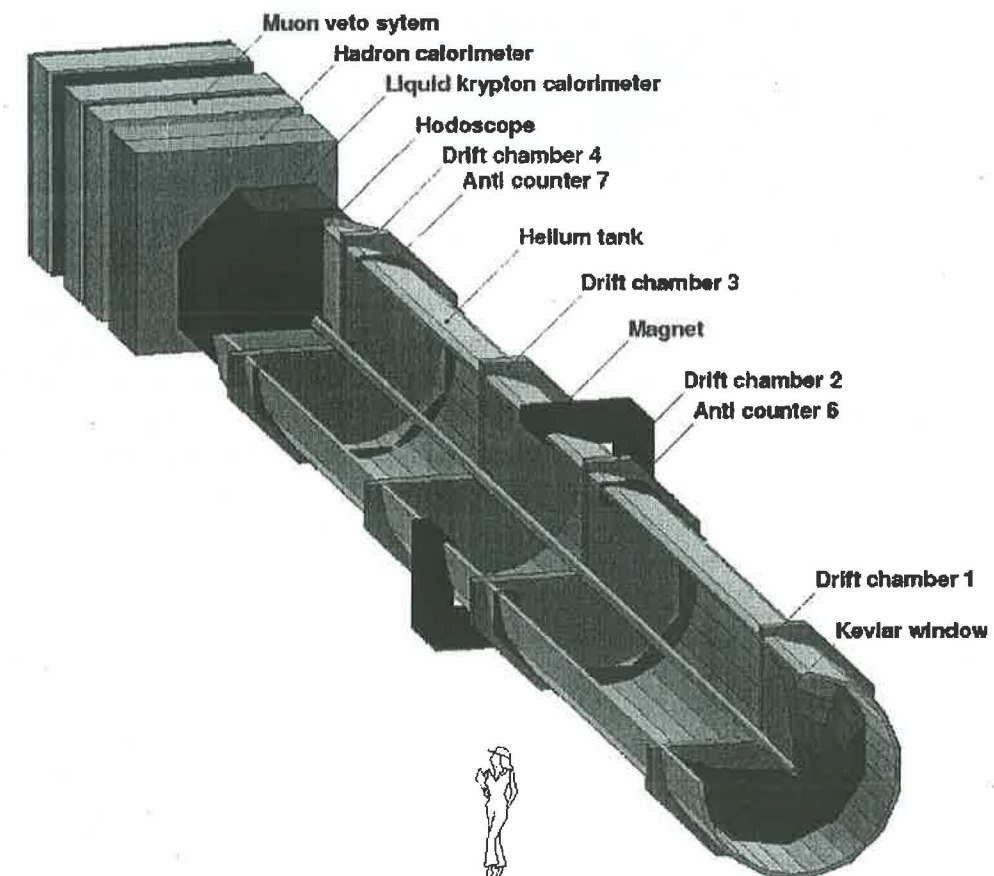


Figure 3.2: The NA48 detector

3.2.2 The Magnetic Spectrometer DCH

The design of the chambers is such as to allow high precision measurements; hits can be reconstructed with a resolution of $\approx 100\mu m$. Each octagonal-shaped chamber consists of 8 planes of sense wires, oriented in four views. Each view has two staggered² planes of 256 wires. The chambers are filled with an Argon/Ethane mixture, the volume between the chambers is filled with Helium.

A precise knowledge of the geometry and position of the DCH is required to achieve a relative precision of 10^{-4} .

The magnet has a field integral of .83 Tm with a current of 1200 A; the polarity is changed in regular intervals to avoid and check for systematics. The magnetic field leads to a kick of the transversal momentum of the charged particles of 265 MeV/c along the x-axis.

The momentum resolution of the DCH is given by

$$\frac{\sigma_P}{P} = 0.0048 \oplus \frac{0.00009P}{\text{GeV/c}} \quad (3.1)$$

where the first term is due to multiple scattering and the second originates from the position resolution of the chambers.

3.2.3 The Electromagnetic Calorimeter LKR

The LKR [28] is a quasi-homogeneous electromagnetic calorimeter filled with liquid Krypton as active medium (for its small Molière radius of 4.7 cm as compared to e.g. 7.2 cm for Argon, leading to well-contained showers). The depth of 127 cm of the calorimeter corresponds to 27 radiation lengths, transversally it is divided by electrodes in 13212 cells³, $2 \times 2 \text{ cm}^2$ each.

The whole calorimeter is enclosed in a vacuum insulated cryostat, which leads to a small energy loss before the particles enter the active volume. The signals from the electrodes are pre-amplified inside the cryogenic system to minimize noise.

The energy resolution of the LKR is given by

$$\frac{\sigma_E}{E} = \frac{3.2\%}{\sqrt{E/\text{GeV}}} \oplus \frac{100 \text{ MeV}}{E} \oplus 0.5\% \quad (3.2)$$

$$\sigma_t \approx 260 \text{ ps} \quad (3.3)$$

$$\sigma_{x,y} \approx 1 \text{ mm} \quad (3.4)$$

²to resolve left-right ambiguities

³that is 128x128 minus some cells missing in the corners due to the octagonal shape

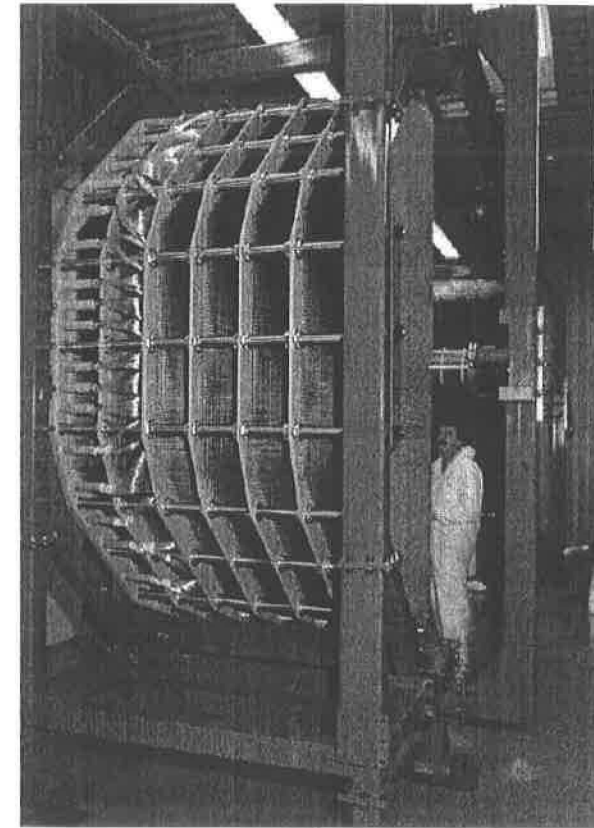


Figure 3.3: the electromagnetic calorimeter, before filled with liquid Krypton (LKR)

3.2.4 Veto Systems (MUV, AKL, AKS)

There are three major veto systems relevant to Ke4: The *muon veto system* (MUV) and the *AKL and AKS anticounter systems*. The muon veto system is intended to reject events containing a muon and is useful for Ke4 to reject $K\mu 3$ background. The AKL is used to reject events when there are hits outside the fiducial region of the decay tube; the AKS is located behind the K_S target and is used to define the begin of the fiducial decay volume, and is also important for the energy calibration.

For Ke4, it is useful to reject events where particles may be lost. So the logical *NOT* of MUV and AKL is used, symbolized by $\overline{\text{MUV}}$ resp. $\overline{\text{AKL}}$.

The efficiency of the MUV has been studied in [36] and was determined to be $99.76 \pm 0.017\%$. The AKL turned out to be less efficient, and in the end was not used for the ϵ'/ϵ analysis. Unfortunately, it was part of the trigger used to collect Ke4 data, and thus caused some problems that will be discussed later.

3.2.5 The Clock System

Being a high-rate experiment, NA48 has to rely on an accurate clock system to be able to relate signals from different detectors to a specific event. The clock frequency is 960 MHz, but only the tagger system operates with this speed; the other detectors use a downscaled signal of 40 MHz. The delay between the different detector subsystems is carefully measured and corrected for, so that eventually each bit of information gets a proper time stamp.

3.3 The NA48 Trigger System

A high-rate experiment also requires a sophisticated trigger system with ideally no dead-time, very high efficiency (as similar as possible for charged and neutral decays) and selectiveness against background (mainly three body decays) to obtain manageable amounts of data.

It is divided in the so-called *charged trigger* for $K \rightarrow \pi^+\pi^-$ and the *neutral trigger* (NUT) for $K \rightarrow \pi^0\pi^0$, which are both described in more detail below. A trigger supervisor (TS) [27] combines their results and issues a final trigger decision.

There is also a third stage of trigger (L3) which is a pure software trigger that is in principle able to reconstruct the events more accurately than the charged trigger and to reject bad events before they are permanently stored on tape; however, to be on the safe side, L3 was never used in this "cutting mode", but only marked events which passed its filters.

3.3.1 Charged Trigger

The charged trigger [31] is designed as a 2-level trigger; at first level (L1) coincidences can be built of signals from the hodoscope, the drift-chambers, and the total energy (ETOT) from the two calorimeters (LKR and HAC).

The second level (L2) consists of the so-called massbox (MBOX), an array of fast processors that are able to perform a simple online reconstruction of events (i.e. calculate vertices and invariant masses).

3.3.2 Neutral Trigger

The neutral trigger (NUT) [29] is based on LKR information and is fully pipelined, which means that there is a trigger decision for each time slice⁴.

Since the number of cells in the LKR is much too large to be all processed by a trigger that fast, two projections of the signal in X- and Y-direction are built, each with a resolution of 64 stripes. From this information the NUT determines the number of peaks and the first three moments of the

⁴of course, this decision is only available some time slices later, but still there is one decision every 25 ns

energy distribution; using these moments and the 2π -hypothesis for the decay, it subsequently also calculates the longitudinal vertex position, the kaon energy and lifetime and the center of gravity. All this information may be combined in a freely programmable way to derive a trigger decision.

3.4 NA48 Data Acquisition and Processing

The NA48 Central Data Recording Infrastructure

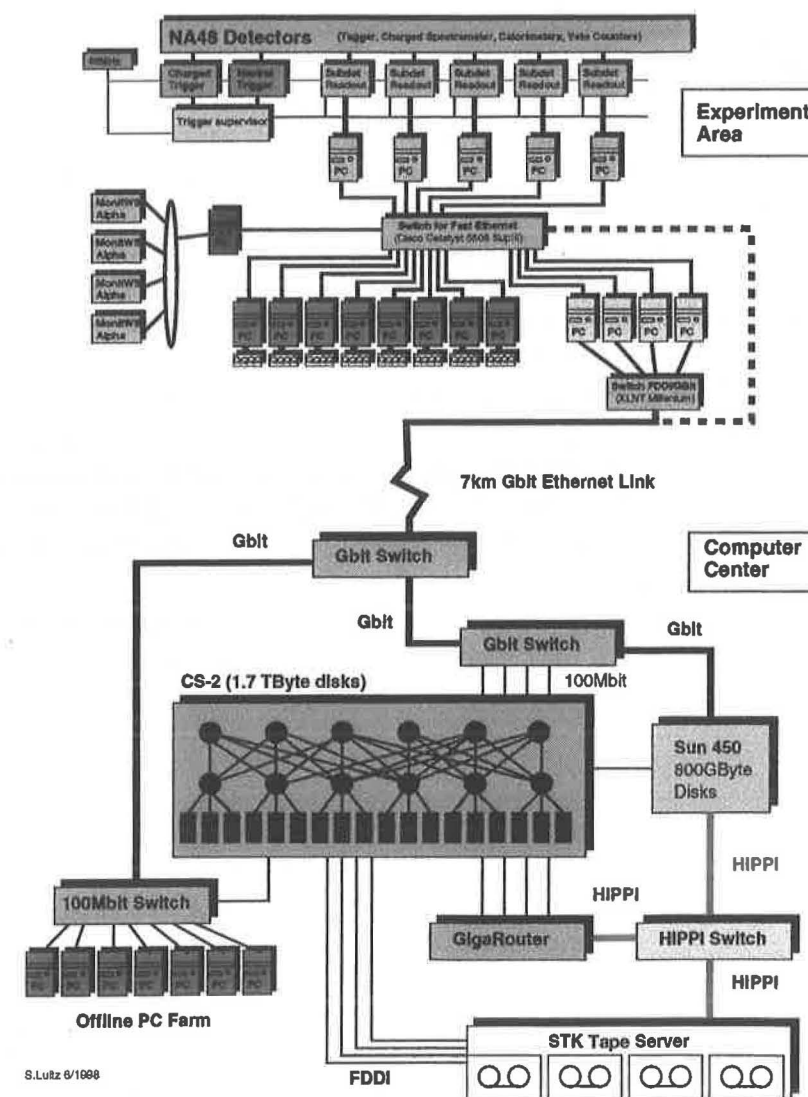


Figure 3.4: the NA48 data acquisition system as in 1997; the CS2 has later been replaced completely by a PC Farm

The NA48 data acquisition system (DAQ) was continuously upgraded to cope with the increasing amounts of data (e.g. 40 TB in 1998, 100 TB in 1999). It uses a farm of industry-standard PCs which are divided in sub-detector PCs responsible to read-out a specific subdetector, and a number of event-building PCs that collect the data from the various subdetector PCs. The completed events are transmitted via a gigabit ethernet link to the central data recording system (CDR) which is located 7 km away on the opposite side of the SPS accelerator where the data is buffered on large disks to finally be written to tape.

The raw data, which contains all the digital information from the detectors, is processed to so-called *compact* data, where the detector information has already been partly transformed to physical information (e.g. the number of clusters in the LKR and their energies instead of the digitized image of the clusters); also some calibrations and corrections are performed at this stage. This processing is done in parallel with the data-taking run, but is also repeated later when calibration constants and all corrections have been finalized (the *reprocessing*).

Since the size of *compact* data is still very large, and most analyses do not need the full amount of information provided by it, there is also a second step of processing to *super-compact* which mainly keeps purely physical information. Initially, *super-compact* was only foreseen for the e'/e -analysis, but later it was extended to be used also for other, mainly rare decay studies.

This Ke4 analysis was first started using *compact* data, then switched to *super-compact*. This way, the (already filtered) data could be copied to a single PC's hard disk, which provided much faster (and independent) access.

Chapter 4

The Ke4 Trigger

This chapter describes the development and implementation of a dedicated Ke4 Trigger. It also gives the results of the efficiency measurement and explains the difficulties involved

4.1 Trigger Development

When developing a trigger, several technical restrictions have to be faced. The main technical restriction is time - you have to decide in a few nanoseconds whether to take a close look at a detected event or not, consequently you can only apply some rather simple selection criteria. But too simple trigger conditions won't be selective enough, so you also have to apply some less simple criteria after the most simple criteria are passed - so you end up with a 3-level trigger as described above (cf. chapter 3.3, page 68). In the following, the Ke4 trigger conditions at each level are discussed.

4.1.1 Pretrigger

As "root" trigger, $Q2$ is used, which is defined as two hits in the hodoscope (corresponding to two charged particles). It is combined with the requirement of no hits in the K_L -Anticounters (AKL) and no hits in the muon veto (MUV). To reduce the rate for the higher levels of the trigger, the pretrigger is downscaled by a factor of 6:

$$Q2 * \overline{AKL} * \overline{MUV} / 6 \quad (4.1)$$

4.1.2 Trigger Level 1 - L1

One advantage of Ke4 is that it both contains charged and neutral particles, which allows to make full use of the NA48 trigger capabilities. The trigger

consequently consists of a charged and neutral part, the logical *AND* of which is the final trigger L1 decision.

Charged Trigger

At L1, this consists solely of the *2tracks*-condition of the spectrometer (two distinctive tracks of charged particles seen in the drift-chambers).

Neutral Trigger

The neutral part of the L1 trigger should be sensible to the π^0 which is detected in the LKR as two clusters, corresponding to the two gammas the π^0 decayed to¹. A third cluster in the LKR corresponds to the electron, and with some probability also the pion leaves a fourth cluster. So the natural trigger condition would be to require three or more LKR clusters.

However, there are some complications. It is not possible to trigger on the actual number of clusters, because the determination of this number is a far too complex task for such a fast trigger. Instead, as described above (cf. chapter 3.3, page 68), *projections* of the clusters in *x*- and *y*-direction are computed in the neutral trigger, and it is possible to trigger on the number of peaks in these projections.

The complication now is, that with some probability overlays may occur in one or both projections, so that not all clusters are seen in the projections. So events with actually 3 clusters may only show 2 peaks. Requiring 3 peaks in any projection (*x* or *y*) will reject some Ke4 events where clusters happen to be too close in both projections. On the other hand, requiring just 2 peaks would increase background dramatically (mainly Ke3, see discussion of background below).

So it has to be checked how many Ke4 events are lost. The plots (see fig. 4.1, page 73) show the number of events as a function of the number of *x*- and *y*-peaks for MC data of Ke4 and some major background modes². Obviously, loosening the condition to 2 peaks is no option; the 3 peaks condition rejects the events marked in the plots.

As the plots show, the 3peaks cut is quite effective to reduce Ke3 and $K\mu 3$ background, leaving only $K\pi 3$.

Besides the conditions above, it is also necessary to apply a constraint on the minimum energy seen in the LKR in order to reject low-energy accidentals. Additionally, a carefully chosen minimal energy value can improve the signal/background ratio while not introducing much bias, because the major background modes tend to leave less energy in the LKR (the $K\pi 3$ because of the lower percentage of energy a pion leaves in the LKR, and Ke3

¹the mean life time of a π^0 is much too short to be observed itself

²the number of peaks for an event is computed assuming a resolution of the neutral trigger projections of 10 cm. This number has been chosen from experimental experience

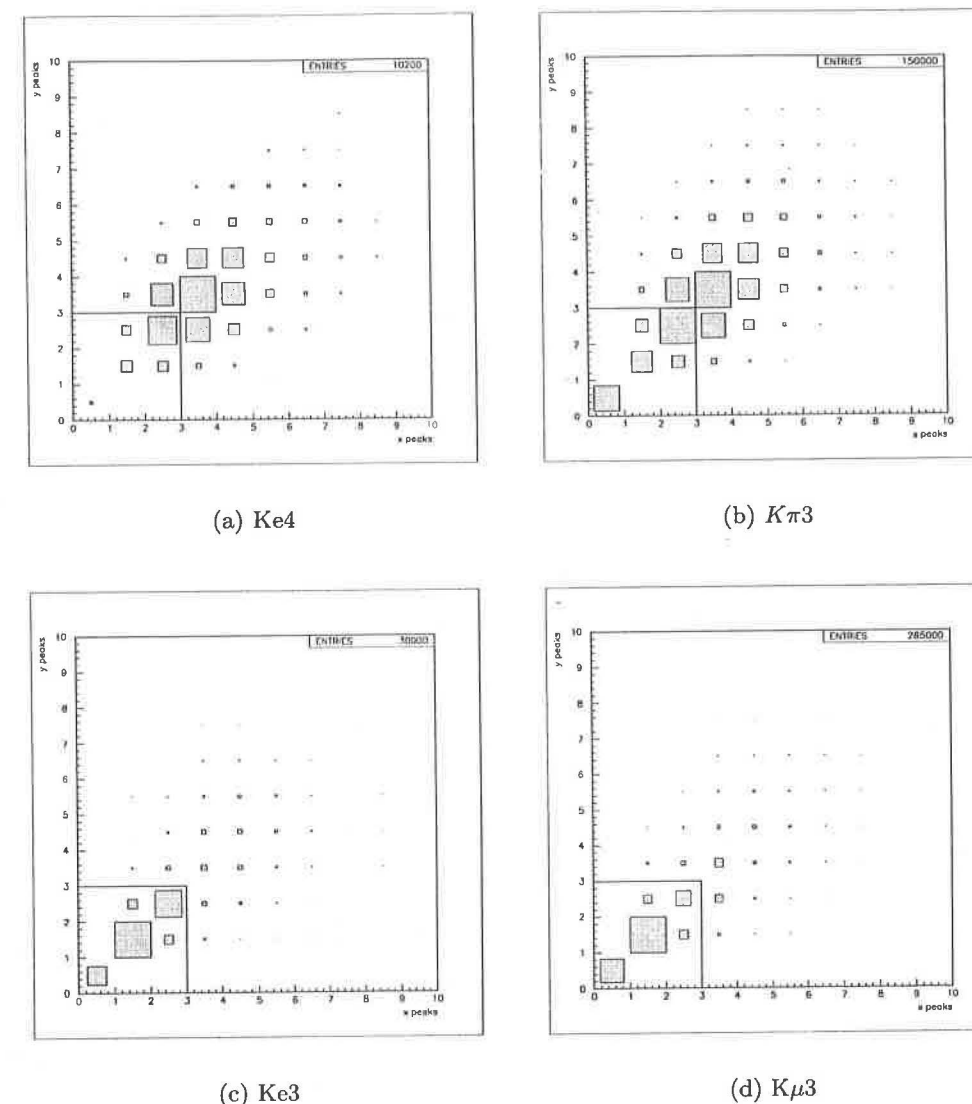


Figure 4.1: Peaks in *x* vs *y* projection for MC-Ke4 and some major background modes (the boxes indicate the 3 peaks condition)

because more energy is lost to the neutrino).

Plot 4.2 shows the ratio of acceptances for Ke4 and the major background modes.

Obviously, the signal/background ratio improves up to rather high cut values. So the choice for optimal E_{LKR} is not determined by best background suppression, but by a compromise between this and Ke4 acceptance (which of course drops at higher values, reducing the total number of Ke4 events).

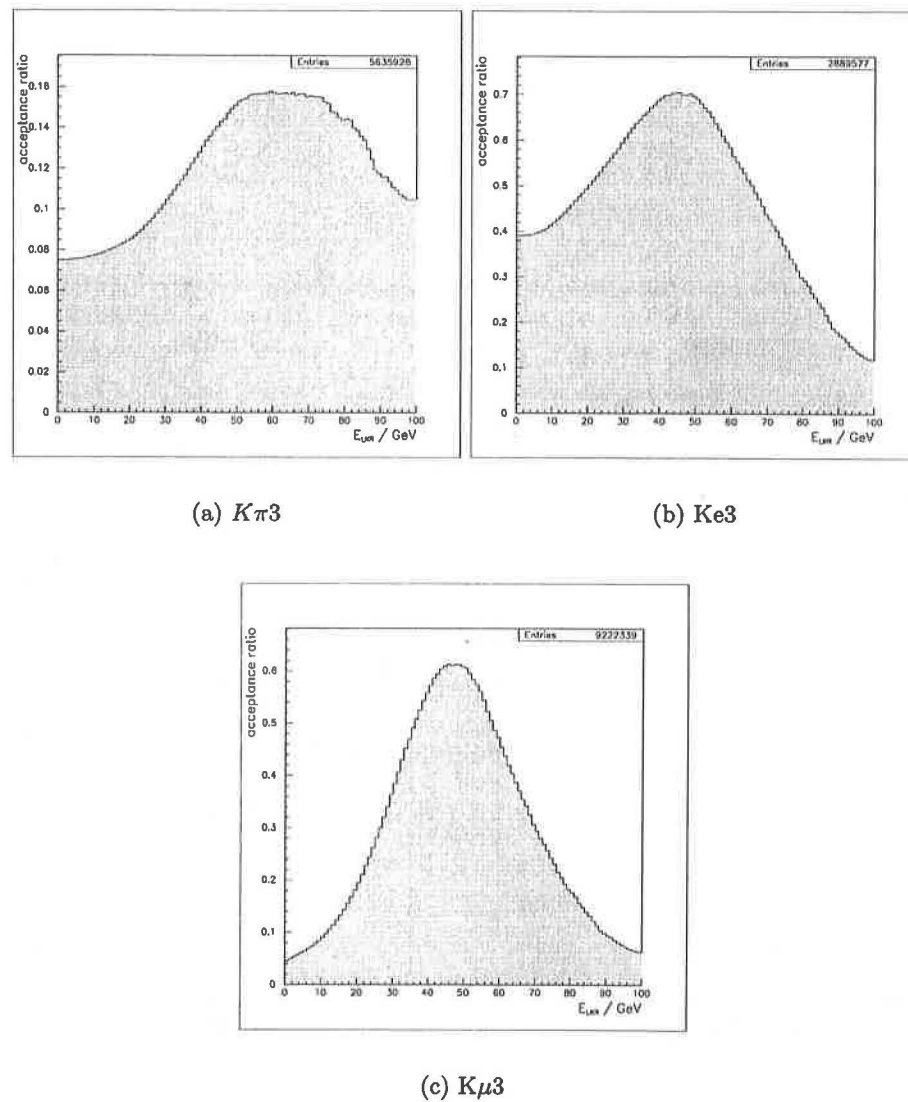


Figure 4.2: acceptance ratios for major backgrounds as a function of E_{LKR} cut value

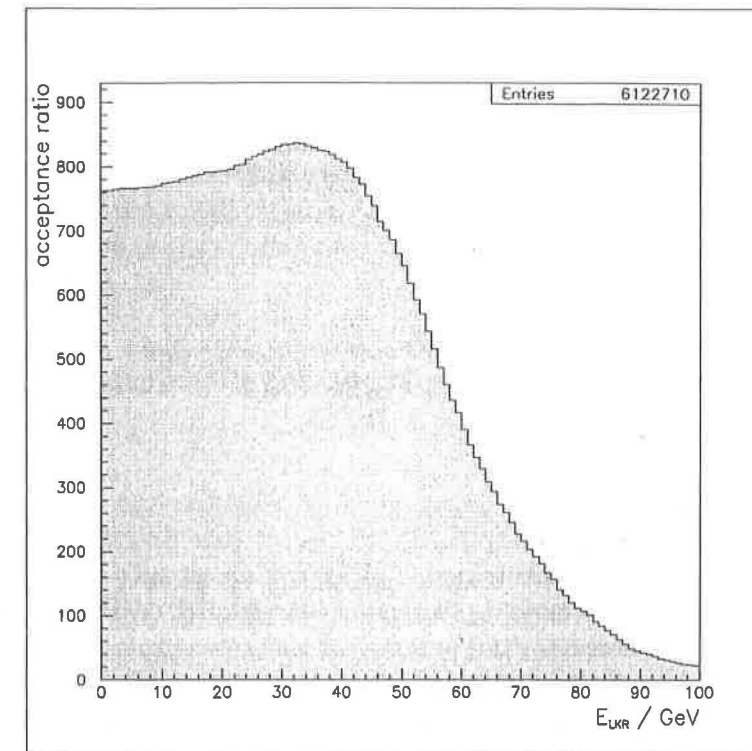


Figure 4.3: acceptance-weighted acceptance ratio for $K\pi3$ background as a function of E_{LKR} cut value

As far as $Ke4$ acceptance is concerned, one may try to optimize the cut value by looking at an acceptance-weighted ratio³ as a measure. For the most dangerous $K\pi3$ background, the result is shown in fig. 4.3.

As the best compromise, and considering the fact that the trigger cut value should be somewhat lower than the final analysis cut, a value of $E_{LKR} = 25$ GeV is chosen.

Summary L1 Trigger

The complete L1 trigger now is:

$$2tracks * KE4NT \quad (KE4NT = 3peaks * ELKR > 25) \quad (4.2)$$

4.1.3 Trigger Level 2 - L2

At second trigger level L2 it is possible to calculate invariant masses, lifetimes and vertices for the charged particles with the *massbox* (MBX). Thus

³that is, $\frac{acc_{Ke4}}{acc_{bg}} \times acc_{Ke4}$

events with no definite charged vertex or invariant masses out of the range expected for Ke4 can be rejected.

Since it turned out that no substantial fraction of events could be rejected using a mass-cut at trigger level, this possibility was not used in order to keep the trigger simple. However, this trigger was used to select events with good vertices.

Summary L2 Trigger

The internal algorithms of the massbox are not given here, we refer to [26]. After the massbox trigger a second downscaling of 10 is applied:

$$\text{KE4MB}/10$$

4.1.4 Trigger Level 3 - L3

L3, being a purely software trigger and having an already highly reduced input rate, can do some more sophisticated calculations, combining information from different detectors and performing a more accurate reconstruction.

So the number of gamma candidates can be determined combining drift-chamber and LKR information, and also a loose cut on the resulting π^0 mass can be included.

Summary L3 Trigger

To summarize, the cuts applied at L3 are:

- basic quality cuts (minimal distance of clusters, ...)
- at least 2 gamma candidates
- m_{π^0} between 115 - 155 MeV

4.2 Trigger Implementation and Rates

The trigger as described above was implemented in the experiment in early June 1999. The typical rates of the trigger were

$$\begin{array}{ll} 120 \text{ Ke4-Triggers/burst} & \text{at L2} \\ 40 \text{ Ke4-Triggers/burst} & \text{at L3} \end{array}$$

4.3 The ETOT Trigger

For efficiency measurements, an independent reference trigger is needed. For this purpose, the ETOT trigger was chosen, a trigger already in use for the ϵ'/ϵ measurement. As a minimum-bias trigger, it is quite simple:

$$\text{ETOT: } 2\text{tracks} * (\text{ETOT} > 30 \text{ GeV})$$

where ETOT is the sum of LKR- and HAC-energy. Obviously, this trigger overlaps highly with the Ke4 trigger. However, also this trigger is downscaled by a factor of 60 (same as the Ke4-trigger), so overlaps in the recorded data are roughly reduced by this factor. Nevertheless, the ETOT trigger can not only be used as a reference trigger, but also to get some additional Ke4 data.

4.4 Trigger Efficiency

4.4.1 Introduction

The determination of the trigger efficiency is both important and demanding. Important, because trigger conditions like *3peaks* might introduce efficiency differences between the Ke4 decay mode and the reference decay mode $K\pi^3$. And this task is demanding because statistics are naturally rather small in the case of the Ke4 mode (both the Ke4 trigger and the reference ETOT trigger are downscaled by a factor of 60, which substantially reduces the number of overlapping events).

Although some of the studies can be done with the $K\pi^3$ events alone (where much more events are available), the rather big error margins of Ke4 trigger efficiency enter unavoidably into some final results.

It is also an important point that the efficiency of the ETOT trigger is expected not to differ significantly for Ke4 and $K\pi^3$ modes⁴.

4.4.2 Method and Problems of Efficiency Measurement

The method for measuring the Ke4 trigger efficiency is quite straightforward: first, ETOT events of the respective decay mode are selected using the cuts described later (cf. chapter 8.1.1, page 103). Then, the percentage of those events which have also been selected by the Ke4 trigger is determined.

The problems are in the details:

First, only every 60th event which would have been selected by the Ke4 trigger is actually taken (because of the downscaling). Of course, the downscaling is known and the result can therefore be corrected in respect of that, but the number of events is simply too small this way. Fortunately, there is some additional information about the trigger in the so-called *pattern-unit* (PU), which records the state of all triggers for each event, no matter which trigger caused the event to be saved. So looking at those bits allows to find out whether the Ke4 trigger fired for a certain event or not, independent of the downscaling applied by TS (trigger supervisor). However, since also the

⁴the ETOT trigger is a *minimum-bias* trigger whose efficiency should hardly depend on the differences between the two decay modes

Ke4 pretrigger downscales by a factor of 6 *before* the information enters the pattern unit, only every 6th event can be selected using PU information, for 5 out of 6 events it is not known whether the trigger was efficient or not.

A second problem is rather hidden and was only discovered due to contradictory cross-checks: the ETOT-triggered events are read-out with a different (higher) energy threshold for the LKR cells than the Ke4 triggered events. This means that the coincidence of a Ke4- and an ETOT trigger is not totally independent of the LKR cell energies, but that the triggers are correlated in the sense that for small energies there is a higher probability for a coincidence than for large energies (because the smaller the energy, the higher the probability that the event has only been read-out because there was a coincident Ke4 trigger, leading to a low threshold - when there is no coincident Ke4 trigger, the high threshold is applied, too few cells are read-out and the event is lost during selection).

To demonstrate this effect, the downscaling factor for Ke4 L2 (which was set to 10) has been measured as a function of e_{LKR} which is the *minimal LKR cluster energy*. To do so, simply the number of events which have the Ke4-bits in the PU set are divided by the number of events which have the Ke4-trigger bit set. As there is no doubt that every 10th event with the PU-bits set leads to a Ke4 trigger, the result should be a flat line at 10. The result is quite different (see fig. 4.4).

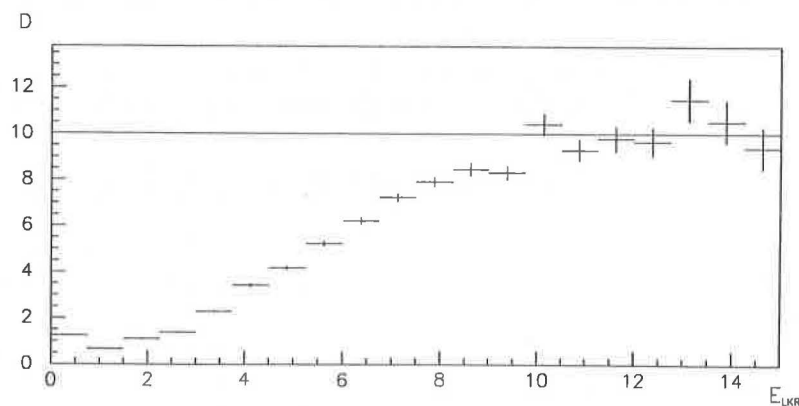


Figure 4.4: Measurement of the downscaling for Ke4 triggered events as a function of minimal LKR cluster energy e_{LKR}

Obviously the downscaling decreases to 1 for small energies, which means that every ETOT event was also selected by the Ke4 trigger (and only for that reason was read-out with the lower LKR threshold and hence could be reconstructed). Unfortunately, the effect does not vanish rapidly with higher LKR energies, but gets smaller rather slowly. Only for values of e_{LKR} larger than 10 GeV, the correct result of downscaling 10 is reached.

So one solution would be to cut on e_{LKR} and allow only events with $e_{LKR} > 10$ GeV. But as most of the events have energies smaller than that, this is not practicable.

The method of choice is to apply a *correlation correction* by explicitly excluding events with the Ke4 trigger bit set. This way, one can be sure that only the higher threshold was applied. Of course, one has to correct for the excluded events. Fortunately, this is not too difficult, since the downscaling factor is known to be 10, so obviously 1/9 of all selected Ke4-PU events have to be added both in the numerator and denominator when calculating the efficiency:

$$\epsilon_{uncorr.} = \frac{N_{PU}}{N_{ETOT}} \rightarrow \epsilon_{corr.} = \frac{\frac{10}{9}N_{PU}}{N_{ETOT} + \frac{1}{9}N_{PU}} \quad (4.3)$$

An important cross-check of this method is to compare the corrected results with the uncorrected ones when applying an $e_{LKR} > 10$ GeV cut.

In the following subsection, the results of both methods are compared.

4.4.3 Results of Efficiency Measurement

Efficiency for the Ke4 and $K\pi^3$ Decay Modes

The efficiency measurement is based on ETOT triggered events. To select Ke4 or $K\pi^3$ events, the cuts described later (cf. chapter 8.1.1, page 103) are applied. For the case of $K\pi^3$ events, the measured efficiencies have also been plotted in dependence of the minimal cluster energy e_{LKR} in fig. 4.5.

As expected, the uncorrected measurement shows increasing⁵ efficiencies for smaller energies due to the correlation between the triggers as explained above. It levels off for energies above 10 GeV, hence to determine the true⁶ efficiency the cut $e_{LKR} > 10$ GeV is applied.

The correlation corrected efficiencies show a much flatter behaviour; however, instead of an increase, the efficiencies decrease for energies below 4 GeV. The reason for this is the fact that for this measurement only events read-out with high energy threshold are used. However, the Ke4-trigger data is read-out with the lower threshold, so the correct value for the efficiency is determined by applying a $e_{LKR} > 4$ GeV cut.

The final results for the efficiencies for both methods are shown in the table 4.1. Obviously there is good agreement between both methods. As the correlation corrected method can make use of considerably more statistics, these results are used.

⁵in fact the values even exceed the theoretical limit of 1; this is a strong hint for correlation in the triggers

⁶"true" here means the efficiency of the low-threshold Ke4 trigger, which can only be measured by the high-threshold ETOT by restraining the sample to high energies where the thresholds do not bias the results

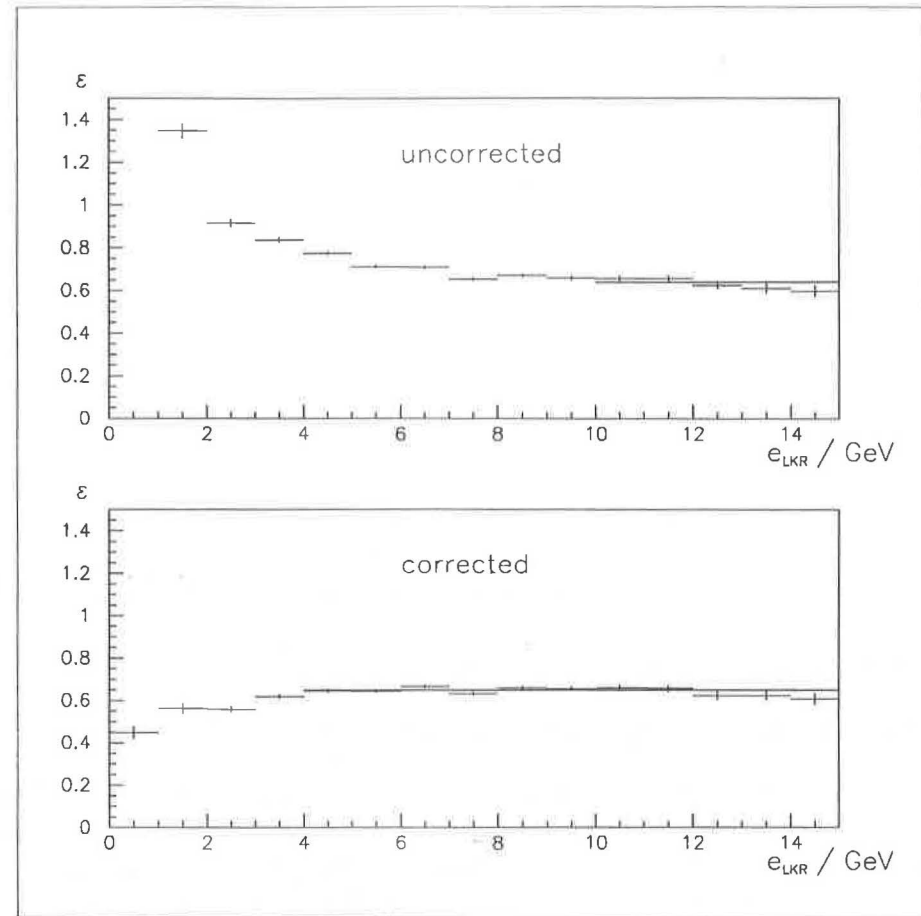


Figure 4.5: comparison of the efficiency for $K\pi 3$ without and with correlation correction, as a function of minimal LKR cluster energy e_{LKR}

method	Ke4	$K\pi 3$
biased ETOT, $e_{LKR} > 10$ GeV cut	$(56 \pm 13)\%$	$(63.56 \pm 0.89)\%$
unbiased ETOT, $e_{LKR} > 4$ GeV, corrected	$(61.3 \pm 5.2)\%$	$(64.00 \pm 0.63)\%$

Table 4.1: Comparison of the two methods to measure Ke4 trigger efficiency

Efficiency for Different Components of the Ke4 Trigger

As the efficiency of the Ke4 trigger is not very good, it is interesting to learn more about the source of the inefficiency. For this purpose, again PU information is used to measure efficiencies for different parts of the trigger. The results are listed in table 4.2.

The conclusion of this measurement is that KE4NT, thus also the 3peaks-condition, is quite efficient, KE4MB causes some losses (probably due to algorithms optimized for ϵ'/ϵ -analysis), but the main inefficiency comes from

trigger	efficiency Ke4	efficiency $K\pi 3$
$Q2 * \overline{MUV} * \overline{AKL}/6$ (Pretrigger)	$(67 \pm 4)\%$	$(71.4 \pm 0.8)\%$
KE4NT	$(99.1 \pm 0.1)\%$	$(98.6 \pm 0.2)\%$
KE4MB	$(92 \pm 3)\%$	$(90.9 \pm 1.4)\%$

Table 4.2: Efficiency for different parts of the Ke4 trigger

the pretrigger. As it turned out, the AKL-veto was rather a random veto.

Of course, these results gave rise to a number of improvements for the Ke4 trigger; unfortunately, no data taken with such an improved trigger could be used for this work, because due to an accident which caused the destruction of the drift chambers it has not been possible to perform a second Ke4 run in 2000.

This also means that the error of the branching ratio given in this work is mainly determined by the error of efficiency measurement.

Part III

Analysis



Eskimo

Chapter 5

Data Samples

This short chapter describes the data samples used for the Ke4 analysis

The origin of data is the 1999 run of the NA48 experiment. From all data, only events with either the KE4- or ETOT-Trigger are selected. Additionally, a special Ke4-Filter was applied to reduce the size of the samples. This filter downscales events with $E/p < .8$ for the electron candidate - which are mainly $K\pi3$ events - by a factor of 10. It also requires the number of photon candidates¹ to be exactly 2.

Depending on how "clean" the sample is required, an according number of runs had to be excluded from the samples for different purposes. For instance, the final version of the Ke4 trigger was implemented only a few weeks after the start of the 1999 run. Also, there have been periods where the Ke4 trigger did not work properly. Whenever such data was used, it was checked for possible biases.

The question which trigger (KE4 or ETOT) to use is not easily answered because each trigger has certain advantages and disadvantages. While KE4 provides a better energy-resolution for the calorimeter read-out, its efficiency is worse (and cannot be measured very accurately due to lack of statistics). ETOT is a minimum-bias trigger, where the efficiency is high and can at first order be assumed to be the same for both the Ke4 and $K\pi3$ decay

¹LKR clusters that are not associated to a charged track, have a minimal energy of 2 GeV and a timestamp that is not more than 3 ns from that of the charged tracks

<i>sample</i>	<i>selection criteria</i>	<i>size</i>
K	KE4-trigger	109299 events
E	ETOT-trigger	296172 events

Table 5.1: list and description of Ke4 data samples

modes².

Since no bias of phase space distributions of ETOT trigger data could be seen within errors, this data will be used for the analysis. Of course, the results for both triggers are compared during the checks of systematics.

²the latter is used as normalization channel for the BR measurement

Chapter 6

Ke4 Events in the Detector

This chapter deals with the detector response to Ke4 events; it is shown how the momenta of the particles can be reconstructed from detector information. The occurrence of two physical solutions for the (undetectable) neutrino, and how one deals with this situation, is also discussed.

To study the detector response to Ke4 one uses Monte Carlo (MC). In NA48, there are two different MC programs in use: *NASIM* [24] and *NMC* [25].

While *NASIM* is the more accurate (and therefore slower) one, which even simulates shower evolution in the calorimeter, *NMC* is the faster choice, using pre-simulated shower libraries. For most of our purposes, *NMC* is sufficiently accurate; however, also *NASIM* generated data samples were used where accuracy was more important than high MC statistics. For the studies below a sample of *NASIM* events has been used.

6.1 Minimal Requirements for Ke4 Reconstruction

As an upper limit for the acceptance of Ke4 events in the NA48 detector, the portion of events fulfilling the following minimal requirements for reconstruction are studied:

- two charged tracks in the DCH (representing the electron and the charged pion) forming a good vertex
- three clusters in the LKR (one for the electron, two for the gammas of the neutral pion decay)

These minimal requirements make sure that all products of the Ke4 decay except the neutrino are seen in the detector - without these requirements,

requirement	acceptance
2track	47%
3clus	87%
both	44%
also good vertex	43%

Table 6.1: acceptances for minimal requirements (numbers based on MC)

reconstruction would not make sense. Owing to them, one can in principle reconstruct the 4-momenta of all particles and even identify the electron (see below).

6.2 Particle Identification

After having seen the required number of tracks in the detector, it is necessary to identify the particles. Charged particles are handled first, because they may leave a cluster in the LKR. All remaining clusters that cannot be related to a charged track are photon candidates. Exactly two photon candidates should remain¹ (to avoid ambiguity) that originate from a π^0 (reducing the minimal requirements acceptance to 26%).

In the following, identification techniques are reviewed and optimized.

6.2.1 Identification of $e^\pm$ and $\pi^\pm$

To distinguish a pion from an electron, the momentum as determined in the spectrometer (through the kick in the magnet) is compared to the energy of the corresponding LKR cluster².

Electrons are stopped inside the LKR and ideally leave all their energy there, so the ratio $\frac{E}{p}$ should be equal to 1 if measured in GeV and GeV/c, respectively:

$$\frac{E}{p} = \frac{mc^2\gamma[\text{GeV}]}{mc\beta\gamma[\text{GeV}/c]} = \frac{1}{\beta}[1] \approx 1 \quad (6.1)$$

Because of finite experimental resolution, there will be actually a distribution around 1.

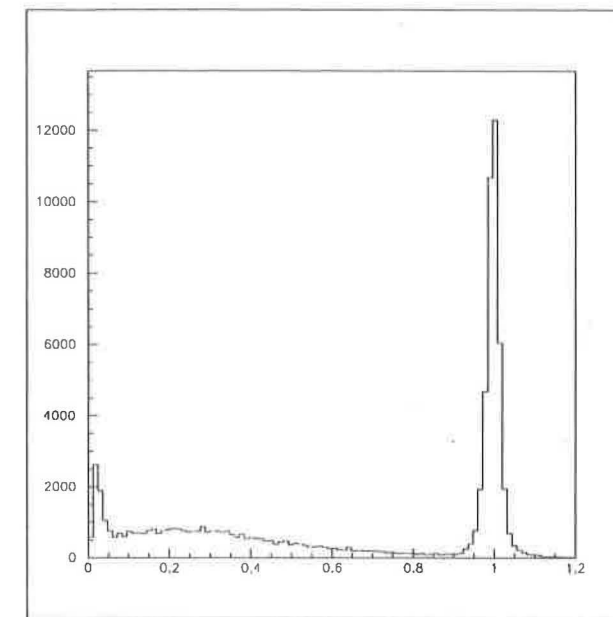
Pions usually pass the LKR and are stopped in the HAC. Thus, the energy deposit in the LKR is only some fraction, and $\frac{E}{p}$ will be smaller than one:

$$\frac{E}{p} < 1 \quad (6.2)$$

¹clusters with an energy less than 2 GeV are ignored
²if there is no cluster, the energy is defined to be zero

The result for $\frac{E}{p}$ is a wide distribution between 0 and 1 corresponding to showering pions, as well as a peak near 0 for minimum-ionizing pions.

An example for a typical $\frac{E}{p}$ histogram is given in fig. 6.1. Because there are both electrons and pions in Ke4, there is the electron peak around 1 as well as the pion distribution below.

Figure 6.1: $\frac{E}{p}$ for MC Ke4 events

To decide whether a charged particle is a pion or an electron, a cut-value $R_{\pi e}$ is chosen such that every $\frac{E}{p}$ smaller than this value is identified as a pion³; everything else as an electron:

- Electron if $\frac{E}{p} > R_{\pi e}$
- Pion if $\frac{E}{p} < R_{\pi e}$

In order to find the best value for $R_{\pi e}$, the acceptance of MC-Ke4 is plotted as a function of this variable (see fig. 6.2, page 92).

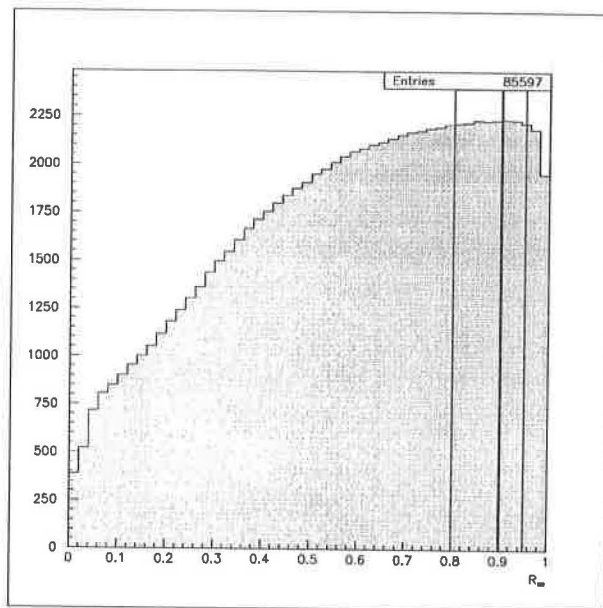
As the plot shows, the best value is

$$R_{\pi e} = 0.85 - 0.95 \quad (\text{acceptance} = 24\%) \quad (6.3)$$

6.2.2 Identification of π^0

The neutral pion is identified through its invariant mass. Therefore, the momenta of the two photon candidates have to be reconstructed and added.

³of course also muons will be identified as pions this way

Figure 6.2: Ke4 acceptance as a function of $R_{\pi e}$

The invariant mass is given by

$$m_{\pi^0}^2 = (p_1 + p_2)^\mu (p_1 + p_2)_\mu = 2p_1^\mu p_{2\mu} \quad (6.4)$$

In order to reconstruct a gamma momentum, one needs to know, besides the energy which is given by the LKR, also the *direction* of the photon. For a Ke4, the photons originate from the same vertex as the charged particles⁴. Using the vertex as reconstructed from spectrometer information, the photon directions are determined and subsequently the invariant mass can be calculated. For a large distance between LKR and vertex (compared to the scale of the LKR), the mass is given by:

$$m \approx \sqrt{E_1 E_2} \frac{|\vec{r}_1 - \vec{r}_2|}{d} \quad d \ll r_{1,2} \quad (6.5)$$

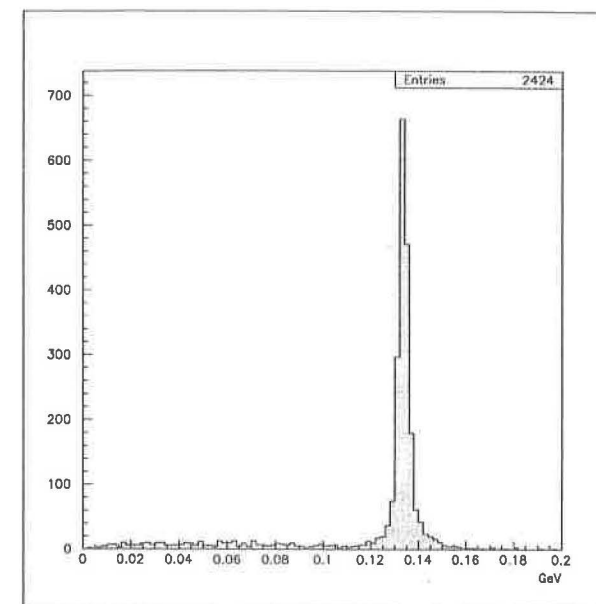
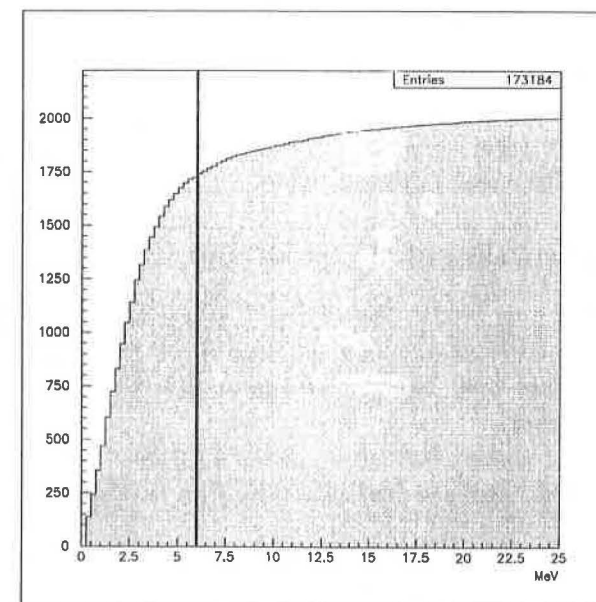
where $E_{1,2}$ are the cluster energies, d the distance from the LKR to the vertex, and $\vec{r}_{1,2}$ 2-dimensional vectors in the LKR plane pointing from the center (where the beam crosses) to the clusters.

Typically, the histogram of this approximated mass looks like fig. 6.3.

The cut to identify a pion is now defined through a parameter w_{π^0} which determines the accepted width of the mass peak:

$$\bullet \pi^0 \text{ if } |m_{\gamma\gamma} - m_{\pi^0}| < w_{\pi^0}$$

⁴the difference between the Ke4 decay vertex and the π^0 decay vertex is far below experimental resolution

Figure 6.3: $m_{\gamma\gamma}$ for MC Ke4 eventsFigure 6.4: Ke4 acceptance vs w_{π^0}

The plot (see fig. 6.4, page 93) shows a knee at a cut value of about 6 MeV (acceptance 19%). Obviously, the acceptance would be best for no cut at all, but neutral pion identification is necessary for background rejection (see below, where w_{π^0} is optimized in respect of this).

requirement	acceptance
exactly 2 gammas	26%
e/ π identification	24%
π^0 identification	19% at knee

Table 6.2: Ke4 acceptances for particle identification

6.3 Reconstruction of Ke4 Events

Having the particles identified, the final reconstruction of the event (that is, the determination of the momenta of all particles) is a rather straightforward task:

- for the charged pion and electron, the 3-momentum is measured in the spectrometer and the energy is just given by $E^2 = p^2 + m^2$.
- for the neutral pion, the energy is measured in the calorimeter, the direction is given by the position of the two gamma clusters in the calorimeter and the (common) decay vertex known from charged particles, thus the 3-momentum can be calculated.
- finally, the neutrino momentum is reconstructed using the conservation laws for energy and momentum. As the original kaon momentum is not known, this leads to an ambiguity (2 physical solutions for the neutrino), the treatment of those two solutions is not a priori clear, and will be discussed in detail below.

6.3.1 Reconstruction of the Neutrino

As will become obvious below, the neutrino reconstruction suffers from the lack of knowledge of the energy and momentum of the decaying kaon. This leads to two different solutions for the neutrino (in association with two different kaon energies).

In the kaon rest frame, the three visible particles ($\pi^\pm, e^\mp, \pi^0$) are combined to one effective particle with the following properties:

$$E_{vis} = E_{\pi^\pm} + E_{e^\mp} + E_{\pi^0} \quad (6.6)$$

$$\vec{p}_{vis} = \vec{p}_{\pi^\pm} + \vec{p}_{e^\mp} + \vec{p}_{\pi^0} \quad (6.7)$$

$$m_{vis} = \sqrt{E_{vis}^2 - p_{vis}^2} \quad (6.8)$$

where m_{vis} is limited to the range

$$m_{vis} = [(m_{\pi^\pm} + m_{e^\mp} + m_{\pi^0}), m_{K^0}] \approx 275 - 500 \text{ MeV} \quad (6.9)$$

In the kaon rest frame, one has (see fig. 6.5, page 95)

$$E_\nu = m_{K^0} - E_{vis} \quad (6.10)$$

$$\vec{p}_\nu = -\vec{p}_{vis} \quad (6.11)$$

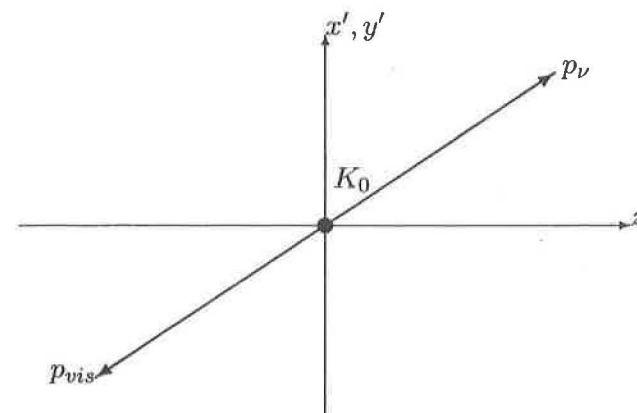


Figure 6.5: neutrino and visible momentum in the Kaon rest-frame

Since the neutrino is massless ($E_\nu = |\vec{p}_\nu|$), the neutrino energy follows as

$$E_\nu = \frac{m_{K^0}^2 - m_{vis}^2}{2m_{K^0}} \quad (6.12)$$

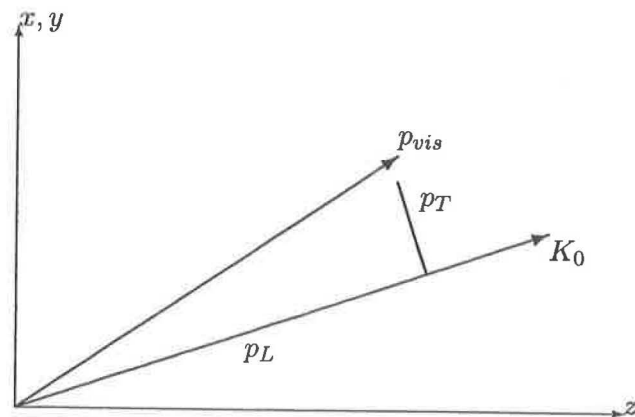
The neutrino energy thus only depends on one measured quantity, namely m_{vis} . However, to fully reconstruct the neutrino, also the direction of the neutrino is needed. Considering (6.11), the direction is opposite to the visible momentum, but as the kaon energy (and therefore the kaon rest frame) is not known, only the momentum *transversal* to the direction of the kaon movement, p_T , is known (p_T is invariant under any boost along the kaon direction and thus does not depend on the unknown kaon energy).

To measure p_T in the lab frame, in principle the direction of the kaon has to be known. Since both the origin of the kaon (the K_L -Target) and its decay vertex is known, the direction is defined by the vector pointing from the target to the vertex. p_T is consequently the component of the visible momentum transversal to this direction, p_L the longitudinal part (see fig. 6.6, page 96).

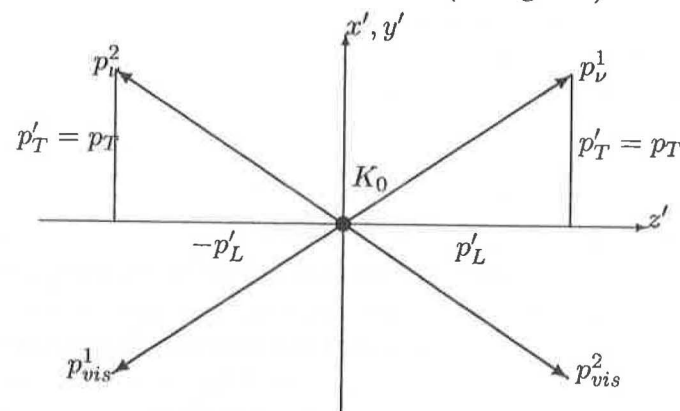
For this analysis, though, it was correct within errors to simply assume kaons going exactly in beam (z -) direction.

In the kaon rest frame, p'_T is obviously equal to p_T (as discussed above). Knowing p'_T , also the longitudinal component p'_L can be calculated in the rest-frame as

$$p'_L = \pm \sqrt{p^2 - p_T^2} \quad (6.13)$$

Figure 6.6: the definition of p_T and p_L in the lab frame

where p is equal to p_ν as given in (6.12). As the sign of p'_L (or the orientation of the neutrino with respect to the kaon in the rest frame) is not known, both solutions have to be taken into account (see fig. 6.7).

Figure 6.7: the 2 solutions for the neutrino in the kaon rest-frame (the kaon has been assumed to move in z -direction in the lab-frame)

In the lab frame, the measured value of p_L is of course different from p'_L . Knowing both p_L and p'_L , one can now calculate the parameters of the boost that connects them:

$$\gamma = \frac{E'_{vis} E_{vis} - p'_L p_L}{E'^2_{vis} - p'^2_L} \quad (6.14)$$

$$\beta\gamma = \frac{E'_{vis} p_L - E_{vis} p'_L}{E'^2_{vis} - p'^2_L} \quad (6.15)$$

Depending on the chosen sign of p'_L , one gets two different boosts, and therefore also two different kaon energies ($E_{K^0} = m_{K^0} \gamma$). As most of the

time both solutions are possible physical solutions, this ambiguity cannot be resolved with the given experimental setup⁵.

6.3.2 Dealing with the 2 Neutrino Solutions

As demonstrated above, there are two solutions for the neutrino, both of which are kinematically and physically allowed. There is generally no way to decide which solution is the correct one, so some method to deal with these 2 solutions has to be found.

Since there is no special direction in the kaon rest frame, both solutions have equal probability of 50 %, so there is no favored solution. However, the two solutions also differ in the resulting kaon energy; as the shape of the kaon energy spectrum is known, one could derive a probability from there.

In principle, there are a couple of possibilities:

1. always use the 1st solution
2. always use the 2nd solution
3. use the mean of both solutions
4. use both solutions with a weight that sums up to 1

The consequences of choosing wrong solutions is discussed in detail in the appendix (page 193). The conclusion is that the best choice is the last one in the list above: use both solutions with an appropriate weight.

6.3.3 Reconstruction Quality

To check the reconstruction quality, true (i.e. generated) values are compared with the reconstructed ones in MC for the five Cabibbo variables (see fig. 6.8).

As $m_{\pi\pi}$ is the only variable that does not depend on the chosen neutrino solution, the plot shows the pure detector resolution; all other plots also show the neutrino ambiguity (both solutions are plotted). The reconstruction of $m_{e\nu}$ is not very good, so it is not used for analysis. ϕ shows some tendency to be reconstructed as $-\phi$.

⁵for a proposed future charged Ke4 experiment, there will be a narrow kaon momentum range together with a beam spectrometer (KABES), which allows to distinguish between the two solutions

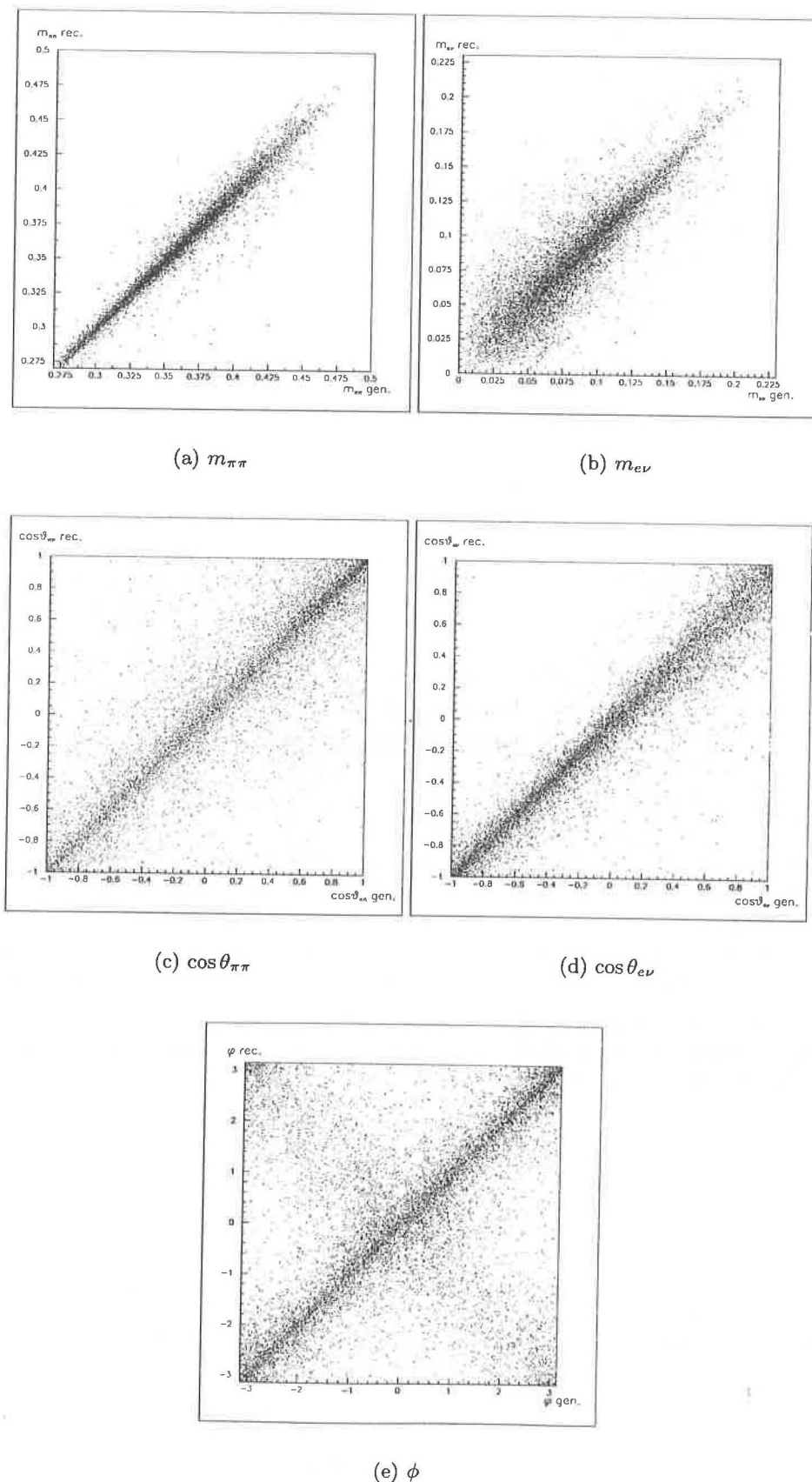


Figure 6.8: comparison reconstructed vs generated values for the five Cabibbo variables

Chapter 7

Sources of Background

In this chapter, all decay modes of K_L and K_S are reviewed with respect to their possible contribution to Ke4 background, in order of their branching ratio (modes with very small branching ratios are not considered). No methods of how to remove these backgrounds are given here, they will be discussed in later sections.

Generally, all modes with charged particles can contribute to background, purely neutral modes only if there are charged accidentals, or charged subdecays (decays of decay products). Actually, there are only three unstable particles involved ($\mu, \pi^\pm, \pi^0$). The μ lives 50 times longer than a K_L , therefore its decay can be safely neglected. The $\pi^\pm$ has a mean lifetime comparable to K_L , but it decays most likely to a μ which is detected in the detector's muon veto system¹. So the only important subdecay is that of the π^0 - it decays instantly². The "default" decay is $\pi^0 \rightarrow \gamma\gamma$ with a BR of 98.8 %. Since this also happens for Ke4, this is no additional source for background, but rather the signature of a π^0 in NA48. However, with a probability of 1.2 %, a Dalitz-decay $\pi^0 \rightarrow e^+e^-\gamma$ occurs; in this case, a formerly neutral decay mode shows charged particles and becomes a possible source of background.

Another particle which has to be considered is the Λ -baryon. Its lifetime is comparable to that of the K_S , and it is also neutral. Thus, the kaon beam also contains a fraction of Λ 's, mostly decaying in the first meters of the decay region. The major decay modes are $\Lambda \rightarrow p\pi^-$ and $\Lambda \rightarrow n\pi^0$. The former mode could contribute to the background if combined with two accidental gammas (the latter only via a Dalitz-decay, suppressed by small BR, and an additional gamma); however, in order to contribute to Ke4

¹if it is not, than it will be most likely misidentified as a pion again

²that is, it does not propagate a significant distance during its lifetime at the typical energy scale of NA48

background, also a charged particle has to be misidentified as an electron.

In the following list all decay modes for both K_L and K_S with reasonably high BR are reviewed with this in mind)³.

7.1 K_L Decay Modes

$K_L \rightarrow \pi^\pm e^\mp \nu_e$ or $Ke3$ (BR 38.78 ± 0.28 %) Although compared to $Ke4$ the π^0 is missing, this is a serious source of background because of the high BR and accidental gammas that at low trigger level cannot be distinguished from the 2 gammas coming from the decaying π^0 (even at higher trigger level it is serious, since there are well enough $Ke3$ to fake a π^0 by accident). Besides the accidental gammas, there can be also gammas due to internal or external bremsstrahlung. When discussing the $Ke3$ background, all of these possibilities are included.

$K_L \rightarrow \pi^\pm \mu^\mp \nu_\mu$ or $K\mu3$ (BR 27.18 ± 0.25 %) This is a far less serious background compared to $Ke3$ because muons can be identified very efficiently using the NA48 muon veto system (already at low trigger level).

$K_L \rightarrow \pi^0 \pi^0 \pi^0$ (BR 21.13 ± 0.27 %) Since this is a purely neutral mode, it is no primary source of background. However, one or more π^0 may decay to $e^+ e^- \gamma$ (*Dalitz decay*). The probability of a Dalitz decay being about 1%, the effective BR of $\pi^0 \pi^0 \pi^0$ as a background for $Ke4$ is reduced to roughly 2×10^{-3} , which is of the order of CP-violating decay modes of K_L . It is not important at low levels of triggering, but has to be taken into account for final $Ke4$ event selection.

$K_L \rightarrow \pi^+ \pi^- \pi^0$ or $K\pi3$ (BR 12.55 ± 0.20 %) This is clearly the most serious background, since the only difference to $Ke4$ is the second pion instead of an electron and the missing neutrino. Charged particle identification is only possible at higher trigger level (it requires combination of information from two detectors), the neutrino is not detected anyway, and because of the range of possible kaon momenta, it is not possible to recognize missing energy or momentum at low trigger level. At the same time, the BR of $\pi^+ \pi^- \pi^0$ is rather high. So any $Ke4$ trigger will unavoidably contain a large fraction of $\pi^+ \pi^- \pi^0$ events.

$K_L \rightarrow \pi^+ \pi^-$ (BR 0.2056 ± 0.0033 %) This CP-violating decay is of course negligible at low trigger level because of its rather small BR. Still, at least in final analysis it has to be considered to make sure that the probability of an accidental fake π^0 and a misidentification of one pion is small enough.

³branching ratios as in [34]

$K_L \rightarrow \pi^0 \pi^0$ (BR 0.0927 ± 0.0019 %) Another CP-violating mode which can be neglected at low trigger level, but can in principle be a background because of a Dalitz-decay of one π^0 (or charged accidentals). It is not expected to really contribute significantly to the background, but is listed as being at the borderline of modes to be considered.

7.2 K_S Decay Modes

Generally, the K_S modes are less important since NA48 is set up to provide K_S decays at a rate comparable to that of the K_L CP-violating ones. So everything said above for these K_L modes also applies for K_S and is not repeated here. For $Ke4$ Analysis, there is no need to determine whether a background comes from K_S or K_L , thus subsequently K_S modes will not be explicitly mentioned.

7.3 Conclusion: List of Major Background to $Ke4$

To conclude, the major background modes for $Ke4$ (in rough order of importance) are:

- $K_L \rightarrow \pi^+ \pi^- \pi^0$ ($K\pi3$)
- $K_L \rightarrow \pi^\pm e^\mp \nu_e + 2\gamma$ ($Ke3$)
- $K_L \rightarrow \pi^0 \pi^0 \pi_D^0$ (one Dalitz decay)

Chapter 8

Selection of Ke4 Events

This chapter shows how Ke4 events are selected and background is rejected. After the definition of cuts it is also shown how the remaining background is subtracted.

In order to select Ke4 candidates, a code in C was written as part of the NA48 analysis program *COMPACT*. In this code, each event is tried to be reconstructed as Ke4. Only clean events are accepted, ambiguous or incomplete events are rejected.

The algorithm in detail is:

- apply L3 filter (optional, for use with MC; real data has been already L3 filtered)
- require 1 vertex, 2 tracks
- calculate mean LKR time of tracked particles
- count number of γ candidates with at least 2 GeV in LKR and no more than 3 ns from above mean time
- reject events that do not have exactly 2 gamma candidates

8.1 Cuts

8.1.1 Ke4 Cuts

The cuts to select Ke4 events are (the definition and motivation for some special cuts is given below):

- event reconstructable as described above (i.e. sufficient number of charged tracks, etc.)

- exactly four clusters in the LKR calorimeter
- total energy in LKR calorimeter at least 30 GeV (to remove low-energy chunk and get rid of problems with trigger inefficiencies for low energies)
- all clusters at least 15 cm radial distance from beam (to cut inefficiencies of the LKR calorimeter near the beam hole)
- E/p of electron candidate between .95 and 1.05
- E/p of pion candidate below .8
- the radius of the $K\pi 3$ -ellipse $R > 20$ (definition see below)
- the ratio p_T/E_ν in the range between 0 and 0.02 (details on this cut see below)
- distance between a gamma-cluster and the extrapolated position of a forward bremsstrahlung photon larger than 5 cm (to reject background from Ke3 with bremsstrahlung)
- distance between a gamma-cluster and the pion cluster larger than 15 cm (to reject background from Ke3 with a multiple pion cluster)

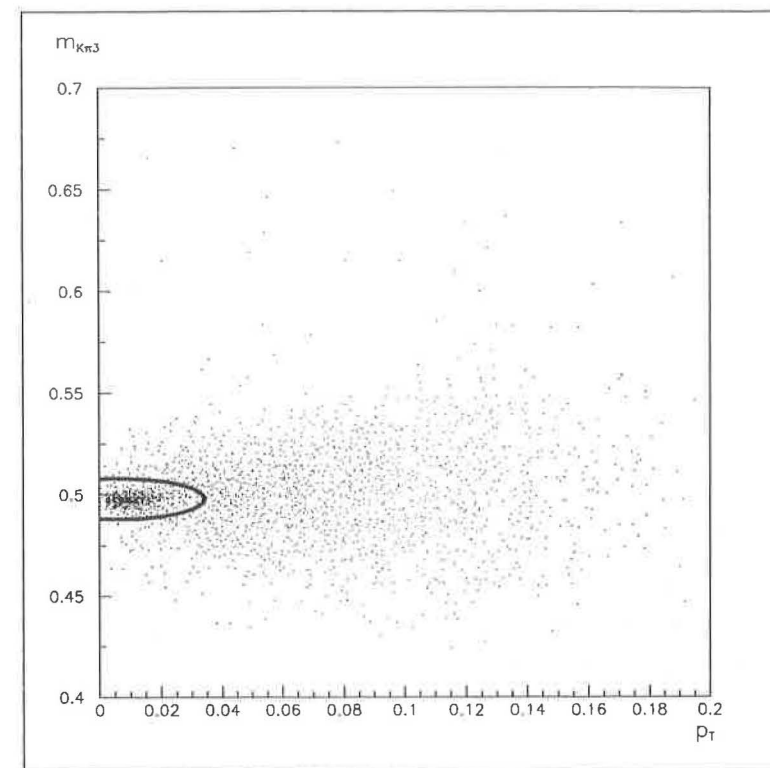
Other cleaning cuts (e.g. minimal distance to dead calorimeter cells) are not listed here since they have no significant effect on event selection.

The $K\pi 3$ Ellipse

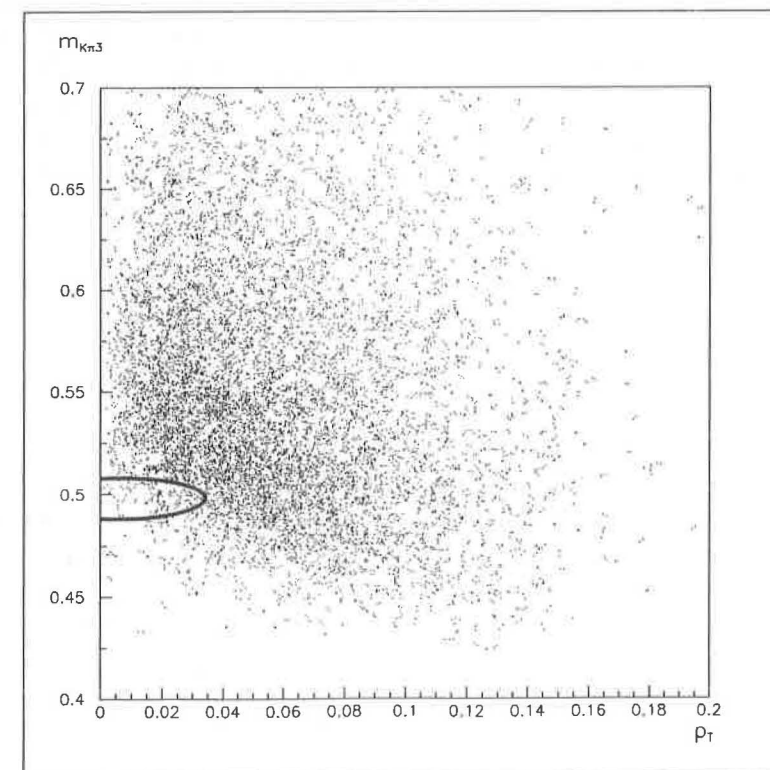
$K\pi 3$ events have typically a small missing transversal momentum p_T (since all particles are observed), and, using the right mass hypothesis, give an invariant mass $m_{K\pi 3}$ close to 498 MeV (mass of the neutral kaon). To cut these events, we define an ellipse around $p_T = 6$ MeV and $m_{K\pi 3} = 498$ MeV parameterized by a radius R defined as

$$R^2 := \left(\frac{p_T - 6 \text{ MeV}}{7 \text{ MeV}} \right)^2 + \left(\frac{m_{K\pi 3} - 498 \text{ MeV}}{2.5 \text{ MeV}} \right)^2 \quad (8.1)$$

where the numbers are chosen from MC experience. Fig. 8.1 shows the ellipse for $R = 4$ and both Ke4 and $K\pi 3$ MC events.



(a) $K\pi 3$ MC



(b) Ke4 MC

Figure 8.1: $K\pi 3$ ellipse for $K\pi 3$ and Ke4 Monte Carlo (axes in units of GeV)

p_T/E_ν

The ratio p_T/E_ν turns out to be a powerful variable to separate Ke4 from $K\pi 3$ and $Ke3$. The proper definition is:

- p_T := sum of all observed momenta ($\pi^\pm, e^\pm, \gamma$) in the xy-plane (transversal to beam)
- E_ν := energy of forward-moving neutrino solution in the lab frame as reconstructed from visible momenta and energies, assuming Ke4 decay with kaon moving in positive z-direction (corresponding to larger value for reconstructed kaon energy)

In the case of Ke4 events, p_T is usually rather large because of the missing neutrino, while E_ν is of course positive (unless badly reconstructed). MC gives $p_T/E_\nu \approx .001 - .03$

For $K\pi 3$ events, p_T is usually rather small. Applying the Ke4 hypothesis, the second pion is misidentified as an electron, and an additional neutrino is assumed. The additional particle leads to an overestimation of the kaon energy, and in turn, since the neutrino is assumed moving forward, E_ν will be large. Consequently, the ratio p_T/E_ν can be expected to peak at very small values; MC gives a range of $0 - 0.001$.

A Ke3 event has rather large p_T because of the missing neutrino. As one adds two additional gammas to the Ke3 decay assuming a Ke4, there is in almost all cases kinematically no room for another particle (the neutrino) - in the rest frame, you have to add the energies of the additional gammas to the kaon mass, which forces the neutrino's mass to be negative in order to fulfill momentum conservation. So one often gets unphysical solutions (= negative values) for E_ν . A physical solution is only possible if the reconstructed kaon mass is so low that even with the gamma energies added, there is still room for a positive neutrino energy. If bremsstrahlung is involved, at least one of the gammas is kinematically correlated to the decay, so that the probability of a physical solution for the neutrino increases. This is the reason why bremsstrahlung-Ke3 tend to have positive neutrino energies (as observed in data, see discussion below).

To conclude: What one expects from this are mostly negative values for p_T/E_ν ; if there is a physical solution for the neutrino, the energy should be rather small, leading to large values of p_T/E_ν .

Fig. 8.2 shows the distributions for different kinds of MC.

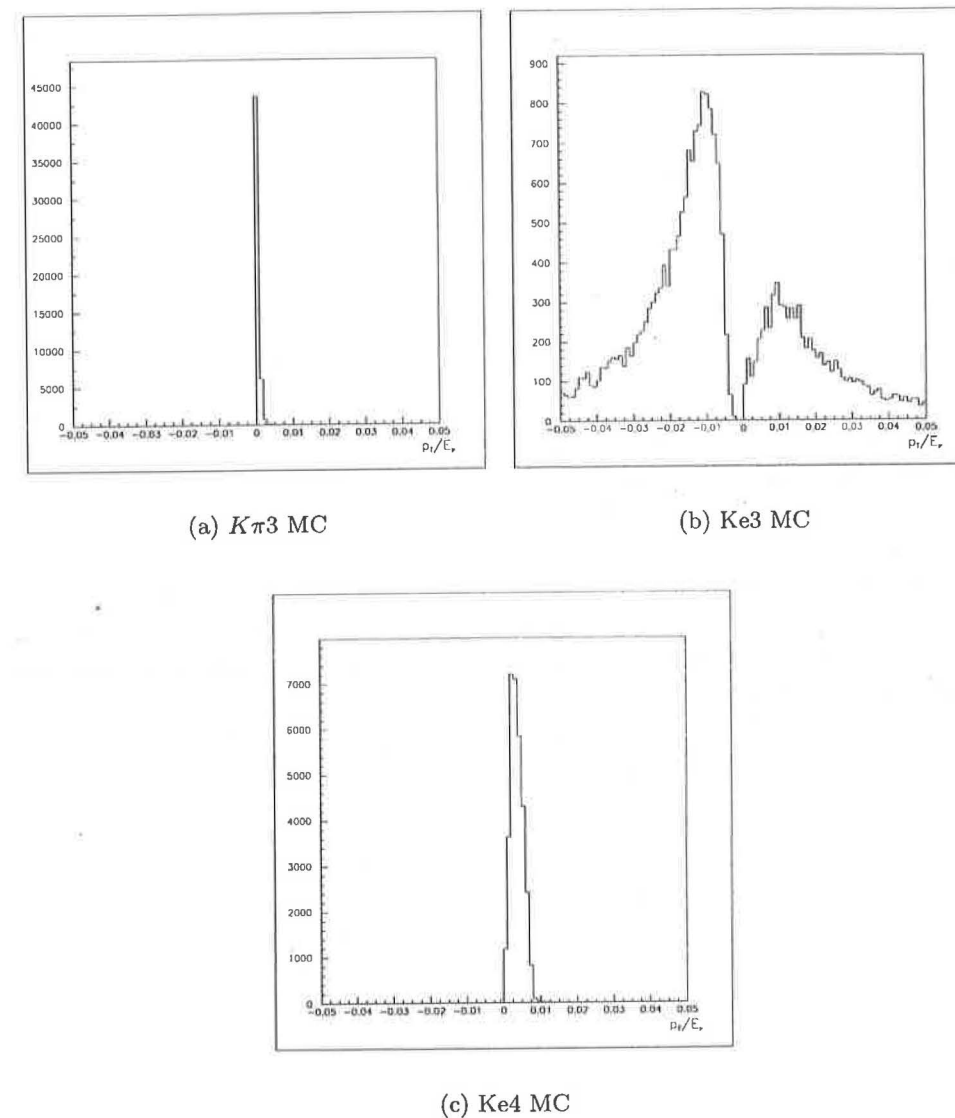


Figure 8.2: p_T/E_ν for $K\pi 3$, Ke3 and Ke4 Monte Carlo

8.1.2 $K\pi 3$ Cuts

$K\pi 3$ are rather easy to select, since most of the triggered events are in fact of this kind, and there is practically no relevant background to this decay. Thus the cuts can be kept simple:

- event reconstructable as described above (i.e. sufficient number of charged tracks, etc.)
- exactly four clusters in the LKR calorimeter
- total energy in LKR calorimeter at least 30 GeV (to remove low-energy chunk and get rid of problems with trigger inefficiencies for low energies)
- E/p of both charged particles between 0 and 0.85
- the radius of the $K\pi 3$ -ellipse $R < 5$

Some characteristic plots are shown in fig. 8.3 (these distributions are discussed in more detail below).

The $K\pi 3$ events selected this way do not reflect those that could become background to Ke4 since they are easily rejected by the E/p cuts; to get a better impression of the $K\pi 3$ as background, an alternative selection uses the same cuts as above, but requires $R > 5$, thus choosing $K\pi 3$ events which are for some reason badly reconstructed¹. The result of this selection is also shown in fig. 8.3.

¹due to the E/p cuts requiring two pions, there are only very few other decay channels selected (mainly $K_{L,S} \rightarrow \pi^+\pi^-$)

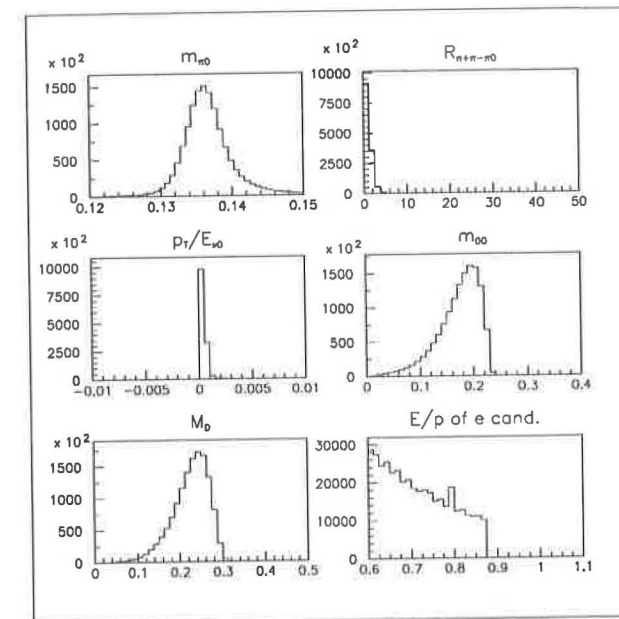
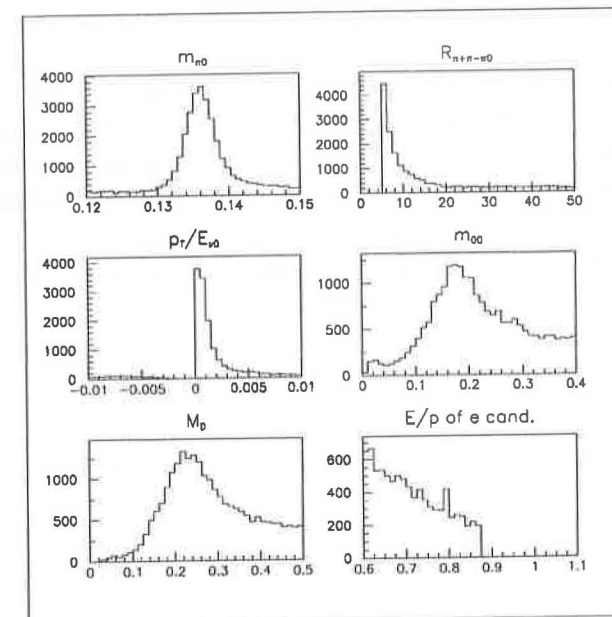
(a) $K\pi 3$ (from data)(b) $K\pi 3$ with $R > 5$ (from data)

Figure 8.3: 6 characteristic histograms for two types of $K\pi 3$ background (selected from data)

8.1.3 Ke3 Cuts

An easy way to select Ke3 is to require at least one of the LKR clusters to be out of time, i.e. the largest time difference between two clusters to be greater than 6 ns; the other cuts are chosen similar to the Ke4 cuts (differences to Ke4 in bold face):

- event reconstructable as described above (i.e. sufficient number of charged tracks, etc.)
- total energy in LKR calorimeter at least 30 GeV (to remove low-energy chunk and get rid of problems with trigger inefficiencies for low energies)
- all clusters at least 15 cm radial distance from beam (to cut inefficiencies of the LKR calorimeter due to the beam hole)
- E/p of electron candidate between .95 and 1.05
- E/p of pion candidate between 0 and .8
- the radius of the $K\pi 3$ -ellipse $R > 20$
- **time difference between two clusters greater than 6 ns**

The characteristic distributions are shown in fig. 8.4 (these distributions are discussed in the following section in more detail). This does not, though, represent all Ke3 which make up a possible Ke4 background - there are also Ke3 with bremsstrahlung, which have an additional in-time cluster and therefore somewhat different kinematical properties.

To select Ke3+bremsstrahlung, the following cuts are used (differences to the Ke4 cuts in bold face):

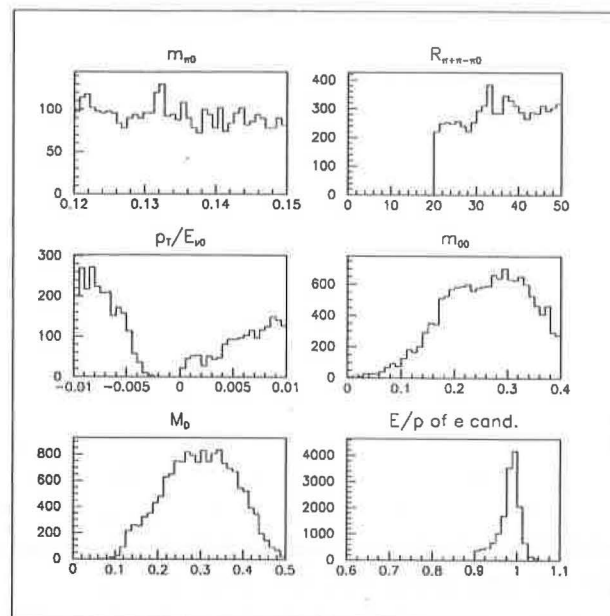
- event reconstructable as described above (i.e. sufficient number of charged tracks, etc.)
- exactly four clusters in the LKR calorimeter
- total energy in LKR calorimeter at least 30 GeV (to remove low-energy chunk and get rid of problems with trigger inefficiencies for low energies)
- all clusters at least 15 cm radial distance from beam (to cut inefficiencies of the LKR calorimeter due to the beam hole)
- E/p of electron candidate between .95 and 1.05
- E/p of pion candidate below .8

- the radius of the $K\pi 3$ -ellipse $R > 20$ (definition see below)
- distance between a gamma-cluster and the extrapolated position of a forward bremsstrahlung photon **smaller than 1 cm** (to select Ke3 + bremsstrahlung background)

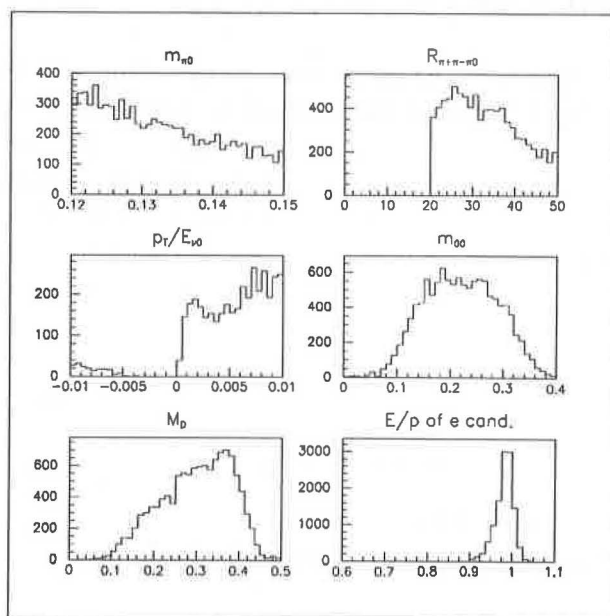
The comparison of this kind of Ke3 with the "normal" one is also seen in fig. 8.4. The most important difference is in the variable p_T/E_ν defined above - while ordinary Ke3 have dominantly negative values, Ke3 + bremsstrahlung usually have positive ones².

As background for Ke4, the Ke3+bremsstrahlung is much more dangerous than pure Ke3, since the latter one can effectively be rejected by requiring all clusters to be in-time, and also by requiring p_T/E_ν to be positive. Nevertheless, pure Ke3 can still contribute to the background if a pion leaves such a large cluster in the LKR that it is reconstructed as two separate ones; that is the reason to require a minimum distance between the pion cluster and others as part of the Ke4 cuts. Ke3+bremsstrahlung is then rejected by the last of the Ke4 cuts listed above.

²the explanation is simply that in the bremsstrahlung's case the additional gammas belong kinematically to the decay, so it is more likely to calculate a physical value for the neutrino energy



(a) Ke3 (from data)



(b) Ke3 + bremsstrahlung (from data)

Figure 8.4: 6 characteristic histograms for various types of Ke3 background (selected from data)

8.2 Checking the Signal

Because of the undetectable neutrino, there is no clear "peak" as evidence for a Ke4 decay. So the only way to make sure that really Ke4 have been selected with the cuts described above, is the comparison of a number of characteristic distributions with the MC of the signal and the known backgrounds, and also selected background from the data.

The list of the plots used for this purpose is:

- the invariant mass m_{π^0} of the π^0 candidate - it is relevant for the $Ke3$ background which does not peak at the pion mass
- the radius R as defined above for the cuts - it is relevant for $K\pi3$ background
- the ratio p_T/E_ν as defined above, relevant for all backgrounds
- the invariant mass m_{00} of the charged particles with mass hypothesis = 0
- the invariant mass m_D of the charged particles interpreted as electrons and a gamma candidate - checks for a Dalitz decay
- the E/p distribution for the electron candidate

A comparison of the selected Ke4 data with MC for Ke4 and the most important backgrounds is shown in figs. 8.5 and 8.6. Obviously the agreement is reasonable; possible remaining background is checked for in the next section.

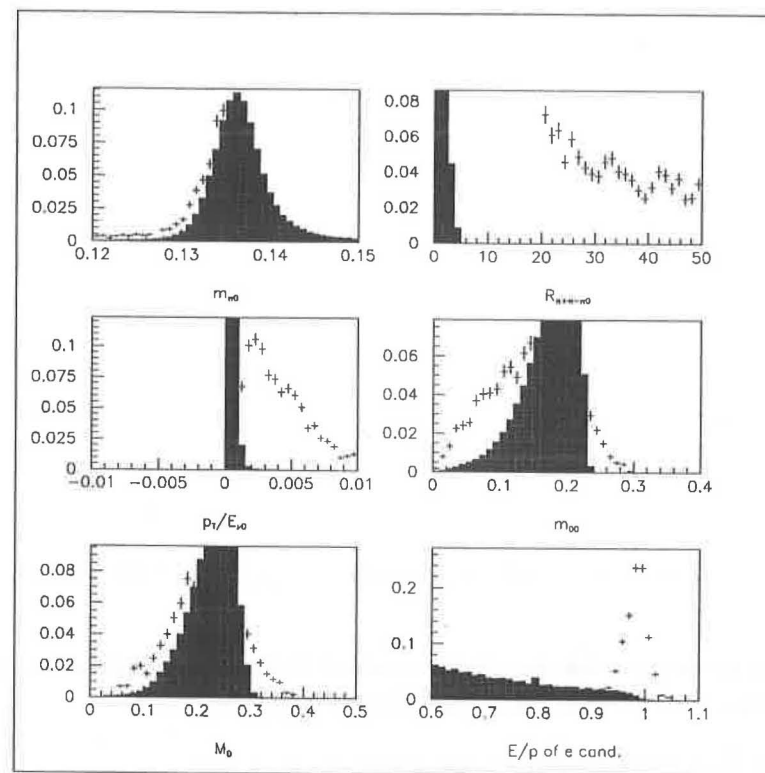
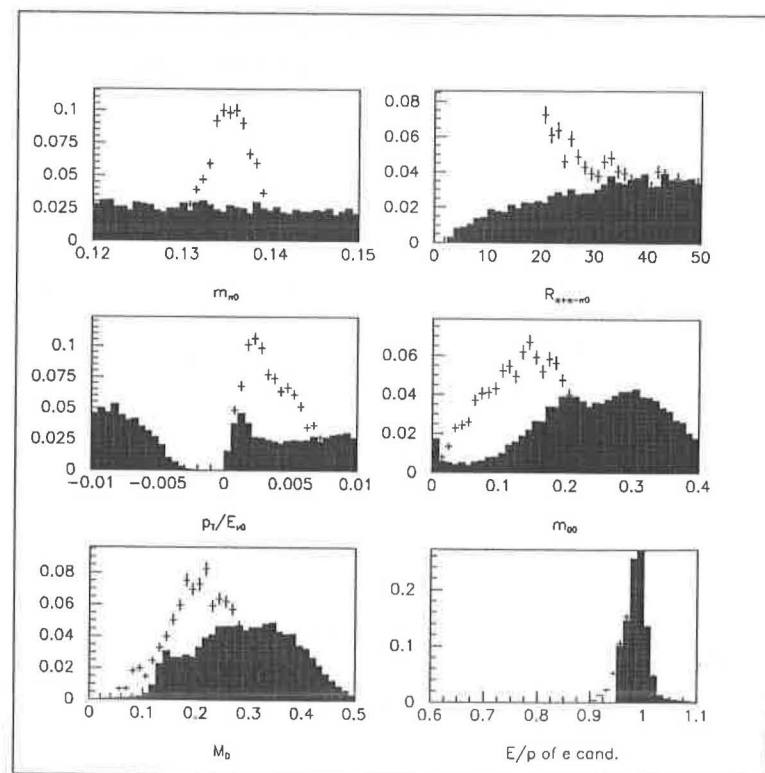
(a) $K\pi^3$ MC(b) Ke^3 MC

Figure 8.5: comparison of 6 characteristic histograms for data (error bars) and various MCs (filled)

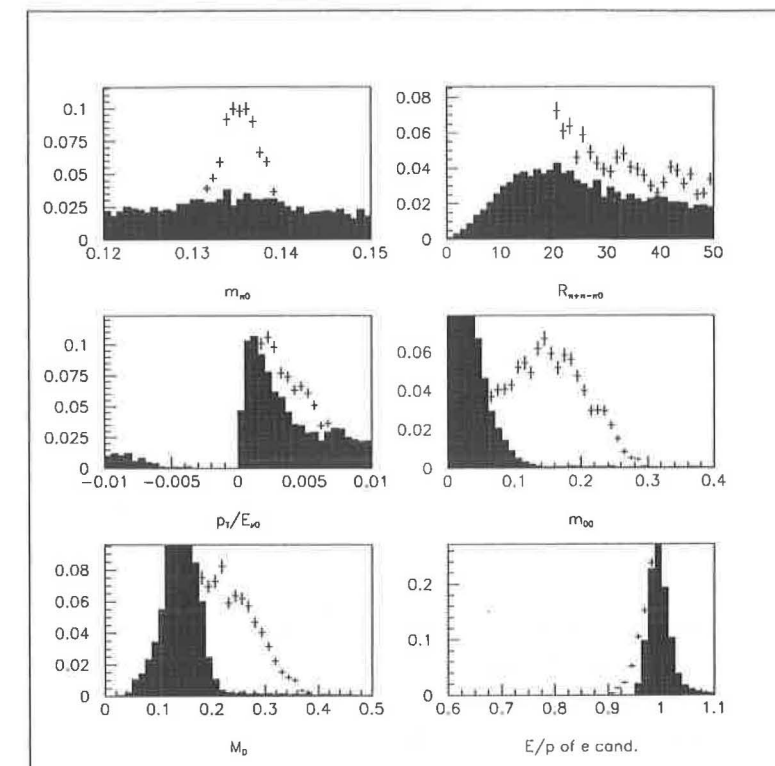
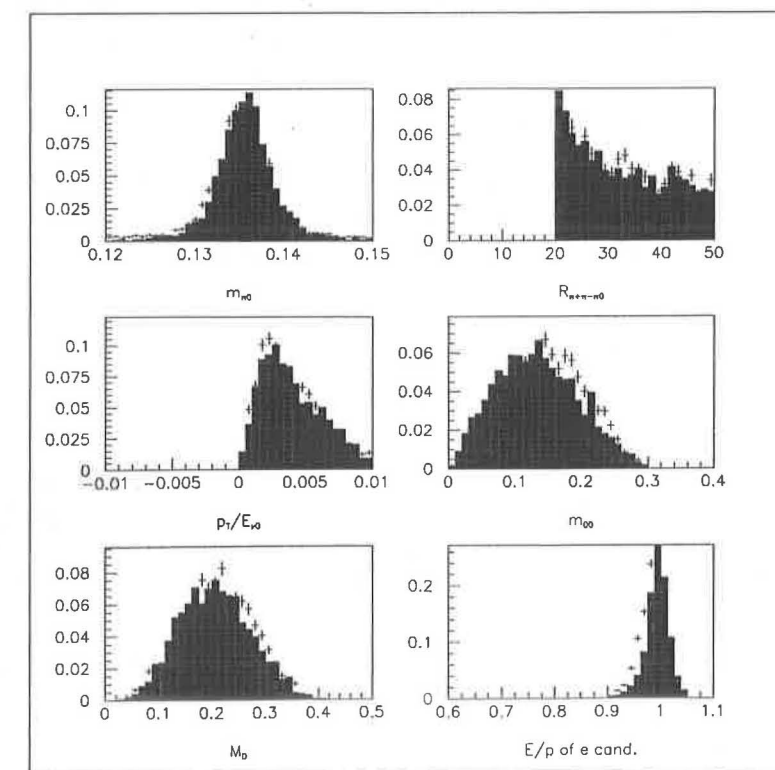
(a) $3\pi^0$ with a Dalitz decay(b) Ke^4 MC

Figure 8.6: comparison of 6 characteristic histograms for data (error bars) and various MCs (filled)

8.3 Background Subtraction

Even with all cuts, there still remains a small amount of background under the signal. However, it is possible to subtract this background once one knows the relative amount of background with respect to the signal. This amount can be estimated looking at certain control regions where the signal (and other backgrounds) are known to be negligible. Normalizing the shape of the background, known from either data or MC, to the observed background in the control region gives the expected background in the signal region, where it can be subtracted.

The question is whether to use data or MC to get the shape of the background. The advantage of data is of course that it does not depend on the quality of a simulation. On the other hand, it is obviously impossible to apply the same cuts to the background as used to isolate the signal, so one has to check that the shape of the background does not (significantly) depend on these cuts. But also for MC, applying the same cuts is a problem: since the cuts are optimized to reject as much background as possible, very large MC statistics are needed to get a reasonable number of events passing the cuts. And since the "standard events" are rejected, only the "pathological" events remain, where one cannot be sure that MC is still correct.

Our conclusion is that it is preferable to use data, and cross-check for the influence of cuts on the shape of background. For this purpose, we define two sets of cuts: one that is optimized to get a large number of pure background events, and another that has as much cuts in common with the Ke4 selecting cuts as possible (of course this has to be a compromise with respect to statistics and pureness of the background events).

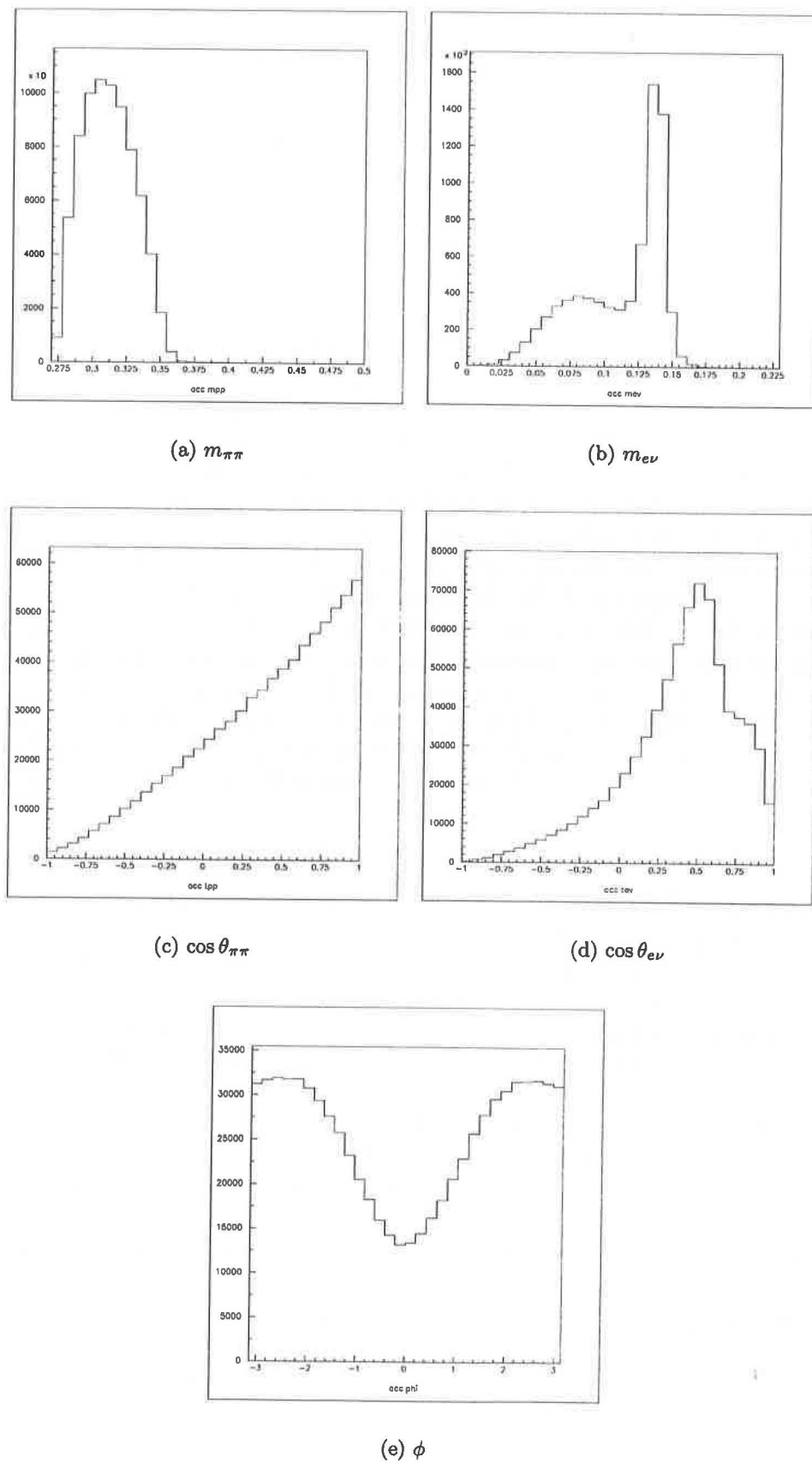
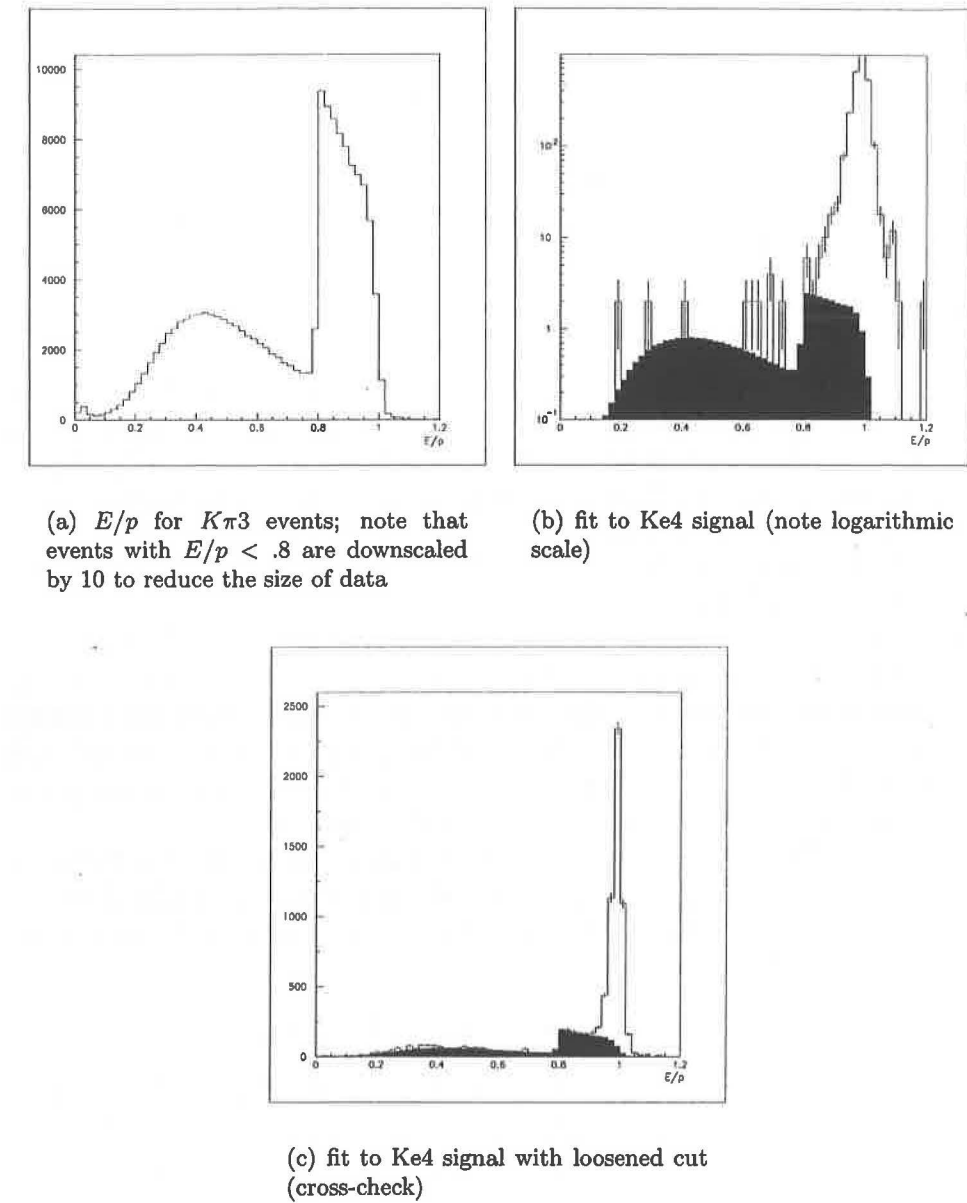
8.3.1 $K\pi 3$ Background

The natural control region for $K\pi 3$ is at small E/p of the electron candidate, since there both the signal and the other important background, $Ke 3$, are negligible. Furthermore, the E/p value of a pion depends on the interaction of the particle in the calorimeter, and is therefore (at first order) independent of the Ke4 decay itself - so one can expect that the shape of the background in the Cabibbo variables does not depend on the E/p value.

To get the shape of the background, we select $K\pi 3$ events using the cuts defined above (see p.108). The result for the distributions of the Cabibbo variables is shown in fig. 8.7. This has been cross-checked by varying the cuts (making it more like the Ke4 cuts) - besides the $\pi^\pm$ -peak in the $m_{e\nu}$ distribution that vanishes for more Ke4-like cuts, the shapes do not change much; as the background in the Ke4 signal is small enough, these differences can be neglected for the available statistics.

Once the shape of the background is known, the amount of $K\pi 3$ events in the Ke4 signal has to be determined. This is done by fitting the E/p

distribution for $K\pi 3$ events to the Ke4 signal with released E/p cut. As there are very few remaining $K\pi 3$ events in the signal, this is also done for loosened Ke4 cuts ($R > 10$ instead of $R > 20$) for a cross-check. The result is shown in fig. 8.8.

Figure 8.7: Shape of $K\pi^3$ backgroundFigure 8.8: Fit of E/p distribution of $K\pi^3$ background to $Ke4$ signal

The plots above show the K-sample of data. Applying the same method also to the E-sample gives the following numerical results:

data sample	Ke4 events after cuts	K π 3 events	% K π 3 events
K (KE4)	$1560 \pm 30_{\text{stat}}$	$3 \pm 1_{\text{stat}}$	$0.19 \pm 0.04_{\text{stat}}$
E (ETOT)	$1435 \pm 30_{\text{stat}}$	$6 \pm 1_{\text{stat}}$	$0.44 \pm 0.09_{\text{stat}}$

Table 8.1: K π 3 background in K- and E-sample of data

This is for both samples well below 1 percent, so it is not necessary to apply a correction to the Cabibbo distributions. However, this background will be considered for the branching ratio measurement.

8.3.2 Ke3 Background

Ke3 can only contribute as a Ke4 background if there is a fake π^0 , so the natural control region seems to be outside of the π^0 mass peak. However, it turns out that Ke3 background is so small that it is hard to distinguish it from tails of a true π^0 peak. In order to avoid an overestimation of the Ke3 background, a more suitable variable has been chosen: p_T/E_ν . The properties of this variable have been discussed above (p. 106); relevant here is that this variable can have negative values for Ke3 events - and only for those events.

To get the shape of the background, we select Ke3 events using the cuts defined above (see p.110). The result for the distributions of the Cabibbo variables is shown in fig. 8.9. This result has again been cross-checked with more Ke4-like cuts - because of the small amount of Ke3 background in the data, the differences are negligible within statistical errors.

To determine the amount of Ke3 background, the p_T/E_ν distribution is fitted in the control region $p_T/E_\nu < 0$. The result is shown in fig. 8.10

The plots above again show only the K-sample of data. Applying the same method to the E-sample gives the following numerical results:

data sample	Ke4 events after cuts	Ke3 events	% Ke3 events
K	$1560 \pm 30_{\text{stat}}$	$69 \pm 7_{\text{stat}}$	$4.5 \pm 0.4_{\text{stat}}$
E	$1435 \pm 30_{\text{stat}}$	$75 \pm 8_{\text{stat}}$	$5.2 \pm 0.5_{\text{stat}}$

Table 8.2: Ke3 background in K- and E-sample of data

These results are used to subtract the Ke3 background in the Cabibbo distributions for the analysis below.

8.3. BACKGROUND SUBTRACTION

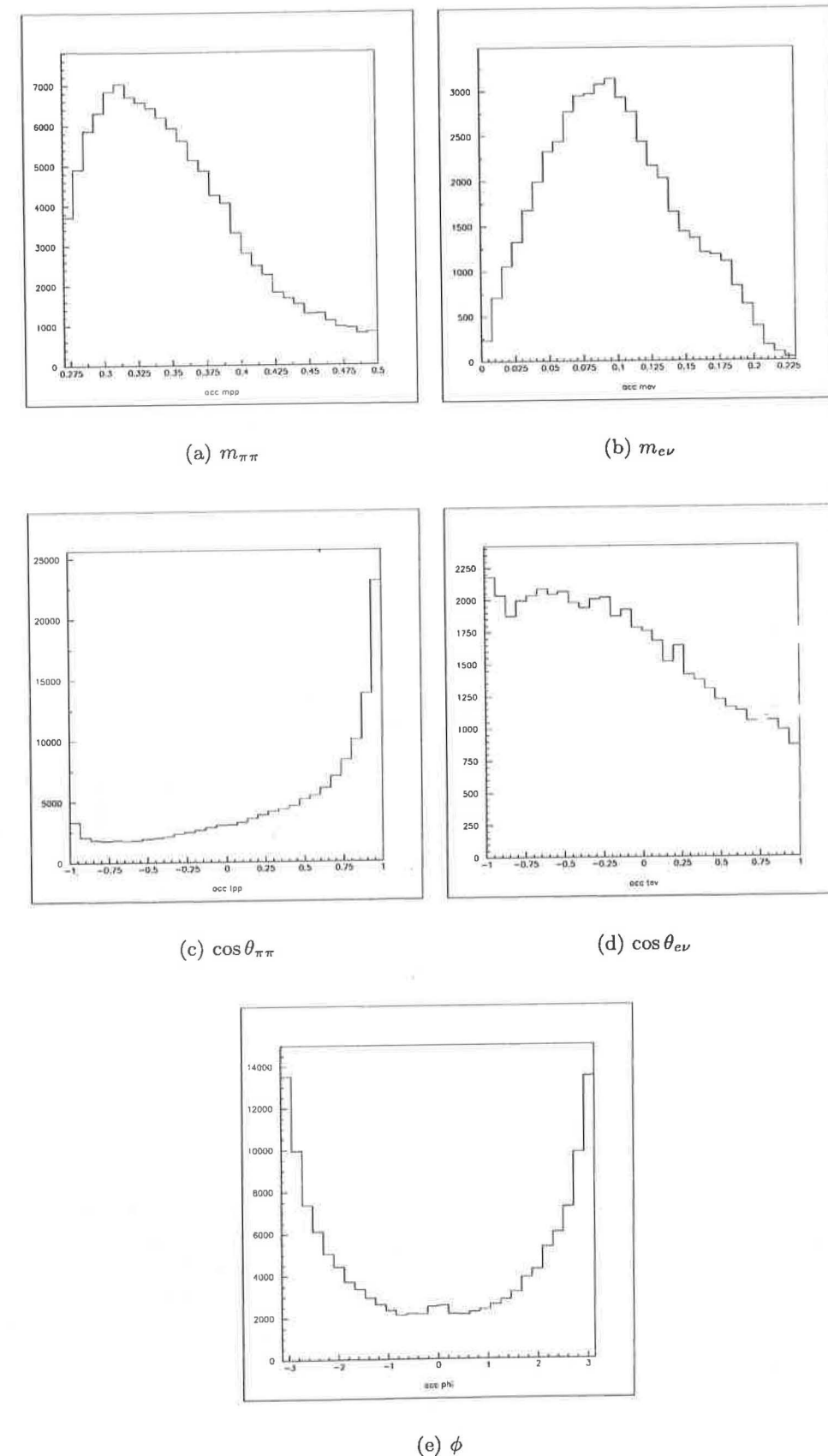
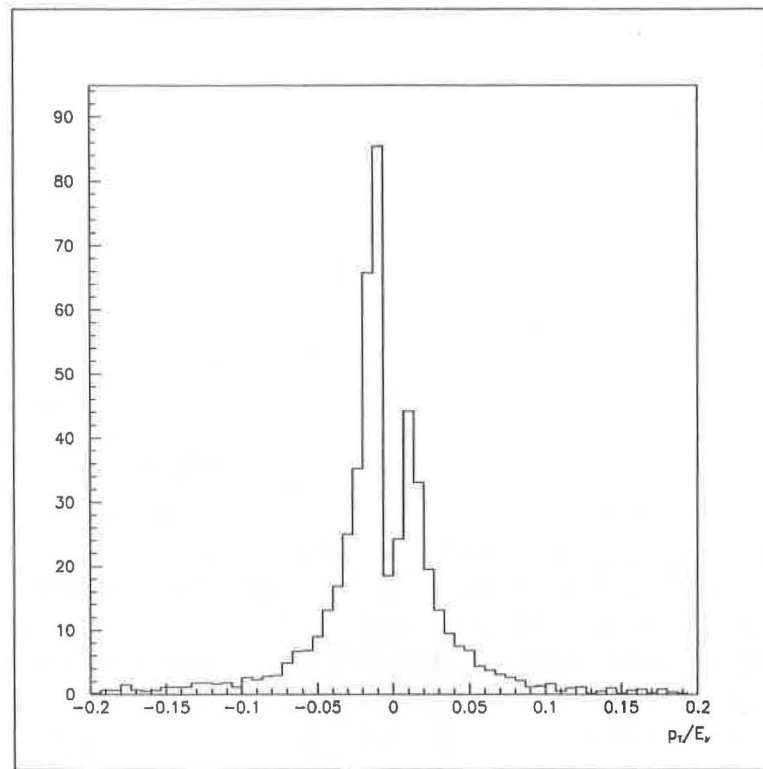
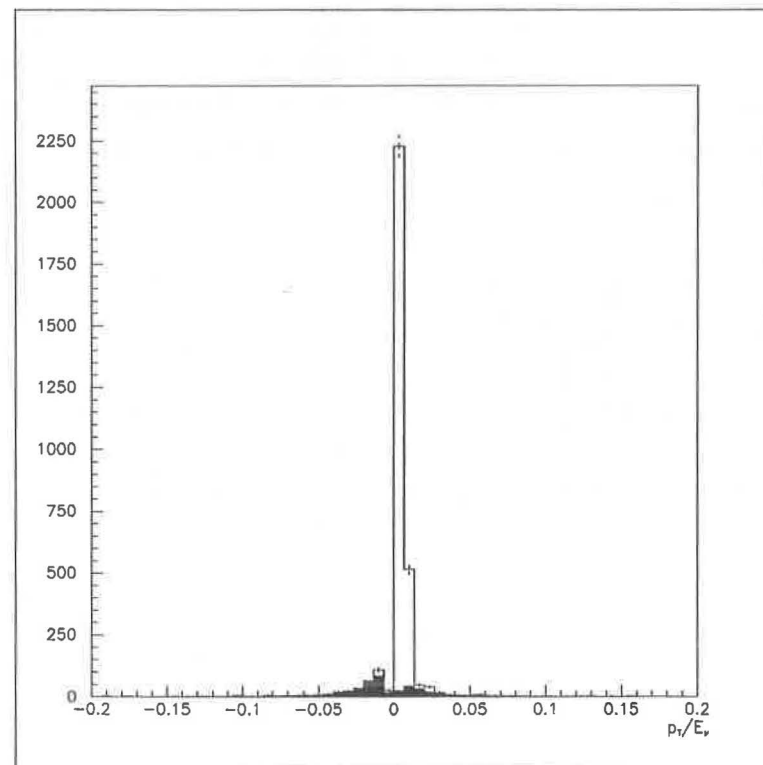


Figure 8.9: Shape of Ke3 background

(a) p_T/E_ν for $Ke3$ events(b) fit to $Ke4$ signalFigure 8.10: fit of p_T/E_ν distribution of $Ke3$ background to $Ke4$ signal

8.4 Final Ke4 Statistics

After all cuts, and subtracting remaining background, the $Ke4$ statistics are listed in the table below.

sample	$Ke4$ after cuts	remaining $K\pi3$	remaining $Ke3$	$Ke4$ after bkg subtr.
K	$1560 \pm 30_{\text{stat}}$	$3 \pm 1_{\text{stat}}$	$69 \pm 7_{\text{stat}}$	$1488 \pm 31_{\text{stat}}$
E	$1435 \pm 30_{\text{stat}}$	$6 \pm 1_{\text{stat}}$	$75 \pm 8_{\text{stat}}$	$1354 \pm 31_{\text{stat}}$
K+E	$2995 \pm 55_{\text{stat}}$	$9 \pm 1_{\text{stat}}$	$144 \pm 11_{\text{stat}}$	$2842 \pm 44_{\text{stat}}$

Table 8.3: Final $Ke4$ statistics for K- and E-sample of data

Chapter 9

Local Acceptance

This chapter shows how the selected Ke4 events are weighted to correct for acceptance.

Of course, the Ke4 selected after all cuts represent a highly biased distribution in phase space. As this cannot be avoided in principle due to the background that must be rejected, the only possible solution is to rely on MC to apply an acceptance correction to the data sample. This requires some care, since generally the acceptance¹ depends on all five Cabibbo variables.

If the distribution in phase space is known (from theory), one need not know the acceptance as a function of the Cabibbo variables, but can simply generate the appropriate distribution in MC and determine a global acceptance correction to get the true total number of events. However, we do not want to be prejudiced this way and aim to measure the distributions instead. We thus need a local acceptance correction.

9.1 Definition of Local Acceptance

The local acceptance α is defined through the following equation:

$$dN_{acc} =: \alpha(m_{\pi\pi}, m_{e\nu}, \cos\theta_{\pi\pi}, \cos\theta_{e\nu}, \phi) \cdot dN_{true} \quad (9.1)$$

where dN_{true} is the true (i.e. generated) number of events in a given element of phase space and dN_{acc} the number of accepted ones (after all cuts).

¹defined as ratio of accepted over generated events for a given element of phase space

9.2 Factorization of Acceptance

In principle, the acceptance α could be determined by simulating a very large number² of Ke4 decays and calculating the ratio of accepted and generated events for a 5-dimensional histogram in all Cabibbo variables. However, considering the limited statistics of experimental data, this seems to be too much of an effort. Alternatively, we factorize the acceptance like this:

$$\alpha(m_{\pi\pi}, m_{e\nu}, \cos\theta_{\pi\pi}, \cos\theta_{e\nu}, \phi) = \alpha_m(m_{\pi\pi}, m_{e\nu}) \cdot \alpha_{\pi\pi}(\cos\theta_{\pi\pi}) \cdot \alpha_{e\nu}(\cos\theta_{e\nu}) \cdot \alpha_\phi(\phi) \cdot \alpha_{corr}(m_{\pi\pi}, m_{e\nu}, \cos\theta_{\pi\pi}, \cos\theta_{e\nu}, \phi)$$

This factorization is obviously not unique - we thus require all α_x but α_m to be normalized:

$$\int dx \alpha_x = 1 \quad (9.2)$$

we further assume that $\alpha_{corr} \approx 1$ - this of course has to be checked with MC: fig 9.1 shows the acceptance as 2-dimensional histograms for each pair of two Cabibbo variables³.

Obviously, the factorization presented above is a good first approximation.

9.3 Local Acceptance Correction

The factorials α_x are extracted from MC data by dividing the corresponding 1- or 2-dimensional histograms of accepted and generated events⁴. To interpolate and smooth the results, the first 20 Fourier components are used (see fig. 9.2).

These functions are used to compute a weight for each event which is the inverse of the acceptance:

$$w(m_{\pi\pi}, m_{e\nu}, \cos\theta_{\pi\pi}, \cos\theta_{e\nu}, \phi) = \frac{1}{\alpha} \quad (9.3)$$

Ideally, weighting all accepted events this way should give the true distribution in phase space. Because of the approximations made, and also

²a rough estimation of the required number of MC events: if we have just 10 bins for each variable (which might be too few), we already have 10^5 bins in total; requiring an statistical uncertainty of 1% for the acceptance, we need 10^4 events in each bin, thus one billion of simulated Ke4 events in total, which is equivalent to roughly 550 years of CPU time; I clearly wanted to finish my thesis earlier

³as this means that it is integrated over the remaining three variables, the results of course depend on the generated distribution in phase space; for this plot, a flat distribution has been used

⁴note that for the accepted events, both neutrino solutions are used with equal weight

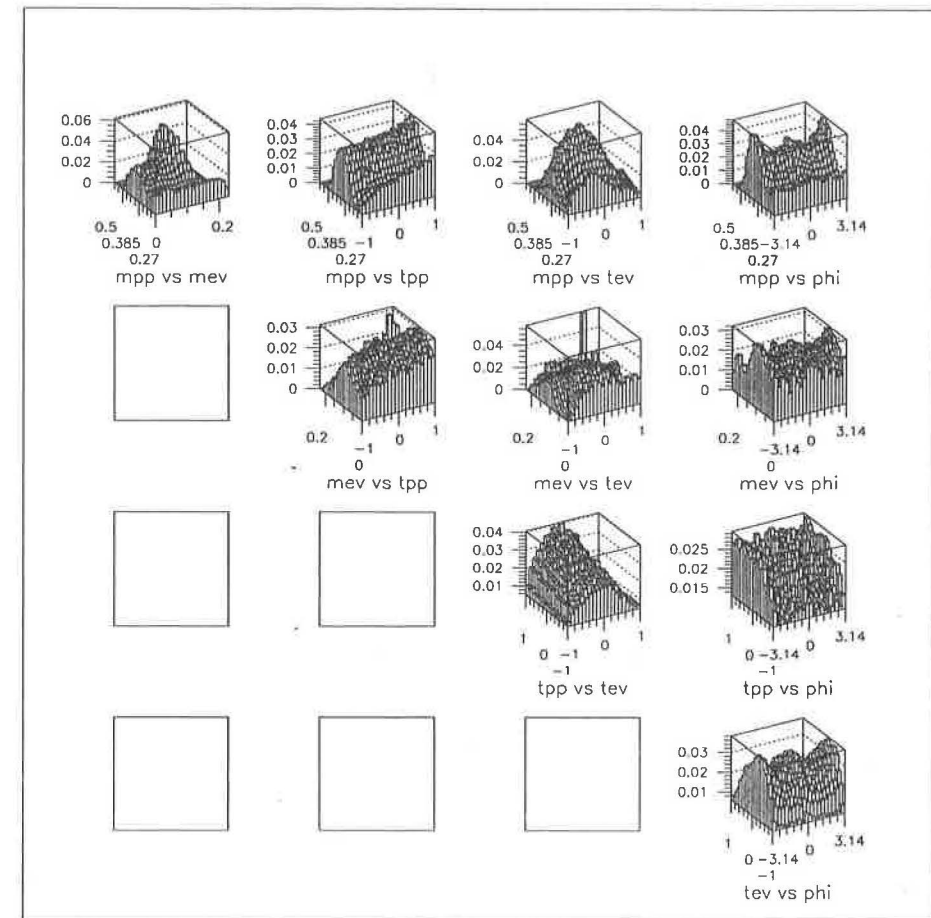


Figure 9.1: acceptance as a function of pairs of two Cabibbo variables, based on MC generated Ke4 decays with a flat distribution in phase space

because of the laws of statistics⁵, there are practical limits. One has also to limit the maximal weight of an event, otherwise events from very-low acceptance regions blow up the error. Fig. 9.3 shows a comparison of generated, accepted and weighted events.

As can be seen from fig. 9.3, for some variables, there are systematic deviations of the acceptance-corrected distributions from the generated ones that are caused by non-factorizeable dependences of the acceptance on the Cabibbo variables. Of course, it would have been easy to use directly the 1d-averaged histograms to correct each variable separately, but the result would clearly depend on the underlying phase space distribution assumed in the MC. So we chose this method, which is (at first order) independent of any assumptions about the phase space distribution, and consider the

⁵in regions of phase space where the acceptance is low, the uncertainty of the true distribution is unavoidably large

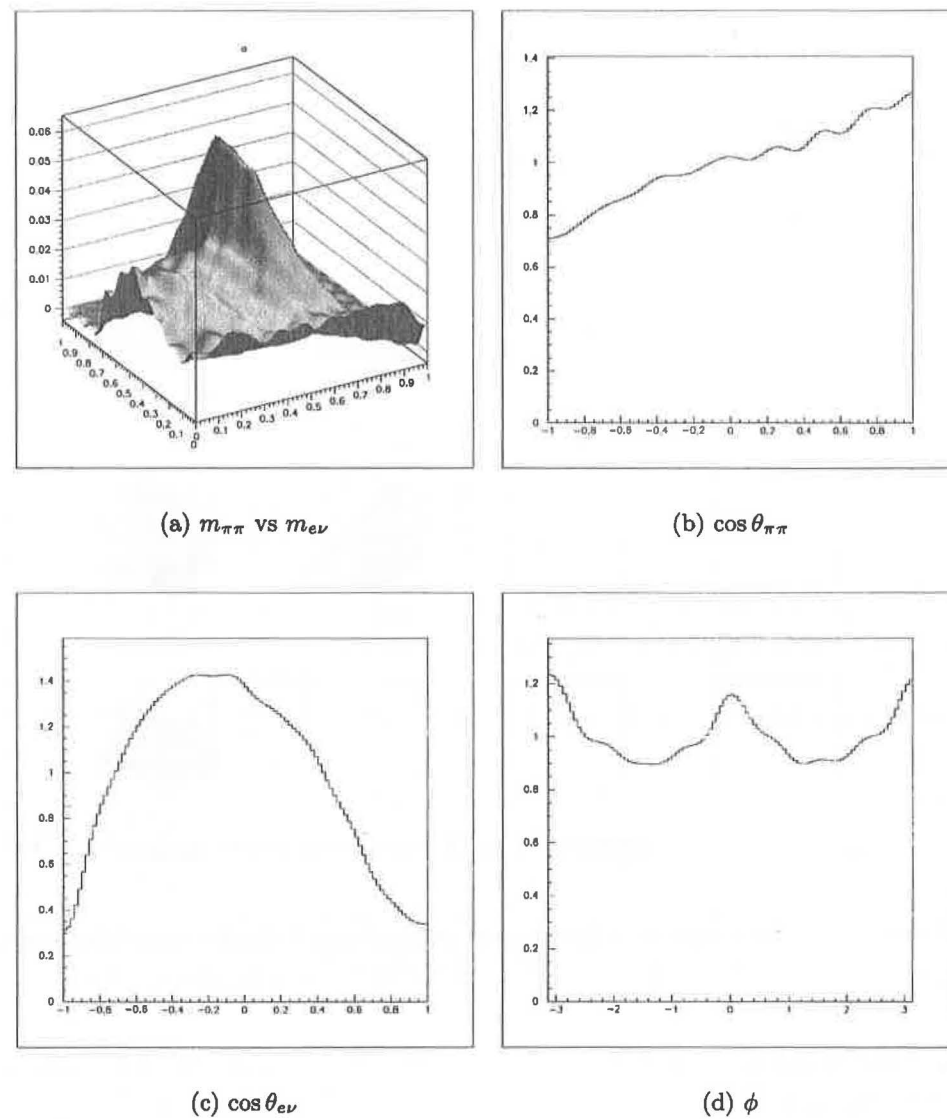


Figure 9.2: acceptances extracted from MC are interpolated and smoothed using the first 20 Fourier components

deviations as systematic errors.

Fig. 9.3 also shows that this systematic error is very small for the most interesting variable $m_{\pi\pi}$, and also quite good for $\cos \theta_{\pi\pi}$ - except for values < -0.75 . The distribution for $\cos \theta_{e\nu}$ is for most values systematically too low, which is only a minor problem since we are at this point more interested in shapes than in the global normalization of acceptances⁶. Nevertheless,

⁶the absolute numbers for acceptance will be derived from a special MC with a phase

these results show the limits of this method due to the neutrino ambiguity. We consider the deviations as systematic errors that are simply defined as the difference between generated and acceptance-corrected distribution.

space distribution based on the results of the form factor analysis

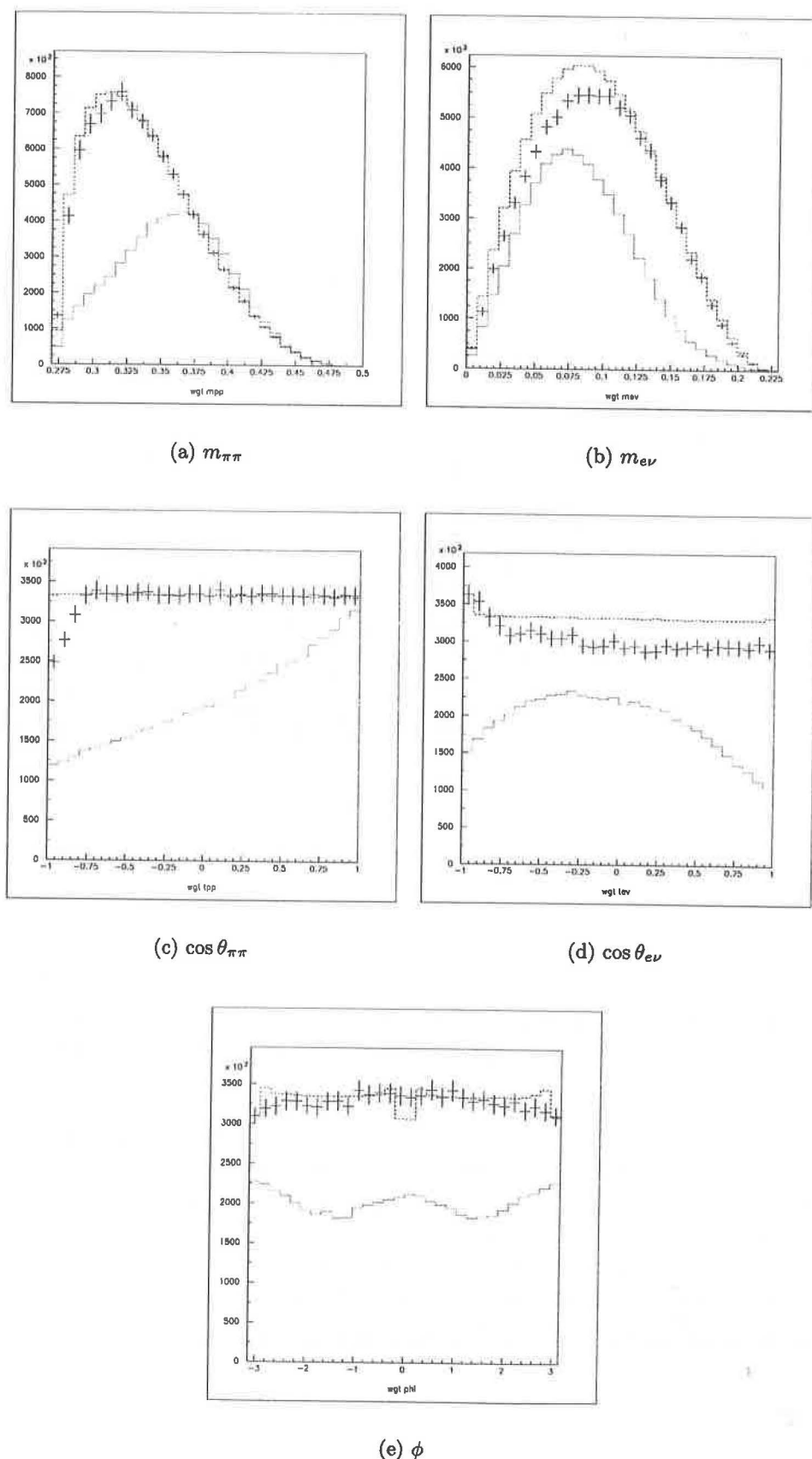


Figure 9.3: Comparison of MC-generated (dashed), accepted (dotted) and acceptance-corrected (with error bars) events

Chapter 10

The Results

This final chapter discusses the results of the fitting of form factors, the Ke4 branching ratio and the L_3 coupling constant

To minimize the number of assumptions concerning the physics of the Ke4 decay, we proceed like this:

First, we assume nothing more than general kinematics to check the predictions originating from symmetry considerations (vanishing s-wave for the dipion). We also study the relative amplitudes of the form factors in a model-independent way. This is done by fitting the distributions of the two most significant Cabibbo variables.

After fixing the phase space distribution up to a constant this way, we generate MC events obeying this distribution to determine a global acceptance correction¹; normalizing to the $K\pi 3$ mode, a result for the branching ratio can be stated, which in turn also gives results for the absolute values of the form factors.

Eventually, the obtained branching ratio is used to determine the chiral L_3 parameter as discussed in the theoretical part of this thesis.

10.1 Ke4 Form Factors

10.1.1 Fit Method

To determine the relative amplitudes of the form factors, we use only the distribution in the $m_{\pi\pi}$ and $\cos \theta_{\pi\pi}$ variables (integrating over the remaining three variables). The reason for this is on one hand the limited statistics, on the other hand the fact that $m_{\pi\pi}$ is the only variable that is totally

¹we do this to be independent of the approximations used for the local acceptance correction

independent of the neutrino solution. However, we will also look at the other variables to compare with theoretical expectations as a cross-check.

Looking at the two dimensional plot of $m_{\pi\pi}$ and $\cos\theta_{\pi\pi}$ (fig. 9.1, p. 127) it is obvious that the correlation of these two variables is small, we therefore prefer to fit two one-dimensional distributions instead of the two dimensional one, which gives smaller statistical errors per bin.

It is also important to note that we do not use local acceptance correction for the fit. This is simply because the local acceptance correction in any case has to rely on MC, so it is more straightforward to directly fit the unweighted data². However, we will use the acceptance-corrected data for a cross-check.

The fit parameters are f_s, f_p, g and λ_g ; we do not fit the form factor h because it is kinematically highly suppressed as derived in the theoretical part of this work³. Still, we will study the influence of h on the fit results and will use the theoretical value to apply a correction to them (see below).

10.1.2 First Cross-Checks

Before looking at the fit, we do a rough comparison of the observed distributions with theoretical predictions for different form factors (fig. 10.1). It is eye-striking that the data is clearly dominated by the G form factor as expected theoretically.

As a cross-check, we apply the acceptance correction to all five Cabibbo variables and compare with the theoretical predictions in the case of pure G form factor (fig. 10.2).

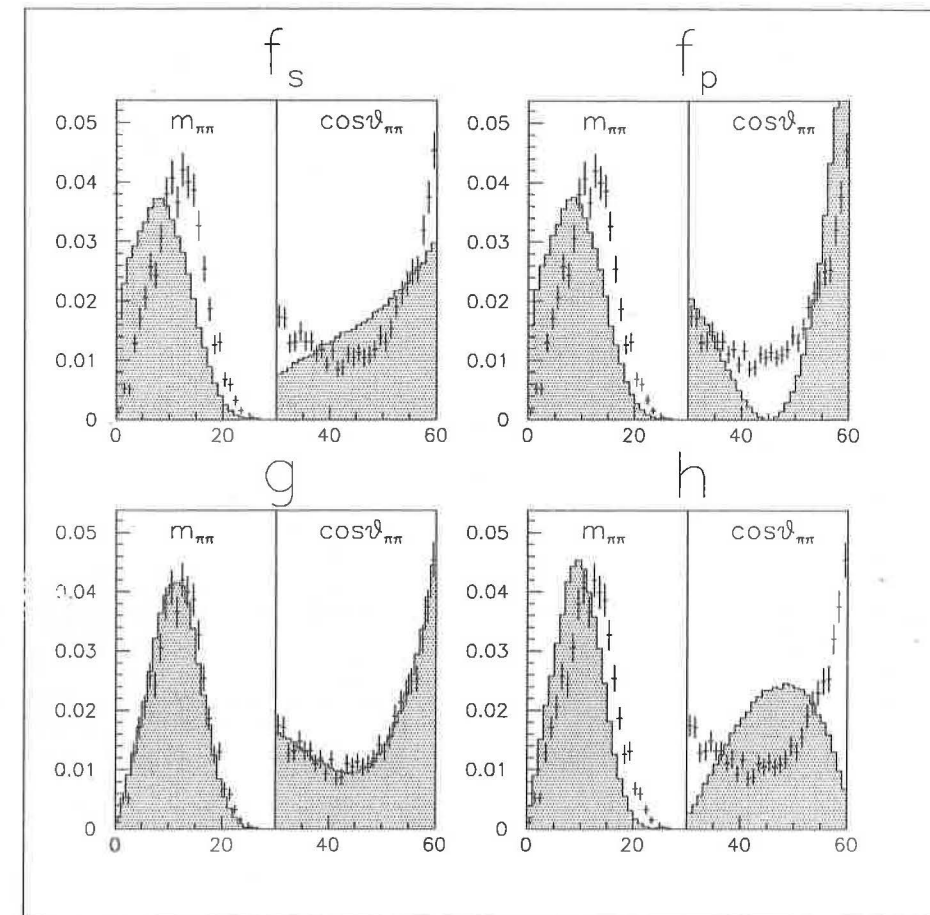


Figure 10.1: comparison of observed distributions in $m_{\pi\pi}$ and $\cos\theta_{\pi\pi}$ (points with error bars) and theoretical distributions for different form factors (shaded regions). Please note that the x-axes show the bin numbers

²which means both neutrino solutions for each event weighted with 1/2 each

³trying to fit h leads to a null result with very large errors of $O(10)$

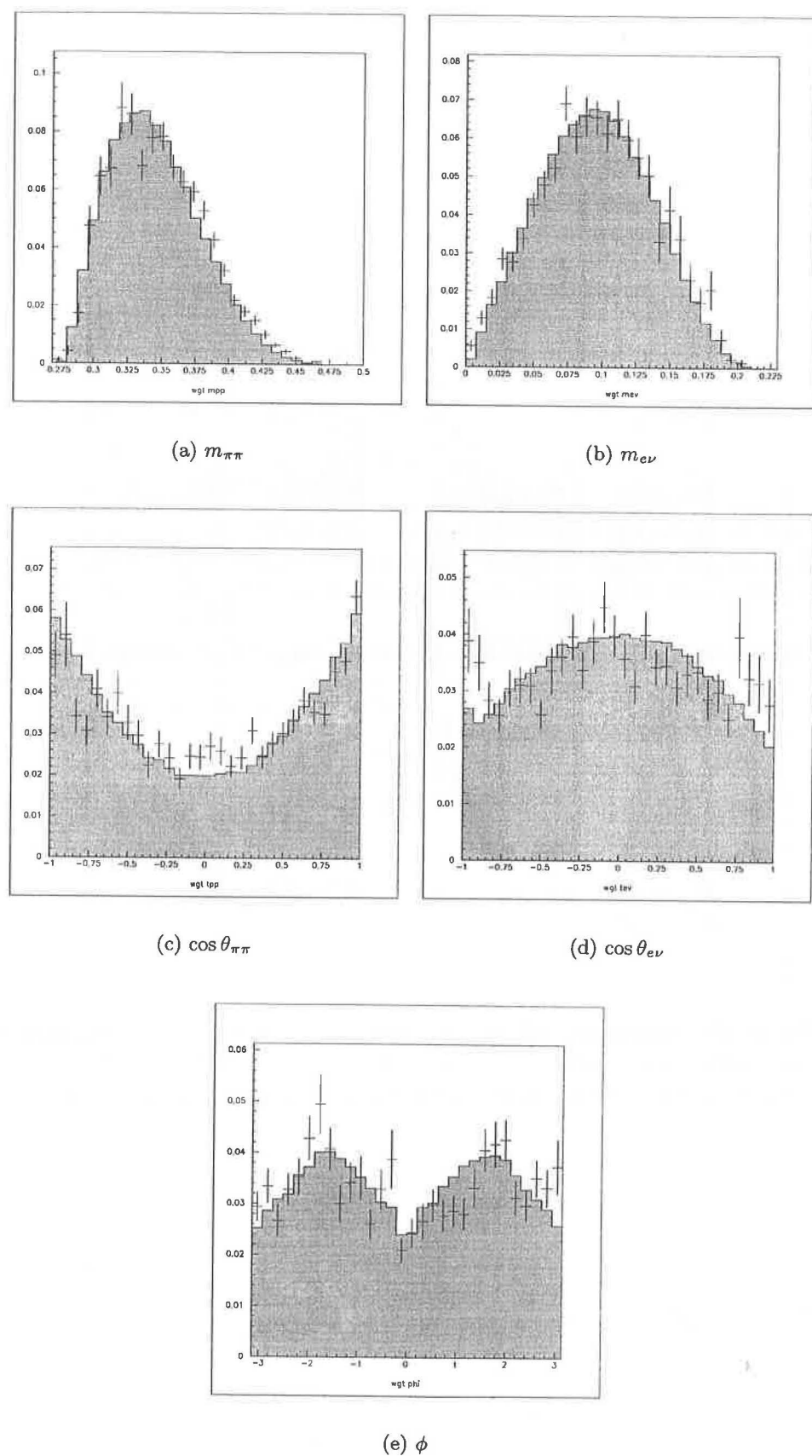


Figure 10.2: comparison of MC-generated (shaded) and acceptance-corrected data (with error bars)

10.1.3 The Fit

Although the agreement between data and theory is already quite nice, for more quantitative statements we perform a non-linear⁴ fit in the four parameters f_s, f_p, g and λ_g . As fit method we use log-likelihood as implemented in MINUIT [37], the (asymmetric) errors are also determined by MINUIT using the e option.

The result of the simultaneous fit of $m_{\pi\pi}$ and $\cos \theta_{\pi\pi}$ (30 bins each) is shown in fig. 10.3.

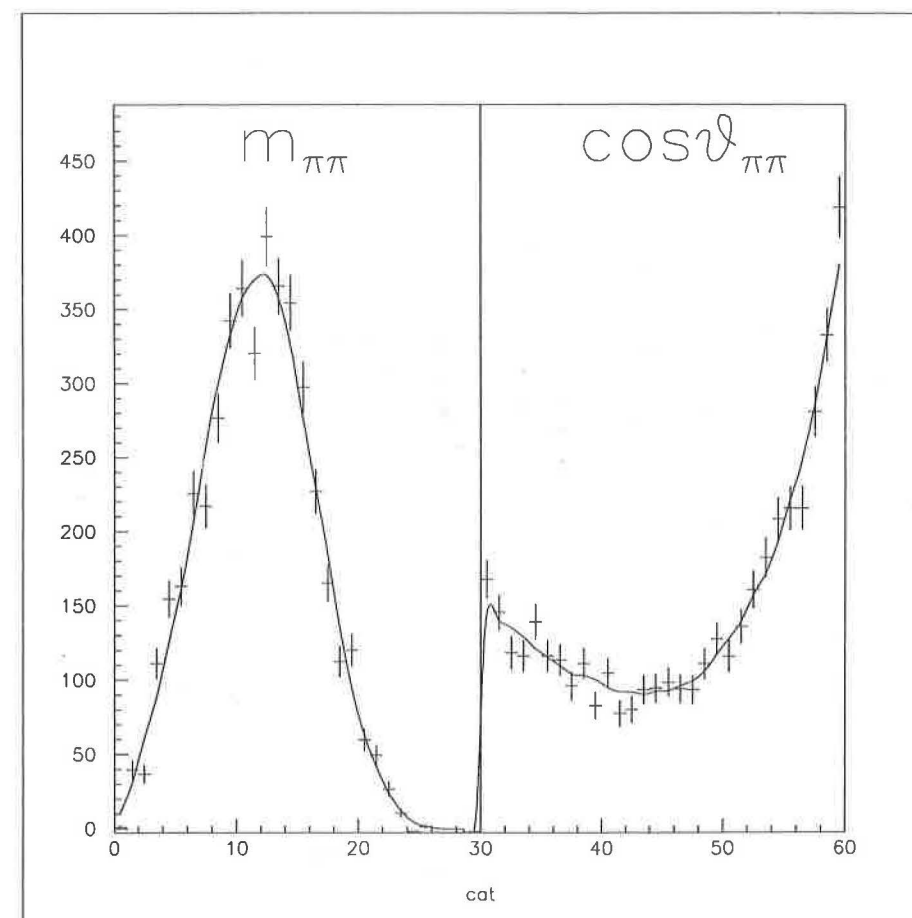


Figure 10.3: Result of the simultaneous fit of $m_{\pi\pi}$ and $\cos \theta_{\pi\pi}$ (30 bins each, bin numbers shown at x-axis)

⁴the fit is non-linear because of the interference between F and G as discussed earlier; the non-linearity has implications on the interpretation of the χ^2 -value, which in this case does not obey a χ^2 -distribution (see discussion below)

10.1.4 Checking the Systematics

The systematics of the fit parameters are checked in figs. 10.4 and 10.5

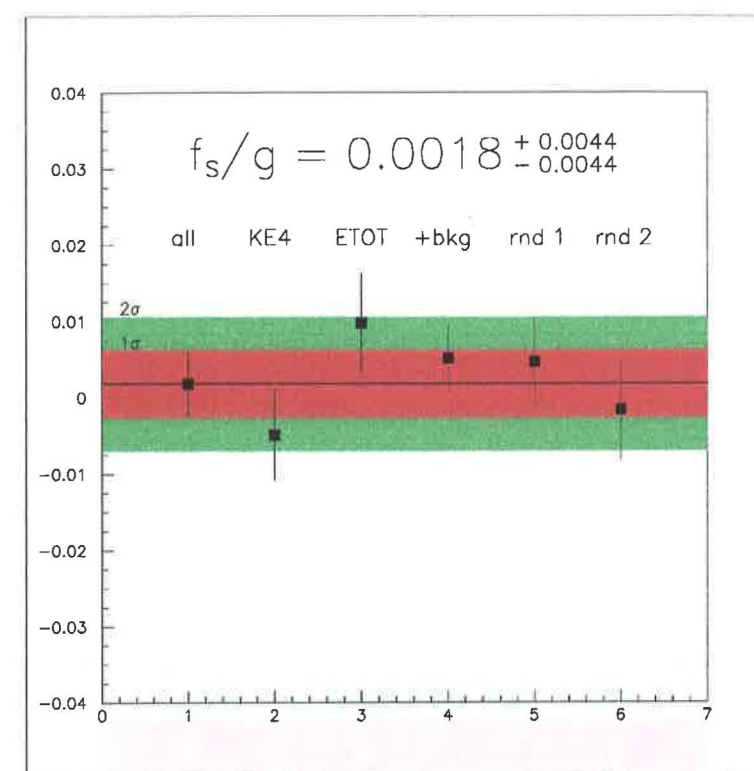
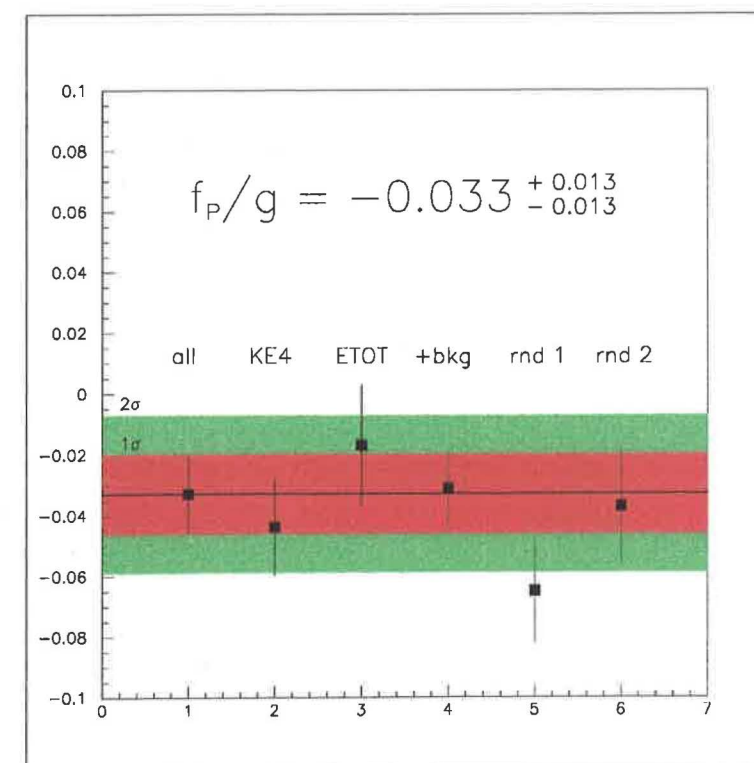
(a) f_s (b) f_P

Figure 10.4: systematics of fit parameters

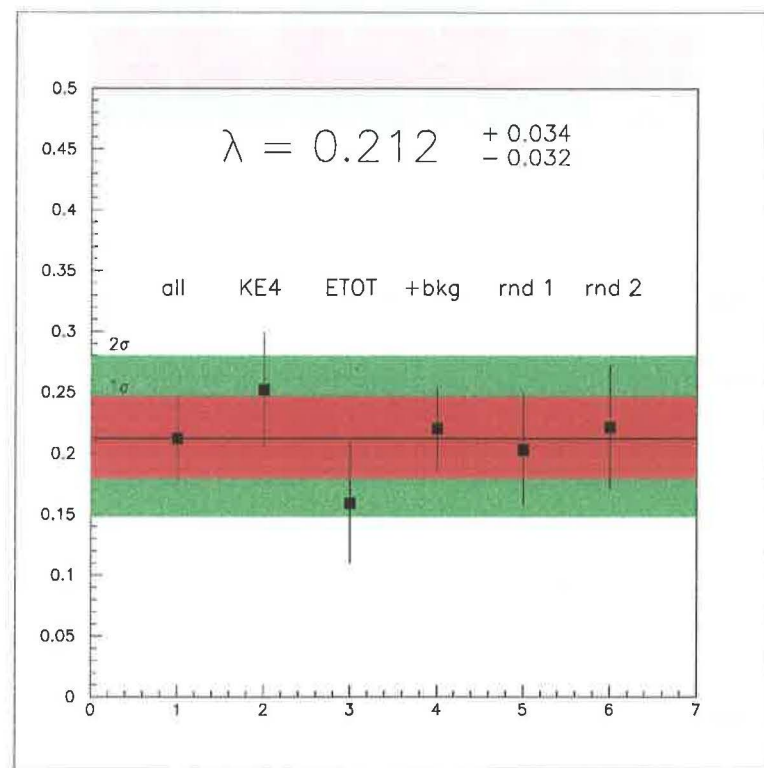
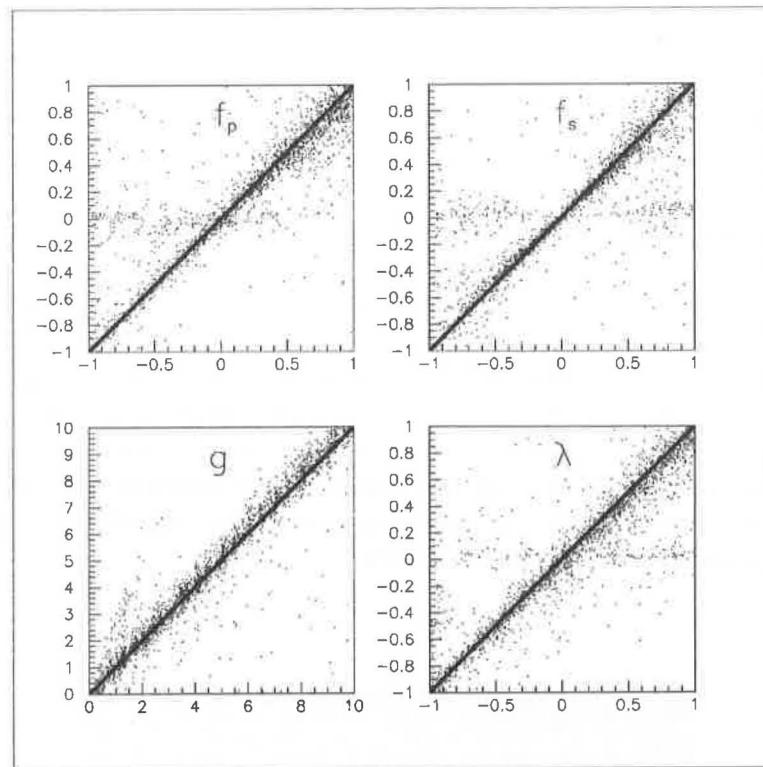
(a) λ_g

Figure 10.5: systematics of fit parameters

Obviously, there are no big systematic effects, and the error is dominated by statistics. To check the fit algorithm itself, simulated data (using the fit results as parameter input, and applying a gaussian smearing according to the expected statistical fluctuations) is fitted many times using exactly the same procedure as for real data. Fig. 10.6 shows the fitted vs. true parameters. Apart from some fits that obviously went wrong⁵, the fit value is correct within errors. Fig. 10.6 shows the (normalized) χ^2 -distribution for those fits, compared with the theoretical χ^2 -distribution for linear fits; the agreement is good for values around 1, but the χ^2 for the non-linear fit has a much bigger tail to higher values.

Finally, also the influence of the form factor h , which was not fitted, has to be studied. As fig. 10.7 shows, the value of h does change the other fit parameters, although within their errors. These systematic shifts can be seen in the residuals, where H was put to the theoretical value of -2.797 (2.96). On the other hand, the distributions of the residuals show that the error is estimated correctly by MINUIT.

⁵to keep the script for this check simple, always the same start values for the parameters were used, although MINUIT sometimes needs some manual intervention to find the correct minimum



(a) fitted vs. true

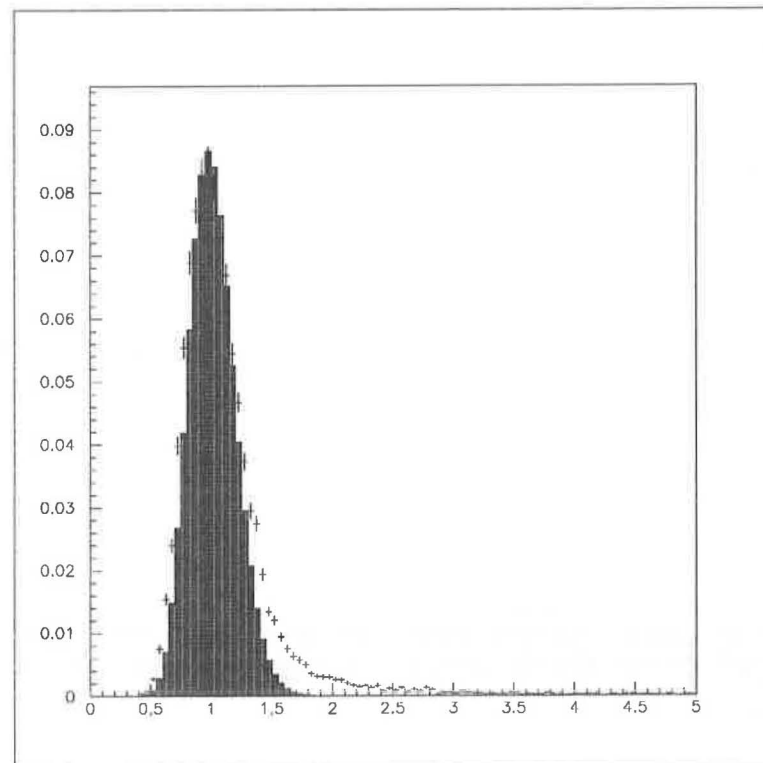
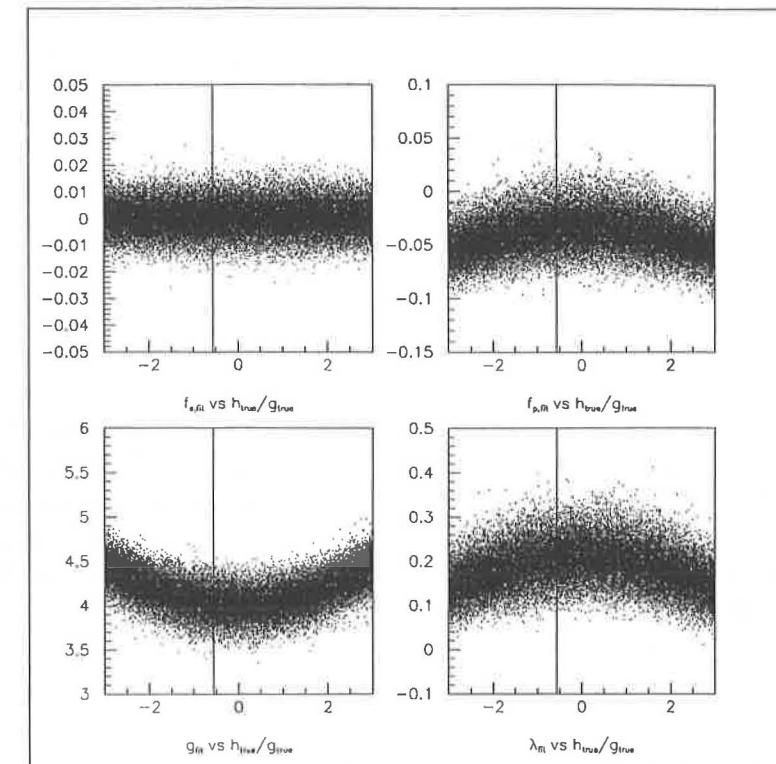
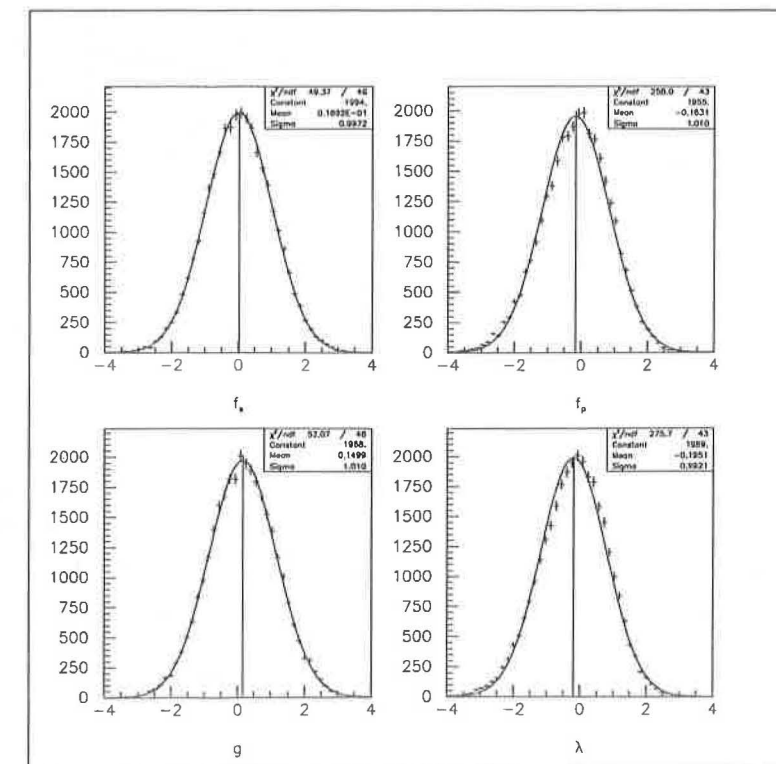
(b) χ^2

Figure 10.6: fit Monte Carlo

(a) dependence on h (b) residuals $\frac{x_{\text{fit}} - x_{\text{true}}}{\sigma_{\text{fit}}}$ Figure 10.7: influence of form factor H on fit parameters

Correcting for these shifts, the final results are shown in table 10.1.

form factor	result w/o correction	result after correction
f_s/g	0.0018 ± 0.0044	0.0017 ± 0.0044
f_p/g	-0.033 ± 0.013	-0.031 ± 0.012
λ_g	0.212 ± 0.033	0.218 ± 0.034

Table 10.1: final results of the form factor fit

10.1.5 Discussion of the Results

As expected, f_s/g is compatible with zero and f_p/g is small. However, λ_g seems to be larger than expected by 1-loop χPT (2.108). To cross-check this result, the G form factor as a function of $m_{\pi\pi}$ has been extracted directly by subtracting the (small) f_p component from the $m_{\pi\pi}$ distribution and dividing by the distribution for flat G ; the result is shown in fig. 10.8. Also here, a clear deviation of the observed slope from the theoretical expectation can be seen.

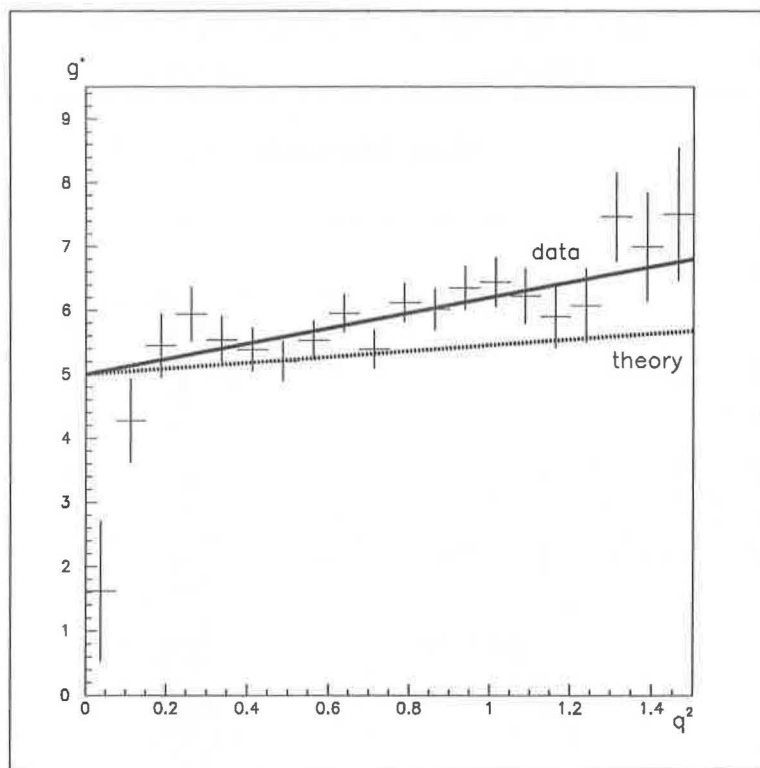


Figure 10.8: slope of the G form factor; comparison between data and theory

Further checks involved fitting of additional background, flat or linear

rising noise in $m_{\pi\pi}$. None of these effects could explain an enlargement of λ_g to this extent within levels consistent with the data. So the given value has to be taken as experimental result.

It has to be remarked that it is well known [11] that higher-order contributions (2-loop) have significant effect on the theoretical form factors, so the experimental result may seem less surprising this way.

10.2 Check of Phase Differences

Although theory does not predict a phase difference between the form factors F and G , data can still be tested for that as described earlier (see chapter 2.5 on page 47). For this method to be valid, also the acceptance correction has to be applied.

The overall result, averaging over both $m_{\pi\pi}$ and $m_{e\nu}$ gives

$$\delta_F - \delta_G = (-5 \pm 9) \text{ mrad} \quad (10.1)$$

This result is well compatible with zero. As a second check, also the (possible) dependence of the phase shift on $m_{\pi\pi}$ is studied; the result is shown in fig. 10.9 - no dependence can be seen within errors.

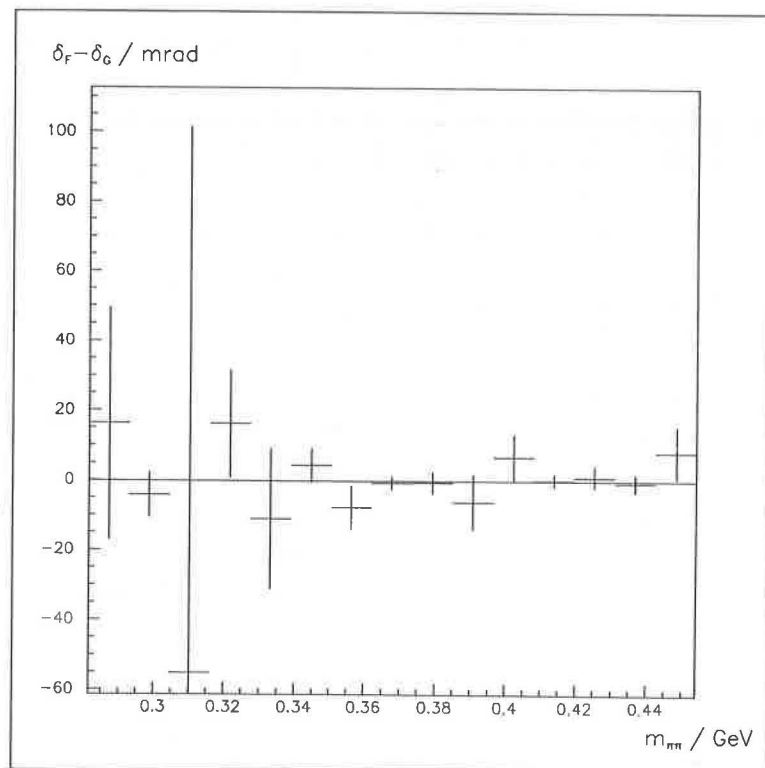


Figure 10.9: dependence of the phase shift

10.3 Ke4 Branching Ratio

10.3.1 Normalization Channel

As normalization channel $K_L \rightarrow \pi^+ \pi^- \pi^0$ with $BR = (12.55 \pm 0.2)\%$ is used. The selection of $K\pi 3$ events is done using essentially the cuts described before, but with no upper bound for the E/p values; this is necessary because of the poor MC modelling of E/p , which would introduce bias in the acceptance correction determined below⁶. In the case of ETOT events, also $K\pi 3$ events with less than 4 clusters (one lost due to low energy resolution) are selected for the same reason.

This $K\pi 3$ normalization channel is taken for several reasons:

- the Ke4 trigger mainly triggers $K\pi 3$ events, as opposed to Ke3 events which are only triggered with low efficiency (due to the *3peaks*-condition, cf. p. 72)
- it can be easily selected and fully reconstructed (no missing neutrino as in the case of Ke3)
- it also contains a π^0 like Ke4, so no correction for Dalitz-decays has to be applied (it simply cancels out)
- and eventually it was also meant to be a cross-check of the analysis of [32] that used Ke3 as normalization channel

The number of $K\pi 3$ events found for both triggers is shown in the table below. Note that the error is affected by the additional downscaling of factor 10 which was applied to $K\pi 3$ data in order to save disk space.

data sample	$K\pi 3$ events after cuts
K (KE4)	$4691800 \pm 6800_{\text{stat}}$
E (ETOT)	$17754000 \pm 13000_{\text{stat}}$

Table 10.2: $K\pi 3$ events in K- and E-sample of data

10.3.2 Global Acceptance Correction

As the MC is not perfect in every detail of the detector (it was optimized for the ϵ'/ϵ -analysis), special care has to be taken in determining the acceptance correction. Although the MC is reasonably accurate for physics variables (see fig. 8.5, p. 114), it is not for more technical variables like event times, or the ratio E/p used to identify charged particles.

⁶since the E/p value is only a detector feature and physically independent of the Ke4 decay, this cut does not harm the other studies

To avoid bias from the wrong shape of the E/p distribution in MC, the corresponding cut is dropped for the determination of the acceptance correction. Secondly, the cleaning cuts in the filters applied to data are *not* applied to MC; it was studied in detail that this only has influence on the (poorly simulated) technical variables, and not on physical variables.

Effects like bremsstrahlung or accidentals are included in the MC and thus need not be corrected for (they are included in the acceptance correction).

For the normalization channel, an MC sample of one million NASIM [24] events was used to find the acceptance for $K\pi 3$ events to be

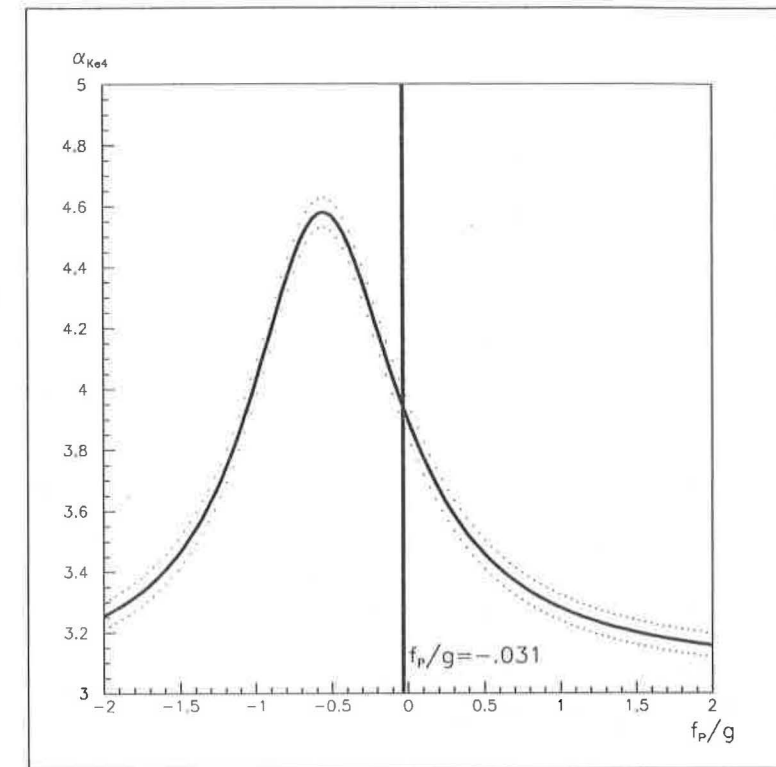
$$\begin{aligned}\alpha_{K\pi 3} &= (5.266 \pm 0.022_{\text{mcstat}}) \% \\ \alpha_{K\pi 3} &= (11.38 \pm 0.14_{\text{mcstat}}) \% \quad (\text{ETOT})\end{aligned}\quad (10.2)$$

where "mcstat" means the uncertainty induced by the MC statistics. Note that the uncertainty is much larger for ETOT, because of imperfections of MC for the lower energy threshold.

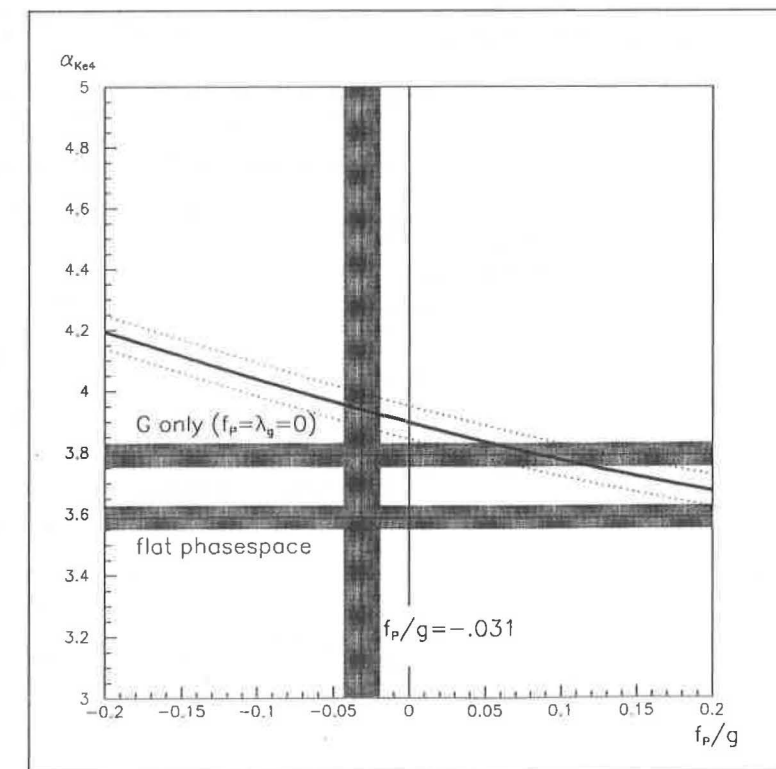
For Ke4 , an MC sample of 237995 NASIM events with a flat distribution in phase space was used. By appropriate weighting also the distributions for a pure G form factor as well as for the observed admixture of the FG -interference and the slope λ_g are generated; the acceptances for these cases are:

$$\begin{aligned}\alpha_{\text{Ke4,flat}} &= (3.588 \pm 0.038_{\text{mcstat}}) \% \\ \alpha_{\text{Ke4,G}} &= (3.792 \pm 0.039_{\text{mcstat}}) \% \\ \alpha_{\text{Ke4,full form factors}} &= (3.939 \pm 0.040_{\text{mcstat}} \pm 0.084_{\text{stat}}) \% \\ \alpha_{\text{Ke4,full form factors}} &= (2.151 \pm 0.068_{\text{mcstat}} \pm 0.046_{\text{stat}}) \% \quad (\text{ETOT})\end{aligned}\quad (10.3)$$

The ratio of G and the interference FG and λ_g are taken as measured above (see table 10.1, p. 142), the second error in the corresponding acceptance arises from the uncertainty of this measurement which is statistically limited; the influence of the admixture FG on the acceptance is shown in fig. 10.10 - obviously the acceptance is sensible to this component.



(a) full range



(b) close up

Figure 10.10: influence of the amount of FG -interference on the Ke4 acceptance (dotted lines show range of uncertainty)

10.3.3 Results and Cross-Checks for the BR

The Ke4 branching ratio (BR) is calculated as

$$BR_{Ke4} = \frac{N_{Ke4}}{N_{K\pi3}} \cdot \frac{\alpha_{K\pi3}}{\alpha_{Ke4}} \cdot \frac{\epsilon_{K\pi3}}{\epsilon_{Ke4}} \cdot BR_{K\pi3} \quad (10.4)$$

this takes into account the following corrections:

- the correction for remaining background (subtracted as described on page 116)
- the correction for π^0 Dalitz decays (cancels with the π^0 in the normalization channel)
- the correction for accidentals and bremsstrahlung (contained in MC)
- the correction for the differences in acceptance
- the correction for differences in trigger efficiency

For the ratio of accepted events, using the numbers from table 8.3, p. 123 and table 10.2, p. 145, one has:

$$\begin{aligned} \frac{N_{Ke4}}{N_{K\pi3}} &= \frac{1488 \pm 31_{\text{stat}}}{4691800 \pm 6800_{\text{stat}}} = (3.171 \pm 0.066_{\text{stat}}) \cdot 10^{-4} \quad (10.5) \\ \frac{N_{Ke4}}{N_{K\pi3}} &= \frac{1354 \pm 31_{\text{stat}}}{17754000 \pm 13000_{\text{stat}}} = (7.63 \pm 0.17_{\text{stat}}) \cdot 10^{-5} \quad (\text{ETOT}) \end{aligned}$$

The ratio of acceptances gives with (10.2) and (10.3)

$$\begin{aligned} \frac{\alpha_{K\pi3}}{\alpha_{Ke4}} &= \frac{5.266 \pm 0.022_{\text{mcstat}}}{3.939 \pm 0.040_{\text{mcstat}} \pm 0.084_{\text{stat}}} = 1.337 \pm 0.015_{\text{mcstat}} \pm 0.027_{\text{stat}} \quad (10.6) \\ \frac{\alpha_{K\pi3}}{\alpha_{Ke4}} &= \frac{11.38 \pm 0.14_{\text{mcstat}}}{2.151 \pm 0.068_{\text{mcstat}} \pm 0.046_{\text{stat}}} = 5.29 \pm 0.18_{\text{mcstat}} \pm 0.011_{\text{stat}} \quad (\text{ETOT}) \end{aligned}$$

Trigger efficiencies (see table 4.1, p. 80) give the following correction factor for KE4-triggered events (ETOT events do not need this correction):

$$\frac{\epsilon_{K\pi3}}{\epsilon_{Ke4}} = \frac{64.00 \pm 0.63_{\text{eff}}}{61.3 \pm 5.2_{\text{eff}}} = 1.044 \pm 0.089_{\text{eff}} \quad (10.7)$$

Finally, the $K\pi3$ branching ratio is taken from PDG [34]:

$$BR_{K\pi3} = (12.55 \pm 0.20)\% \quad (10.8)$$

Inserting all numbers, the final Ke4 branching ratio is given by

$$\begin{aligned} BR_{Ke4} &= (5.55 \pm 0.16_{\text{stat}} \pm 0.11_{\text{syst}} \pm 0.47_{\text{eff}}) \times 10^{-5} \quad (10.9) \\ &\quad \pm 0.51 \\ BR_{Ke4} &= (5.07 \pm 0.16_{\text{stat}} \pm 0.19_{\text{syst}}) \times 10^{-5} \quad (\text{ETOT}) \\ &\quad \pm 0.25 \end{aligned}$$

where the systematic error contains the uncertainties of the $K\pi3$ BR and the acceptances. Since in the case of KE4-triggered events the biggest contribution to the error originates solely from the poorly known trigger efficiencies, this error is stated extra to demonstrate the smallness of the other uncertainties.

Looking closer at the efficiency problem one might argue like such: as shown before (cf. p. 81), the inefficiency originates from problems with the pretrigger mostly. There is no reason for any difference in pretrigger efficiency for the two modes Ke4 and $K\pi3$; consequently, one might assume that the efficiencies are in fact the same (within errors) for both modes, which is also in agreement with the measured efficiencies. As a cross-check, efficiency has been remeasured using looser Ke4 cuts. Although there is consequently more background in the selected Ke4 events, this does not harm the efficiency measurement very much. The result was a Ke4 efficiency of $(63.4 \pm 2.1)\%$, which supports the assumption of negligible efficiency differences. In this case, the efficiency correction can be dropped, and the BR result changes to

$$BR'_{Ke4} = (5.32 \pm 0.15_{\text{stat}} \pm 0.11_{\text{syst}}) \times 10^{-5} \quad (10.10) \quad \pm 0.19$$

Although we are convinced that it is quite reasonable to abstain from the efficiency correction, we still quote both results, which we believe is the more honest procedure.

To cross-check this result, the BR has also been measured for various subsamples, and also one enlarged sample:

- BR for $K_L \rightarrow \pi^0 \pi^- e^+ \nu$ only (e^+ only)
- BR for $K_L \rightarrow \pi^0 \pi^+ e^- \bar{\nu}$ only (e^- only)
- BR for 1st neutrino solution (ν sol. 1)
- BR for 2nd neutrino solution (ν sol. 2)
- BR for relaxed $K\pi3$ -ellipse cut ($R > 10$ instead $R > 20$, leading to $\approx 80\%$ more Ke4 events⁷)

⁷the uncertainty on the increased background is larger than the gain in systematics, that is why this cut is tighter for the fiducial sample of Ke4

The results of this systematics check is shown in fig. 10.11; all but the first results are without efficiency correction

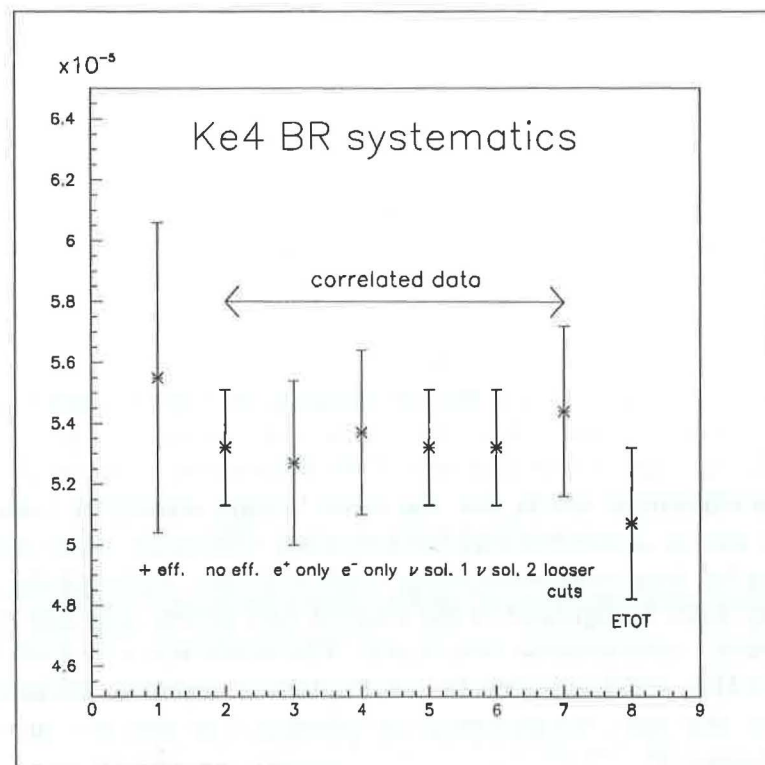


Figure 10.11: systematics check for Ke4 branching ratio

Especially the result with the enlarged sample (80% more events), where independently the acceptance is larger by (almost) the same factor, provides a valuable check of the acceptance correction. No immediate conclusion about the efficiency correction can be drawn, since efficiency might be similar for all checked samples, and still be different for both decay modes. Still it can be noted that the results without efficiency correction look consistent, also with the ETOT result.

The comparison with previous results is shown in fig. 10.12. The new results, both with and without efficiency correction, agree well with the previous measurements. Looking at the new world averages it should be noted that the mean value does not depend much on whether the efficiency correction is applied or not (of course the error margins are different).

The new averages (including the Makoff result as well as both the KE4 and ETOT results) are numerically

$$BR_{\text{average}} = (5.16 \pm 0.18) \cdot 10^{-5} \quad (10.11)$$

$$BR'_{\text{average}} = (5.22 \pm 0.13) \cdot 10^{-5} \quad (10.12)$$

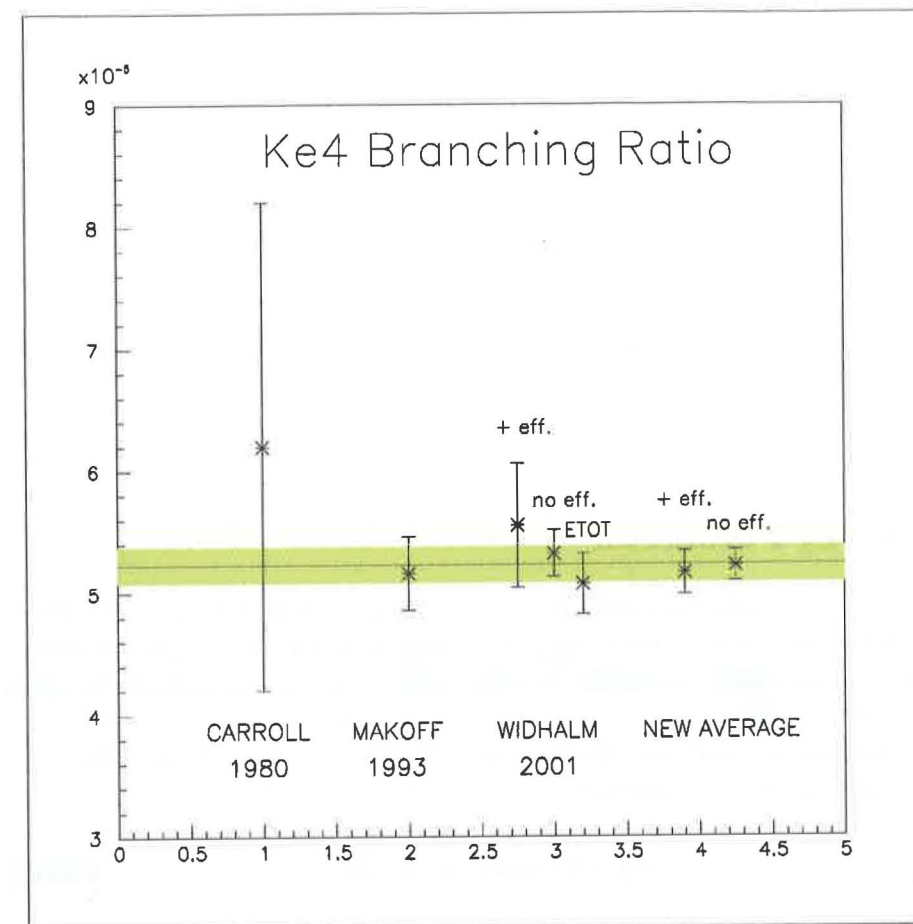


Figure 10.12: comparison of the Ke4 branching ratio with previous measurements; the second result labelled 'no eff' shows the branching ratio without efficiency correction

For all subsequent results, only these two averages are used.

10.4 Ke4 Form Factors - Absolute Values

Form factor G can be calculated using (2.107) and averaging over the $m_{e\nu}$ -dependence, which in turn gives also absolute values for the other form factors (see table 10.1 on page 142):

form factor	with efficiency corr.	without efficiency corr.
g	5.02 ± 0.11	5.057 ± 0.079
f_s	0.009 ± 0.022	0.009 ± 0.022
f_p	-0.156 ± 0.060	-0.157 ± 0.061

Table 10.3: final results for form factors

10.5 L_3 Parameter

It has to be emphasized that L_3 is determined within 1-loop χPT ; the 2-loop corrections are already calculated, and are known to significantly shift L_3 to smaller values. However, it is beyond the scope of this work to apply higher-order χPT .

Using the method described on page 52, one finds that the measured BR is compatible with a value for L_3 of

$$L_3 = (-4.02 \pm 0.35) \cdot 10^{-3} \quad (10.13)$$

with efficiency correction and

$$L'_3 = (-4.14 \pm 0.25) \cdot 10^{-3} \quad (10.14)$$

without efficiency correction.

The value of L_3 can also be derived from the f_p form factor, which is directly proportional to L_3 (2.97). Averaging over the $m_{\pi\pi}$ and $m_{e\nu}$ dependence of the theoretical f_p one has with table 10.3

$$L_3^{fp} = (0.0227 \pm 0.0067) \cdot f_p = (-3.5 \pm 1.7) \cdot 10^{-3} \quad (10.15)$$

$$L'^{fp}_3 = (-3.6 \pm 1.7) \cdot 10^{-3} \quad (10.16)$$

The error is of course large, due to the uncertainty on the (small) form factor f_p , nevertheless it provides an independent⁸ measurement of L_3 and therefore is a valuable cross-check for both the measured values of L_3 and f_p .

⁸it is independent in the sense that it is derived from a different form factor, but of course the Ke4 BR enters both measurements

These results are compared to previous ones in fig. 10.13; for the Makoff results, the correction for the interference term has been applied as described at page 52.

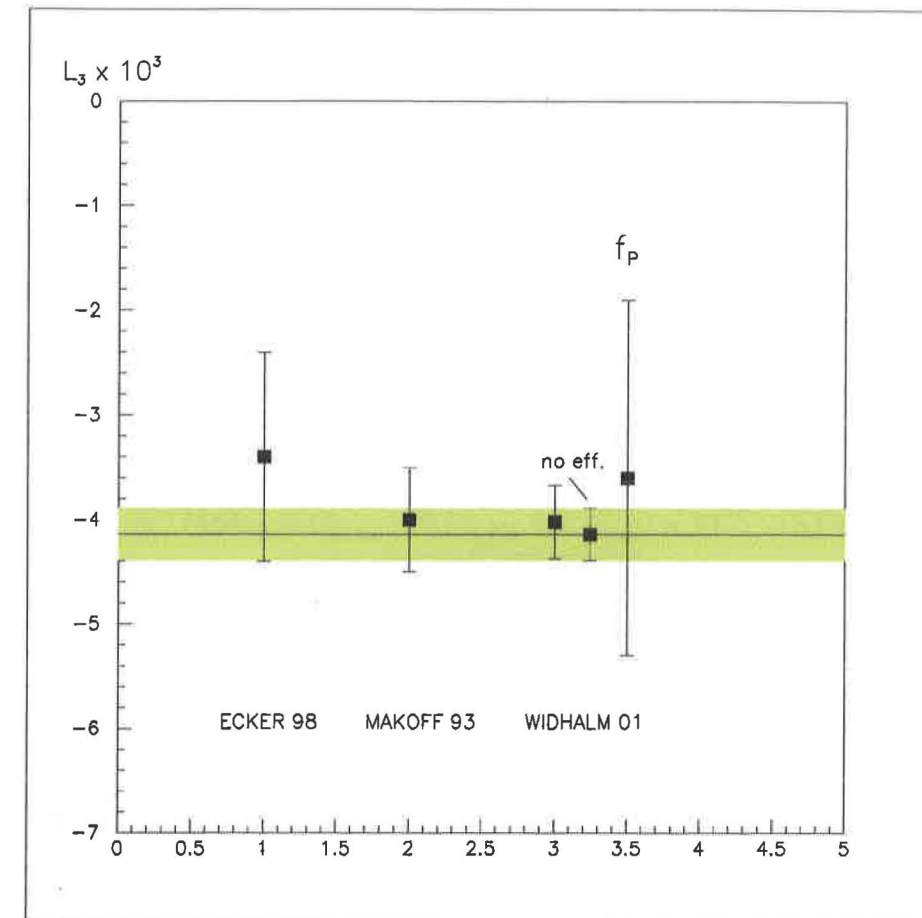


Figure 10.13: comparison of the L_3 measurement with previous results

Chapter 11

Summary and Conclusions

This Ke4 analysis was based on a sample of 2842 ± 44 Ke4 events taken with two different triggers at the CERN NA48 experiment. For the BR measurement, the results are stated for both triggers separately:

$$BR_{Ke4} = (5.55 \pm \underbrace{0.16_{\text{stat}} \pm 0.11_{\text{syst}} \pm 0.47_{\text{eff}}}_{\pm 0.51}) \times 10^{-5} \quad (\text{KE4-trigger})$$

$$BR_{Ke4} = (5.07 \pm \underbrace{0.16_{\text{stat}} \pm 0.19_{\text{syst}}}_{\pm 0.25}) \times 10^{-5} \quad (\text{ETOT-trigger})$$

The dominating contribution to the error for the KE4-trigger result was due to the unfortunately poorly measurable efficiency of the trigger. Nevertheless, the result is in agreement with the present world average of $(5.19 \pm 0.29) \cdot 10^{-5}$. Under the assumption of negligible efficiency correction the result changes to

$$BR'_{Ke4} = (5.32 \pm \underbrace{0.15_{\text{stat}} \pm 0.11_{\text{syst}}}_{\pm 0.19}) \times 10^{-5} \quad (11.1)$$

which is even better in agreement with previous results. The combined result KE4+ETOT then is

$$BR_{Ke4}^{\text{combined}} = (5.23 \pm 0.15) \times 10^{-5} \quad (11.2)$$

Also the Ke4 form factors have been measured by fitting two out of five kinematical variables. The final results for negligible efficiency correction were

$$g = 5.057 \pm 0.079 \quad (11.3)$$

$$f_s = 0.009 \pm 0.022 \quad (11.4)$$

$$f_p = -0.157 \pm 0.061 \quad (11.5)$$

For the form factor G , also a slope has been measured, parameterized by λ_g :

$$\lambda_g = 0.218 \pm 0.034 \quad (11.6)$$

As a cross-check, the phase difference between F and G has also been measured; theory predicts no phase difference, the result of this analysis is

$$\delta_F - \delta_G = (-5 \pm 9) \text{mrad} \quad (11.7)$$

Applying the framework of χPT , the value of the L_3 coupling parameter, which dominantly determines the theoretical Ke4 branching ratio, can be measured. The value for 1-loop χPT compatible with the observed BR is

$$L_3 = (-4.02 \pm 0.35) \cdot 10^{-3} \quad (11.8)$$

For negligible efficiency correction the result is

$$L'_3 = (-4.14 \pm 0.25) \cdot 10^{-3} \quad (11.9)$$

These results are somewhat larger (by absolute value) than the result given in the previous Ke4 analysis by Makoff [32] of $(3.4 \pm 0.4) \cdot 10^{-3}$; however, this is mainly due to an interference between F and G form factors which has been neglected in [32]; this has been discussed in detail in this work.

To conclude, this analysis provided a cross check as well as some improvements to [32], which previously was the only more accurate measurement of neutral Ke4. In general, the results could be confirmed (besides some differences discussed above) using a different experiment, different analysis techniques and finally also a different normalization channel. Theoretical predictions, arising solely from symmetry considerations as well as more predictive ones from chiral perturbation theory, could be confirmed; only one parameter (the slope λ_g) was measured higher than expected by 1-loop χPT , but as higher order corrections are known to be non-negligible, and the slope is a second order effect, this is not too surprising.

[Frigschneck]

Chapter 12

Outlook

The data this thesis is based on was taken during just one year, with a first try of a dedicated Ke4 trigger. From the experiences gained, an improved trigger could be developed that was used during NA48 data taking in 2001 which is expected to at least double the Ke4 statistics, while - most importantly - the efficiency measurement will be much better. Although this new data could not be taken into account for this work, it will be analysed and the combined results, cross-checked with a second analysis group, will be published in a paper to be written in 2002.

Acknowledgements

Trying to figure out who should be included in the acknowledgements, I realize that indeed many more contributed in some sense to this work than those who are aware of it. Obviously, I have to thank Manfred Markytan, my doctor father, who very carefully checked my results. No less I would like to thank Heinz Dibon, who originally invited me to make my thesis at CERN, and did most of the Monte Carlo work. Last but certainly not least is Günther Neuhofer, my "CERN boss", who initially brought me to the NA48 experiment at CERN, and always cared very well for me.

And what would I have done without the continuous help and assistance of my CERN colleagues - Manfred Jeitler, Ivan Mikulec and Günther Fischer - without them, I certainly would not have made it through the booby traps of NA48 detectors and analysis.

Additional help for the theoretical side came from Gerhard Ecker who patiently replied my numerous emails.

I also want to thank the Austrian Academy of Sciences to provide the financial basis for me to complete my thesis.

But there is more than the pure scientific life at CERN; I am grateful to Pierre Schnizer for his shuttle service between CERN and the Geneva airport, but mostly for giving me the opportunity to improve my cooking skills in his flat in Ferney.

Finally, I also want to thank the children of the SOS children village Wienerwald - the time I could spent with them was a very important counterweight to the purely scientific work, and by asking questions only children can ask they sometimes lead me back to the roots of scientific curiosity.

Appendix A

Formulae and Constants

This chapter contains some formulae as well as a list of all constants used in this work; for definitions we refer to the main text, this is only meant to be a quick look-up table

$$-\epsilon_{\mu\alpha\beta\gamma}\epsilon^{\mu abc} = \delta_\alpha^a\delta_\beta^b\delta_\gamma^c + \delta_\alpha^b\delta_\beta^c\delta_\gamma^a + \delta_\alpha^c\delta_\beta^a\delta_\gamma^b - \delta_\alpha^b\delta_\beta^a\delta_\gamma^c - \delta_\alpha^a\delta_\beta^c\delta_\gamma^b - \delta_\alpha^c\delta_\beta^b\delta_\gamma^a \quad (\text{A.1})$$

$$P_n(x) = \frac{1}{n!2^n} \frac{d^n}{dx^n} (x^2 - 1)^n \quad (\text{A.2})$$

$$1s = 1.51926760(33) \cdot 10^{24} \text{GeV}^{-1} \quad (\text{A.3})$$

$$\tau_{K_L} = (7.86 \pm 0.06) \cdot 10^{16} \text{GeV}^{-1} \quad (\text{A.4})$$

$$x := \frac{m_{\pi\pi}}{m} \quad y := \frac{m_{e\nu}^2}{m^2} \quad (\text{A.5})$$

$$\sigma_{xy} := \frac{2}{m^2} \sigma = \sqrt{(1 - x^2 - y)^2 - 4x^2y} \quad (\text{A.6})$$

$$\sigma_x := x\sigma_\pi = \sqrt{x^2 - c_\pi + \frac{\epsilon^2}{x^2}} \quad (\text{A.7})$$

$$q^2 := \frac{m_{\pi\pi}^2 - 4m_\pi^2}{4m_\pi^2} \quad (\text{A.8})$$

$$\epsilon = (5.09192 \pm 0.00064) \cdot 10^{-3} \quad (\text{A.9})$$

$$c_\pi = 0.304416 \pm 0.000038 \quad (\text{A.10})$$

$$\Gamma^0 = (7.10 \pm 0.15) \cdot 10^{-20} \text{GeV} \quad (\text{A.11})$$

$$x_0 = 0.551662 \pm 0.000034 \quad (\text{A.12})$$

$$G_0 = 3.808 \pm 0.011 \quad (\text{A.13})$$

$$u_F = 0.005 \quad (\text{A.14})$$

$$u_G = 0.34 \quad (\text{A.15})$$

$$c_G = -0.018 \quad (\text{A.16})$$

Appendix B

Kinematics of a 4-Body Decay

B.1 Definitions

In this chapter, kinematic properties and relations for a 4-body decay are derived and discussed. The momenta of the four particles shall be given by p_i , ($i = 1..4$), their invariant masses by m_i . For discussion of their kinematics, the particles are grouped in two pairs, (i, j) and (k, l) (of course $i \neq j \neq k \neq l$), with invariant masses m_{ij} and m_{kl} , respectively. The total momentum and invariant mass of the four particles shall be p and m , referred to as momentum and mass of the decaying particle.

It is convenient to define momenta that are symmetric and antisymmetric with respect to exchange of the two particles of a pair:

$$\begin{aligned} P_{ij} &:= p_i + p_j & (P_{ij} = P_{ji}) \\ Q_{ij} &:= p_i - p_j & (Q_{ij} = -Q_{ji}) \end{aligned} \quad (\text{B.1})$$

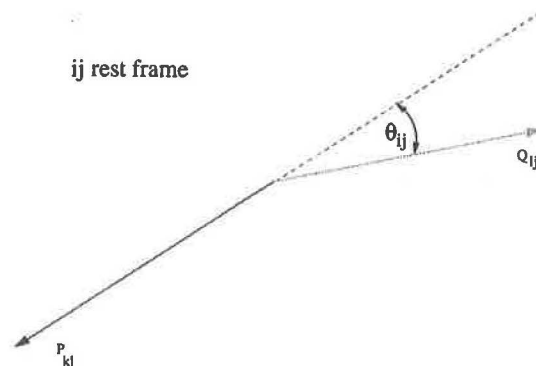
We now define the angles θ_{ij} as the angle between $\vec{p}_i$ in the rest frame of the diparticle¹ and the momentum of the diparticle $\vec{p}_i + \vec{p}_j$ in the rest frame of the decaying particle²; this is equivalent to the angle between $\vec{p}_i$ and the *negative* 3-momentum of the decaying particle, both taken in the diparticle's rest-frame. θ_{ij} can be related to the 3-vector parts of P and Q as such:

First, one notes that in the diparticle's rest frame, there is $\vec{Q}_{ij} = \vec{p}_i - \vec{p}_j = 2\vec{p}_i \parallel \vec{p}_i$ and the 3-momentum of the decaying particle is $\vec{p} = \vec{p}_i + \vec{p}_j + \vec{p}_k + \vec{p}_l = 0 + \vec{P}_{kl}$. Then θ_{ij} can be expressed as

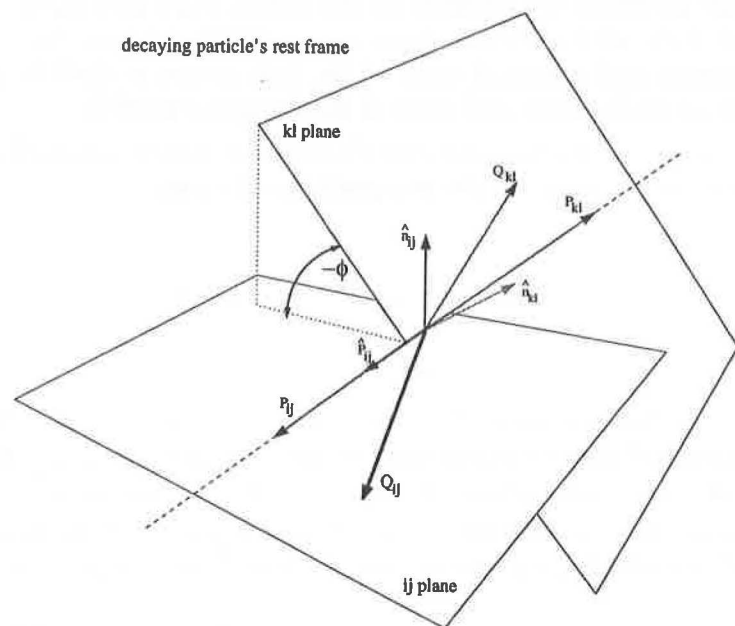
¹defined by $\vec{p}_i + \vec{p}_j = 0$

²defined by $\sum_{i=1}^4 \vec{p}_i = 0$

$$\cos \theta_{ij} = -\frac{\vec{Q}_{ij} \cdot \vec{P}_{kl}}{|\vec{Q}_{ij}| |\vec{P}_{kl}|} \quad (\text{B.2})$$

Figure B.1: definition of θ_{ij}

Obviously, θ_{ij} is *antisymmetric* in ij . Also note that $\cos \theta_{ij}$ is unambiguous with respect to θ_{ij} in the range $[0, \pi]$; it is possible to restrict θ_{ij} to this range in combination with the yet to be defined angle ϕ :

Figure B.2: definition of ϕ

We define the angle ϕ as the (oriented) angle between the planes formed by the two pairs of 3-momenta in the decaying particle's rest frame (see fig. B.2). Oriented means that the angle is related to the direction of the first

pair's momentum $\vec{p}_i + \vec{p}_j = \vec{P}_{ij}$, which is due to $\vec{P}_{ij} + \vec{P}_{kl} = \vec{p} = 0$ parallel to the line of intersection of the two planes. This provides a proper definition of ϕ in the whole range of $[-\pi, \pi]^3$:

$$\begin{aligned} \hat{P}_{ij} &:= \frac{\vec{P}_{ij}}{|\vec{P}_{ij}|} \\ \hat{N}_{ij} &:= \frac{\vec{P}_{ij} \times \vec{Q}_{ij}}{|\vec{P}_{ij} \times \vec{Q}_{ij}|} \\ \cos(\phi) &:= \hat{N}_{ij} \cdot \hat{N}_{kl} \quad \sin(\phi) := (\hat{N}_{ij} \times \hat{N}_{kl}) \cdot \hat{P}_{ij} \end{aligned} \quad (\text{B.3})$$

B.2 General Relations

Instantly one finds the following relations:

$$P_{ij}^2 = (p_i + p_j)^2 = m_{ij}^2 \Rightarrow p_i p_j = \frac{1}{2}(m_{ij}^2 - m_i^2 - m_j^2) \quad (\text{B.4})$$

$$Q_{ij}^2 = (p_i - p_j)^2 = m_i^2 + m_j^2 - 2p_i p_j = 2(m_i^2 + m_j^2) - m_{ij}^2 \quad (\text{B.5})$$

$$P_{ij} Q_{ij} = (p_i + p_j)(p_i - p_j) = m_i^2 - m_j^2 \quad (\text{B.6})$$

B.3 Relations in the Decaying Particle's Rest Frame

In the decaying particle's rest frame, there is

$$P_{ij}^0 + P_{kl}^0 = m \quad \vec{P}_{ij} + \vec{P}_{kl} = 0 \quad (\text{B.7})$$

which instantly gives

$$(P_{ij}^0)^2 - m_{ij}^2 = |\vec{P}_{ij}|^2 = |\vec{P}_{kl}|^2 = P_{kl}^{02} - m_{kl}^2 = (m - P_{ij}^0)^2 - m_j^2 \quad (\text{B.8})$$

and leads to

$$P_{ij}^0 = \frac{\bar{p} + m_{ij}^2}{m} \quad (\text{B.9})$$

$$|\vec{P}_{ij}| = \frac{\sigma}{m} \quad (\text{B.10})$$

³without any orientation, the angle between two planes is unambiguous only in the range $[0, \frac{\pi}{2}]$; introducing an order of the two planes (e.g. by using the vector product of the normal vectors to define the angle), this unambiguous range extends to $[0, \pi]$. If one now also chooses an orientation of the intersection of the two planes, the angle can be defined in an unambiguously way in the whole range $[-\pi, \pi]$. Together with the range $[0, \pi]$ of θ_{ij} this covers the whole phase space

Note the introduction of the kinematic variables $\bar{p}$ and σ :

$$\bar{p} := \frac{1}{2}(m^2 - m_{ij}^2 - m_{kl}^2) \quad (\text{B.11})$$

$$\sigma := \sqrt{\bar{p}^2 - m_{ij}^2 m_{kl}^2} \quad (\text{B.12})$$

B.4 Relations in the Diparticle's Rest Frame

In the rest frame of the diparticle ij there is obviously⁴

$$P_{ij}^0 = m_{ij} \quad \vec{P}_{ij}' = 0 \quad (\text{B.13})$$

for Q_{ij}^0 with $E_i' := p_i^0$ one finds using $|\vec{p}_i| = |\vec{p}_j|$

$$\vec{Q}_{ij}^0 = E_i' - E_j' = \frac{E_i'^2 - E_j'^2}{E_i' + E_j'} = \frac{m_i^2 - m_j^2}{m_{ij}} \quad (\text{B.14})$$

and from this using also (B.5)

$$|\vec{Q}_{ij}'| = \sqrt{E_{ij}^2 - Q_{ij}^2} = \frac{2\sigma_{ij}}{m_{ij}} \quad (\text{B.15})$$

where σ_{ij} analogously to σ is defined as

$$\bar{p}_{ij} := \frac{1}{2}(m_{ij}^2 - m_i^2 - m_j^2) \quad (\text{B.16})$$

$$\sigma_{ij} := \sqrt{\bar{p}_{ij}^2 - m_i^2 m_j^2} \quad (\text{B.17})$$

For later use, we also derive an expression for the component of $\vec{Q}_{ij}'$ orthogonal to $\vec{P}_{kl}'$:

$$|\vec{Q}_{ij}'^\perp| = |\vec{Q}_{ij}' - \frac{(\vec{Q}_{ij}' \cdot \vec{P}_{kl}') \vec{P}_{kl}'}{|\vec{P}_{kl}'|^2}| = |\vec{Q}_{ij}'| \frac{|\vec{Q}_{ij}'|}{|\vec{Q}_{ij}'|} + \cos \theta_{ij} \frac{|\vec{P}_{kl}'|}{|\vec{P}_{kl}'|} \quad (\text{B.18})$$

$$= |\vec{Q}_{ij}'| \sin \theta_{ij} = \frac{2\sigma_{ij}}{m_{ij}} \sin \theta_{ij} \quad (\text{B.19})$$

Note that this component is invariant under Lorentz transformations along $\vec{P}_{kl}'$ and therefore has the same value in the decaying particle's rest frame:

⁴to distinguish the expressions from those in the decaying particle's rest frame, the variables are primed

$$|\vec{Q}_{ij}'^\perp| = |\vec{Q}_{ij}'| \sin \theta_{ij} = \frac{2\sigma_{ij}}{m_{ij}} \sin \theta_{ij} \quad (\text{B.20})$$

If one now evaluates the invariant expression pP_{ij} in both the decaying particle's rest frame and the diparticle's rest frame, one gets the energy p'^0 of the decaying particle in the diparticle's rest frame:

$$pP_{ij} = mP_{ij}^0 = p'^0 m_{ij} \Rightarrow p'^0 = \frac{\bar{p} + m_{ij}^2}{m_{ij}} \quad (\text{B.21})$$

this also gives P_{kl} in the ij rest frame:

$$P_{kl}^0 = p'^0 - P_{ij}^0 = \frac{\bar{p}}{m_{ij}} \quad (\text{B.22})$$

$$|\vec{P}_{kl}'| = \frac{\sigma}{m_{ij}} \quad (\text{B.23})$$

An application of this is the invariant expression $P_{ij}P_{kl}$, evaluated in the diparticle's rest frame:

$$P_{ij}P_{kl} = P_{ij}^0 P_{kl}^0 - 0 = \bar{p} \quad (\text{B.24})$$

B.5 Relations involving θ_{ij}

We evaluate $Q_{ij}P_{kl}$ in the diparticle ij 's rest frame, using the definition of θ_{ij} in (B.2):

$$Q_{ij}P_{kl} = Q_{ij}^0 P_{kl}^0 - \vec{Q}_{ij}' \cdot \vec{P}_{kl}' = Q_{ij}^0 P_{kl}^0 + |\vec{Q}_{ij}'| |\vec{P}_{kl}'| \cos \theta_{ij} \quad (\text{B.25})$$

Inserting the expressions found above, one arrives at

$$Q_{ij}P_{kl} = \frac{m_i^2 - m_j^2}{m_{ij}^2} \bar{p} + \frac{2}{m_{ij}^2} \sigma \sigma_{ij} \cos \theta_{ij} \quad (\text{B.26})$$

If one now adds (B.6), one instantly gets

$$Q_{ij}p = \frac{(m_i^2 - m_j^2)(\bar{p} + m_{ij}^2)}{m_{ij}^2} + \frac{2}{m_{ij}^2} \sigma \sigma_{ij} \cos \theta_{ij} \quad (\text{B.27})$$

which, evaluated in the decaying particle's rest frame, gives:

$$Q_{ij}^0 = \frac{1}{mm_{ij}^2} [(m_i^2 - m_j^2)(\bar{p} + m_{ij}^2) + 2\sigma \sigma_{ij} \cos \theta_{ij}] \quad (\text{B.28})$$

B.6 Relations involving ϕ

We first consider $\cos \phi$; evaluating the expression from (B.3) one finds

$$\cos(\phi) = \frac{(\vec{P}_{ij} \vec{P}_{kl})(\vec{Q}_{ij} \vec{Q}_{kl}) - (\vec{P}_{ij} \vec{Q}_{kl})(\vec{P}_{kl} \vec{Q}_{ij})}{|\vec{P}_{ij} \times \vec{Q}_{ij}| |\vec{P}_{kl} \times \vec{Q}_{kl}|} \quad (\text{B.29})$$

everything to be evaluated in the decaying particle's rest frame. Looking at each term separately one gets:

- $|\vec{P}_{ij} \times \vec{Q}_{ij}|$:

This can be easily calculated using (B.10) and (B.20):

$$|\vec{P}_{ij} \times \vec{Q}_{ij}| = |\vec{P}_{ij}| |\vec{Q}_{ij}^\perp| = \frac{2\sigma\sigma_{ij}}{mm_{ij}} \sin \theta_{ij} \quad (\text{B.30})$$

- $\vec{P}_{ij} \vec{P}_{kl}$:

In the decaying particle's rest frame, there is $\vec{P}_{kl} = -\vec{P}_{ij}$ and therefore using (B.10)

$$\vec{P}_{ij} \vec{P}_{kl} = -\vec{P}_{ij}^2 = -\frac{\sigma^2}{m^2} \quad (\text{B.31})$$

- $\vec{Q}_{ij} \vec{Q}_{kl}$:

Looking at the results from above, $Q_{ij}Q_{kl}$ is the last scalar product of P, Q for which we have not derived an explicit expression yet; obviously, it is this scalar product that is connected with ϕ , and therefore we just write

$$\vec{Q}_{ij} \vec{Q}_{kl} = Q_{ij}^0 Q_{kl}^0 - Q_{ij} Q_{kl} = \frac{Q_{ijkl}}{m^2 m_{ij}^2 m_{kl}^2} - Q_{ij} Q_{kl} \quad (\text{B.32})$$

with

$$\begin{aligned} Q_{ijkl} := & (m_i^2 - m_j^2)(m_k^2 - m_l^2)(\bar{p} + m_{ij}^2)(\bar{p} + m_{kl}^2) + \\ & + 2\sigma\sigma_{ij}(m_k^2 - m_l^2)(\bar{p} + m_{kl}^2) \cos \theta_{ij} + \\ & + 2\sigma\sigma_{kl}(m_i^2 - m_j^2)(\bar{p} + m_{ij}^2) \cos \theta_{kl} + \\ & + 4\sigma^2\sigma_{ij}\sigma_{kl} \cos \theta_{ij} \cos \theta_{kl} \end{aligned} \quad (\text{B.33})$$

- $\vec{P}_{ij} \vec{Q}_{kl}$:

using (B.9), (B.28) and (B.26) one arrives at

$$\vec{P}_{ij} \vec{Q}_{kl} = -\frac{\sigma}{m^2 m_{kl}^2} [\sigma(m_k^2 - m_l^2) + 2\sigma_{kl}(\bar{p} + m_{kl}^2) \cos \theta_{kl}] \quad (\text{B.34})$$

Inserting everything in (B.29) and solving for $Q_{ij}Q_{kl}$ results in

$$Q_{ij}Q_{kl} = \frac{1}{m_{ij}^2 m_{kl}^2} [(m_i^2 - m_j^2)(m_k^2 - m_l^2)\bar{p} + \quad (\text{B.35})$$

$$\begin{aligned} & + 2\sigma\sigma_{ij}(m_k^2 - m_l^2) \cos \theta_{ij} + \\ & + 2\sigma\sigma_{kl}(m_i^2 - m_j^2) \cos \theta_{kl} + \\ & + 4\sigma_{ij}\sigma_{kl}\bar{p} \cos \theta_{ij} \cos \theta_{kl} + \\ & + 4\sigma_{ij}\sigma_{kl}m_{ij}m_{kl} \sin \theta_{ij} \sin \theta_{kl} \cos \phi] \end{aligned} \quad (\text{B.36})$$

The last missing part is now that proportional to $\sin \phi$. Looking closer at the definition of ϕ one realizes that it is actually a pseudo-scalar; this does not affect $\cos \phi$, but does affect $\sin \phi$. So it can be expected that $\sin \phi$ is related to a pseudo-scalar expression in P and Q . The only pseudo-scalar that can be built uses the Levi-Civita-Tensor $\epsilon_{\alpha\beta\gamma\delta}$ contracting all four momenta:

$$\epsilon_{\alpha\beta\gamma\delta} P_{ij}^\alpha Q_{ij}^\beta P_{kl}^\gamma Q_{kl}^\delta \quad (\text{B.37})$$

If one evaluates this in the decaying particle's rest-frame, one finds after some ϵ -tensor gymnastics and considering $\vec{P}_{kl} = -\vec{P}_{ij}$

$$\epsilon_{\alpha\beta\gamma\delta} P_{ij}^\alpha Q_{ij}^\beta P_{kl}^\gamma Q_{kl}^\delta = m \vec{P}_{ij} \cdot (\vec{Q}_{ij} \times \vec{Q}_{kl}) \quad (\text{B.38})$$

On the other hand, evaluating the definition of $\sin \phi$ leads to

$$\sin(\phi) = -\frac{|\vec{P}_{ij}|}{|\vec{P}_{ij} \times \vec{Q}_{ij}| |\vec{P}_{kl} \times \vec{Q}_{kl}|} \vec{P}_{ij} \cdot (\vec{Q}_{ij} \times \vec{Q}_{kl}) \quad (\text{B.39})$$

Consequently, one has

$$\epsilon_{\alpha\beta\gamma\delta} P_{ij}^\alpha Q_{ij}^\beta P_{kl}^\gamma Q_{kl}^\delta = -m \frac{|\vec{P}_{ij} \times \vec{Q}_{ij}| |\vec{P}_{kl} \times \vec{Q}_{kl}|}{|\vec{P}_{ij}|} \sin \phi \quad (\text{B.40})$$

and using (B.10) and (B.20) finally ends up with

$$\epsilon_{\alpha\beta\gamma\delta} P_{ij}^\alpha Q_{ij}^\beta P_{kl}^\gamma Q_{kl}^\delta = -\frac{4\sigma\sigma_{ij}\sigma_{kl}}{m_{ij}m_{kl}} \sin \theta_{ij} \sin \theta_{kl} \sin \phi \quad (\text{B.41})$$

B.7 Scalar Invariants of P and Q

We now systematically collect all Lorentz invariant scalar products of P and Q :

$$P_{ij}P_{ij} = m_{ij}^2 \quad (\text{B.42})$$

$$P_{ij}Q_{ij} = m_i^2 - m_j^2 \quad (\text{B.43})$$

$$P_{ij}P_{kl} = \bar{p} \quad (\text{B.44})$$

$$P_{ij}Q_{kl} = \frac{m_k^2 - m_l^2}{m_{kl}^2} \bar{p} + \frac{2}{m_{kl}^2} \sigma \sigma_{kl} \cos \theta_{kl} \quad (\text{B.45})$$

$$Q_{ij}Q_{ij} = 2(m_i^2 + m_j^2) - m_{ij}^2 \quad (\text{B.46})$$

$$Q_{ij}Q_{kl} = \frac{1}{m_{ij}^2 m_{kl}^2} [(m_i^2 - m_j^2)(m_k^2 - m_l^2) \bar{p} + 2\sigma \sigma_{ij} (m_k^2 - m_l^2) \cos \theta_{ij} + 2\sigma \sigma_{kl} (m_i^2 - m_j^2) \cos \theta_{kl} + 4\sigma_{ij} \sigma_{kl} \bar{p} \cos \theta_{ij} \cos \theta_{kl} + 4\sigma_{ij} \sigma_{kl} m_{ij} m_{kl} \sin \theta_{ij} \sin \theta_{kl} \cos \phi] \quad (\text{B.47})$$

$$\epsilon_{\alpha\beta\gamma\delta} P_{ij}^\alpha Q_{ij}^\beta P_{kl}^\gamma Q_{kl}^\delta = -\frac{4\sigma \sigma_{ij} \sigma_{kl}}{m_{ij} m_{kl}} \sin \theta_{ij} \sin \theta_{kl} \sin \phi \quad (\text{B.48})$$

with the definitions

$$\bar{p} := \frac{1}{2}(m^2 - m_{ij}^2 - m_{kl}^2) \quad (\text{B.49})$$

$$\sigma := \sqrt{\bar{p}^2 - m_{ij}^2 m_{kl}^2} \quad (\text{B.50})$$

$$\bar{p}_{ij} := \frac{1}{2}(m_{ij}^2 - m_i^2 - m_j^2) \quad (\text{B.51})$$

$$\sigma_{ij} := \sqrt{\bar{p}_{ij}^2 - m_i^2 m_j^2} \quad (\text{B.52})$$

B.8 Phase Space Integration

In this section it is shown how the phase space integration is performed for a function of the Cabibbo variables. One starts with

$$F := \int dLIPS(p; p_i, p_j, p_k, p_l) f(m_{\pi\pi}, m_{e\nu}, \theta_{\pi\pi}, \theta_{e\nu}, \phi) := \int \frac{d^3 p_i}{(2\pi)^3 2E_i} \int \frac{d^3 p_j}{(2\pi)^3 2E_j} \int \frac{d^3 p_k}{(2\pi)^3 2E_k} \int \frac{d^3 p_l}{(2\pi)^3 2E_l} (2\pi)^4 \delta^4(p_i + p_j + p_k + p_l - p) f(m_{\pi\pi}, m_{e\nu}, \theta_{\pi\pi}, \theta_{e\nu}, \phi)$$

by insertion of two δ -function integrals like $\delta^4(p_i + p_j - P_{ij})$, this can be split in two symmetric parts corresponding to the diparticles:

$$F = \frac{1}{2^4 (2\pi)^8} \int d^4 P_{ij} \int d^4 P_{kl} \delta^4(P_{ij} + P_{kl} - p) \int_{ij} \int_{kl} f \quad (\text{B.53})$$

with

$$\int_{ij} := \int \frac{d^3 p_i}{E_i} \frac{d^3 p_j}{E_j} \delta^4(p_i + p_j - P_{ij}) \quad (\text{B.54})$$

Since this expression is Lorentz-invariant, it can be evaluated in any frame of reference, so we choose the rest frame of the diparticle ($P_{ij}^0 = m_{ij}$, $\vec{P}_{ij} = 0$). Performing the $d^3 p_j$ integral⁵ using the space part of the δ -function gives

$$\int_{ij} = \int \frac{d^3 p_i}{E_i E_j} \delta(E_i + E_j - m_{ij}) \quad (\text{B.55})$$

where now p_j is fixed via $\vec{p}_j + \vec{p}_i = 0$. The remaining integral can be written in spherical coordinates:

$$\int_{ij} = \int \frac{p_i^2 dp_i d(\cos \theta) d\phi}{E_i E_j} \delta(E_i + E_j - m_{ij}) \quad (\text{B.56})$$

It is possible to choose the spherical coordinates such that $\theta = \theta_{ij}$. ϕ is not (necessarily) the Cabibbo variable of the same name, to avoid confusion it is henceforth called ϕ_{ij} . To get rid of the last δ -function, the dp_i -integration has to be performed. As the argument of the δ -function depends on the integration variable, it is convenient to transform the integral like this:

$$X := E_i + E_j = \sqrt{p_i^2 + m_i^2} + \sqrt{p_i^2 + m_j^2} \quad (\text{B.57})$$

$$dX = \frac{X p_i dp_i}{E_i E_j} \quad (\text{B.58})$$

Inserting this in the integral above the integration can easily be performed:

$$\int_{ij} = \frac{\sigma_{ij}}{m_{ij}^2} \int d(\cos \theta_{ij}) d\phi_{ij} \quad (\text{B.59})$$

Note that the δ -function gives $X = m_{ij}$ and p_i in the diparticle's rest frame is given by one half of (B.15). As a last step the $d\phi_{ij}$ integration can be performed - as f only depends on the relative angle between the diparticle's planes, the integration is independent of f and just gives a factor 2π :

⁵please note that this can be done independent of the function f

$$\int_{ij} = \frac{2\pi\sigma_{ij}}{m_{ij}^2} \int d(\cos\theta_{ij}) \quad (\text{B.60})$$

Now the remaining integral can be studied:

$$\int d^4 P_{ij} \int d^4 P_{kl} \delta^4(P_{ij} + P_{kl} - p) \quad (\text{B.61})$$

with the transformation to $m_{ij}^2 = (P^0)^2 - \vec{P}_{ij}^2$ one arrives at

$$\int m_{ij} dm_{ij} \int m_{kl} dm_{kl} \int \frac{d^3 P_{ij}}{P_{ij}^0} \int \frac{d^3 P_{kl}}{P_{kl}^0} \delta^4(P_{ij} + P_{kl} - p) \quad (\text{B.62})$$

which is formally equivalent to (B.54) and leads to

$$\int m_{ij} dm_{ij} \int m_{kl} dm_{kl} \frac{\sigma}{m^2} \int d(\cos\theta) d\phi \quad (\text{B.63})$$

The spherical coordinates can now be chosen such that ϕ becomes the corresponding Cabibbo-variable, so this integration can not be performed independently of f . Still, f does not depend on θ , so this integration gives

$$\int m_{ij} dm_{ij} \int m_{kl} dm_{kl} \frac{2\sigma}{m^2} \int d\phi \quad (\text{B.64})$$

Using eventually also $m_{ij} dm_{ij} = \frac{1}{2} dm_{ij}^2$ and putting all parts together one gets the final result

$$F = \frac{1}{2^5 (2\pi)^6 m^2} \int dm_{ij}^2 dm_{kl}^2 d(\cos\theta_{ij}) d(\cos\theta_{kl}) d\phi w(m_{ij}, m_{kl}) f(m_{\pi\pi}, m_{e\nu}, \theta_{\pi\pi}, \theta_{e\nu}, \phi) \quad (\text{B.65})$$

with the phase space weight

$$w(m_{ij}, m_{kl}) := \frac{\sigma_{ij}\sigma_{kl}}{m_{ij}^2 m_{kl}^2} \quad (\text{B.66})$$

Remark: The integration over ϕ_{ij}, ϕ_{kl} and θ independent of f corresponds to the freedom to choose an arbitrarily orientated coordinate system - due to isotropy, the physics cannot depend on this choice or on these angles, respectively.

Appendix C

Some Calculations

C.1 The Leptonic Tensor $T_{\mu\nu}$

The tensor $T_{\mu\nu}$ is defined like this:

$$T_{\mu\nu} := \frac{1}{4} \sum_{\text{spins}} L_\mu^+ L_\nu \quad L_\mu := \bar{u}(p_j) \gamma_\mu (1 - \gamma_5) v(p_i) \quad (\text{C.1})$$

$p_{i,j}$ are the momenta of two leptons. Inserting L_μ one has

$$T_{\mu\nu} := \frac{1}{4} \sum_{\text{spins}} (\bar{u} \gamma_\mu (1 - \gamma_5) v)^+ \bar{u} \gamma_\nu (1 - \gamma_5) v = \frac{1}{4} \sum_{\text{spins}} \bar{v} (1 + \gamma_5) \gamma_\mu u \bar{u} \gamma_\nu (1 - \gamma_5) v \quad (\text{C.2})$$

this can equivalently be written as

$$T_{\mu\nu} = \frac{1}{4} \sum_{\text{spins}} \text{Tr}[(1 + \gamma_5) \gamma_\mu u \bar{u} \gamma_\nu (1 - \gamma_5) v \bar{v}] \quad (\text{C.3})$$

using the well known relation

$$\sum_{\text{spins}} u \bar{u} = \not{p} + m \quad \sum_{\text{spins}} v \bar{v} = \not{p} - m \quad (\text{C.4})$$

one gets

$$T_{\mu\nu} = \frac{1}{4} \text{Tr}[(1 + \gamma_5) \gamma_\mu (\not{p}_j + m_j) \gamma_\nu (1 - \gamma_5) (\not{p}_i - m_i)] \quad (\text{C.5})$$

using

$$(1 + \gamma_5) \gamma_\mu = \gamma_\mu (1 - \gamma_5) \quad (1 + \gamma_5)(1 - \gamma_5) = 0 \quad \text{Tr}[AB] = \text{Tr}[BA] \quad (\text{C.6})$$

one shows that all terms proportional to m_i or m_j vanish and the remaining expression is

$$T_{\mu\nu} = \frac{1}{2} p_i^\alpha p_j^\beta \text{Tr}[(1 + \gamma_5) \gamma_\mu \gamma_\beta \gamma_\nu \gamma_\alpha] \quad (\text{C.7})$$

The trace can be evaluated using

$$\text{Tr}[\gamma_\mu \gamma_\beta \gamma_\nu \gamma_\alpha] = 4(g_{\mu\beta} g_{\nu\alpha} + g_{\mu\alpha} g_{\nu\beta} - g_{\mu\nu} g_{\alpha\beta}) \quad \text{Tr}[\gamma_5 \gamma_\mu \gamma_\beta \gamma_\nu \gamma_\alpha] = 4i\epsilon_{\mu\beta\nu\alpha} \quad (\text{C.8})$$

and gives

$$T_{\mu\nu} = 2p_{i\mu} p_{j\nu} + 2p_{i\nu} p_{j\mu} - 2p_i^\alpha p_{j\alpha} g_{\mu\nu} + 2i\epsilon_{\mu\nu\alpha\beta} p_i^\alpha p_j^\beta \quad (\text{C.9})$$

Using the variables P_{ij}, Q_{ij} as defined in (B.1) and the definition (B.16) this can also be written as

$$T^{\mu\nu} = P_{ij}^\mu P_{ij}^\nu - Q_{ij}^\mu Q_{ij}^\nu - 2\bar{p}_{ij} g^{\mu\nu} - i\epsilon^{\mu\nu\alpha\beta} P_{ij\alpha} Q_{ij\beta} \quad (\text{C.10})$$

C.2 Properties of the Integrals $I_{a,b}(x)$

The definition of $I_{a,b}(x)$ is

$$I_{a,b}(x) := \int_0^{(1-x)^2} dy y^a \sigma_{xy}^b \quad a, b \geq 0 \quad (\text{C.11})$$

with

$$\sigma_{xy}(x, y) := \sqrt{(1-x^2-y)^2 - 4x^2y} \quad (\text{C.12})$$

Properties of σ_{xy} are:

$$\begin{aligned} \sigma_{xy}(x, 0) &= 1 - x^2 \\ \sigma_{xy}(x, (1-x)^2) &= 0 \\ \frac{d\sigma_{xy}}{dy} &= -\frac{1+x^2-y}{\sigma_{xy}} \end{aligned} \quad (\text{C.13})$$

Partial integration gives the relation

$$I_{a,b} = \frac{b}{a+1} [(1+x^2)I_{a+1,b-2} - I_{a+2,b-2}] \quad b \geq 1 \quad (\text{C.14})$$

so obviously all $I_{a,b}$ with even b can be iteratively reduced to sums of $I_{a,0}$ that can easily be calculated:

$$I_{a,0} = \frac{(1-x)^{2a+2}}{a+1} \quad (\text{C.15})$$

It can be concluded that for even b $I_{a,b}$ is just a polynomial in x of $O(x^{2(a+b+1)})$

To evaluate the integral for odd b it is useful to study the transformation $y \rightarrow 1+x^2-z$ which leads to

$$I_{a,b}(x) = \int_{2x}^{1+x^2} dz (1+x^2-z)^a \sqrt{z^2-4x^2}^b \quad (\text{C.16})$$

Since all $I_{a,b}$ with odd b can be reduced to sums of $I_{a,1}$ it is sufficient to study only the special case $b=1$; all $I_{a,1}$ can be evaluated as

$$I_{a,1}(x) = \sum_{k=0}^a \binom{a}{k} (1+x^2)^{a-k} (-1)^k \tilde{I}_k \quad (\text{C.17})$$

with

$$\tilde{I}_k(x) := \int_{2x}^{1+x^2} dz z^k \sqrt{z^2-4x^2} \quad (\text{C.18})$$

It is now possible to derive a recursion for $\tilde{I}_k(x)$ by rewriting the integral as¹

$$\begin{aligned} \tilde{I}_k &= \int_{2x}^{1+x^2} dz z^{k-1} (z^2-4x^2) \cdot \frac{z}{\sqrt{z^2-4x^2}} = \\ &= \int_{2x}^{1+x^2} dz z^{k+1} \frac{z}{\sqrt{z^2-4x^2}} - 4x^2 \int_{2x}^{1+x^2} dz z^{k-1} \frac{z}{\sqrt{z^2-4x^2}} \end{aligned}$$

Using the obvious formula

$$\int \frac{z dz}{\sqrt{z^2-4x^2}} = \sqrt{z^2-4x^2} + C \quad (\text{C.19})$$

one computes using partial integration

$$\begin{aligned} \int_{2x}^{1+x^2} dz z^{k+1} \frac{z}{\sqrt{z^2-4x^2}} &= z^{k+1} \sqrt{z^2-4x^2} \Big|_{2x}^{1+x^2} - (k+1) \int_{2x}^{1+x^2} dz z^k \sqrt{z^2-4x^2} \\ &= (1+x^2)^{k+1} (1-x^2) - (k+1) \tilde{I}_k \end{aligned}$$

¹ this is valid because $\sqrt{z^2-4x^2}^2 = |z^2-4x^2| = z^2-4x^2$ for $2x \leq z \leq 1+x^2$

Inserting this in the rewritten integral above two times, and solving for $\tilde{I}_k$ one finds

$$\tilde{I}_k(x) = \frac{1}{k+2}(1-x^2)^3(1+x^2)^{k-1} + \frac{k-1}{k+2}4x^2\tilde{I}_{k-2} \quad (\text{C.20})$$

for $k=1$ follows immediately

$$\tilde{I}_1(x) = \frac{1}{3}(1-x^2)^3 \quad (\text{C.21})$$

and all $\tilde{I}_{2k+1}(x)$ can be recursively reduced to this with (C.20). For even k one needs the starting function for the recursion:

$$\begin{aligned} \tilde{I}_0(x) &= \int_{2x}^{1+x^2} dz \sqrt{z^2 - 4x^2} = \frac{1}{2} \left[z \sqrt{z^2 - 4x^2} - 4x^2 \operatorname{arccosh}\left(\frac{z}{2x}\right) \right]_{2x}^{1+x^2} = \\ &= \frac{1}{2}(1-x^4) + 2x^2 \ln x \end{aligned}$$

The $\tilde{I}_k$ for $k \leq 5$ have been computed and are listed here (see also fig. C.1):

$$\begin{aligned} \tilde{I}_0(x) &= \frac{1}{2}(1-x^4) + 2x^2 \ln x \\ \tilde{I}_1(x) &= \frac{1}{3}(1-x^2)^3 \\ \tilde{I}_2(x) &= \frac{1}{4}(1-x^8) + 2x^4 \ln x \\ \tilde{I}_3(x) &= \frac{1}{5}(1-x^2)^5 + \frac{4}{3}x^2(1-x^2)^3 \\ \tilde{I}_4(x) &= \frac{1}{6}(1-x^4)[(1-x^4)^2 + 3x^2(1+x^4)] + 4x^6 \ln x \\ \tilde{I}_5(x) &= \frac{1}{7}(1-x^2)^7 + \frac{8}{15}x^2(1-x^2)^3(3+4x^2+3x^4) \end{aligned}$$

C.2.1 Structure of $I_{a,b}$

Going back to $I_{a,1}$ one has

$$\begin{aligned} I_{0,1}(x) &= \tilde{I}_0 \\ I_{1,1}(x) &= (1+x^2)\tilde{I}_0 - \tilde{I}_1 \\ I_{2,1}(x) &= (1+x^2)^2\tilde{I}_0 - 2(1+x^2)\tilde{I}_1 + \tilde{I}_2 \\ I_{3,1}(x) &= (1+x^2)^3\tilde{I}_0 - 3(1+x^2)^2\tilde{I}_1 + 3(1+x^2)\tilde{I}_2 - \tilde{I}_3 \end{aligned}$$

Applying the recursion (C.14) to expand $I_{a,3}$ in $\tilde{I}_k$ one finds

C.2. PROPERTIES OF THE INTEGRALS $I_{A,B}(X)$

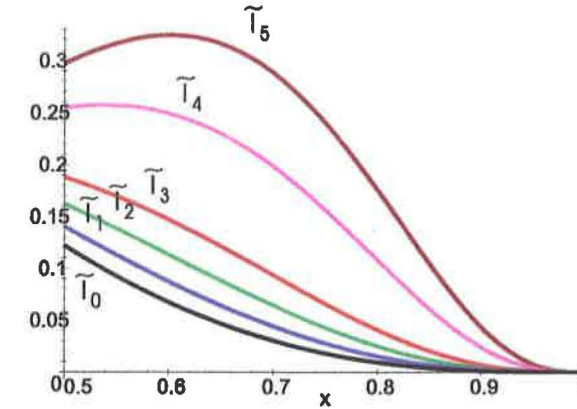


Figure C.1: logarithmic plots of $\tilde{I}_k$ for $0 \leq k \leq 5$

$$\frac{a+1}{3}I_{a,3}(x) = \sum_{k=0}^{a+1} \binom{a+1}{k} (1+x^2)^{a+1-k} (-1)^k \tilde{I}_{k+1} \quad (\text{C.22})$$

which also gives a clear structure:

$$\begin{aligned} \frac{1}{3}I_{0,3}(x) &= (1+x^2)\tilde{I}_1 - \tilde{I}_2 \\ \frac{2}{3}I_{1,3}(x) &= (1+x^2)^2\tilde{I}_1 - 2(1+x^2)\tilde{I}_2 + \tilde{I}_3 \\ I_{2,3}(x) &= (1+x^2)^3\tilde{I}_1 - 3(1+x^2)^2\tilde{I}_2 + 3(1+x^2)\tilde{I}_3 - \tilde{I}_4 \end{aligned}$$

C.2.2 Upper Bound for $I_{a,b}$

As the functions get more and more complicated, it is useful to have an upper bound² for the $I_{a,b}$. Because of $\sigma_{xy} < 1-x^2$ it is possible to estimate

$$I_{a,b}(x) = \int dy y^a \sigma_{xy}^b \leq \frac{1}{a+1} (1-x^2)^b (1-x)^{2a+2} \quad (\text{C.23})$$

The maximum of the right hand side within the x -range is at the lower bound $x_0 = \frac{m_{\pi^+} + m_{\pi^0}}{m} \approx 0.55$ - to get handy expressions, we just use $x_0 = \frac{1}{2}$ - this still gives a valid upper bound for the maximum of $I_{a,b}$:

$$I_{a,b}^{max} \leq \frac{3^b}{(a+1)2^{2(a+b+1)}} \quad (\text{C.24})$$

We also give an estimation for the mean value:

²note that all $I_{a,b}$ are positive in the whole x -range

$$I_{a,b}^{mean} = 2 \int_{\frac{1}{2}}^1 dx I_{a,b}(x) \leq \frac{3^b}{(a+1)2^{b-1}} \int dx (1-x)^{2a+b+2} \quad (C.25)$$

which finally gives

$$I_{a,b}^{mean} \leq \frac{3^b}{(a+1)(2a+b+3)2^{2(a+b+1)}} \quad (C.26)$$

which is yet by $\frac{1}{2a+b+3}$ smaller than the maximum value. This has to be compared to the leading order values. Obviously there is

$$I_{a,b} \leq I_{a-1,b} \quad I_{a,b} \leq \frac{3}{4} I_{a,b-1} \quad (C.27)$$

Therefore the leading order integral is that with lowest (a,b) - with respect to the application of this integrals in this thesis, this is $(0,1)$. The mean value calculates to

$$I_{0,1}^{mean} \approx \frac{1}{2^5} \quad (C.28)$$

The final estimation for $I_{a,b}$, normalized to this leading order contribution, is now

$$I_{a,b}^{mean} \leq \frac{8 \cdot 3^b}{(a+1)(2a+b+3)2^{2(a+b)}} I_{0,1}^{mean} \quad (C.29)$$

C.2.3 Plots of $I_{a,b}$

The $I_{a,b}$ for $0 \leq a \leq 5$ and $1 \leq b \leq 3$ have been plotted; the thin lines show the corresponding upper bound (C.23).

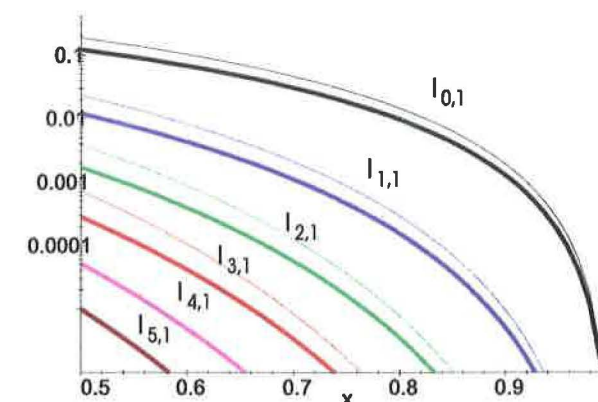
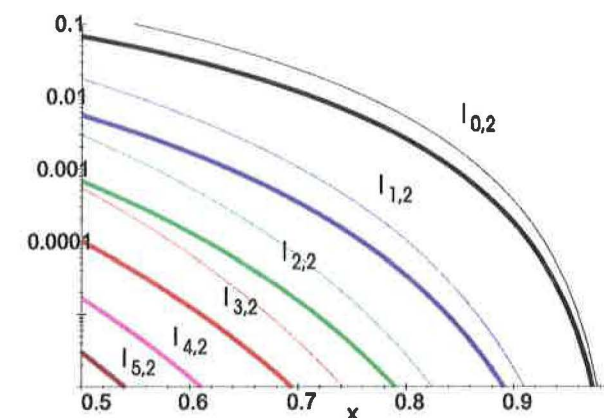
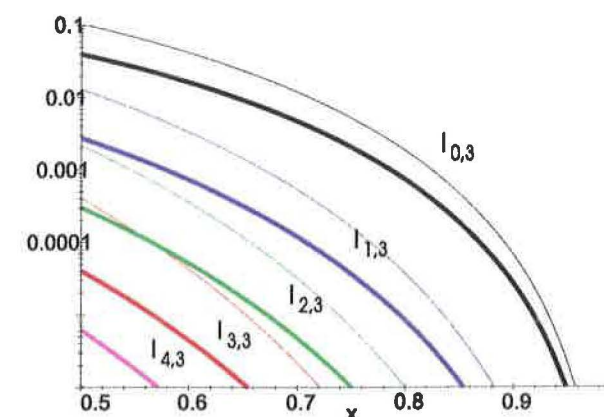
(a) $b=1$ (b) $b=2$ (c) $b=3$

Figure C.2: logarithmic plots of $I_{a,b}$ for $0 \leq a \leq 5$ and $1 \leq b \leq 3$, the thin lines show the estimated upper bounds

Appendix D

Symmetry Properties of the $|\pi\pi\rangle$ -System

The pion is a $I = 1$ particle with the following states (notation $|I; I_z\rangle$):

$$\begin{aligned} |1; -1\rangle &= |\pi^- \rangle \\ |1; 0\rangle &= |\pi^0 \rangle \\ |1; 1\rangle &= |\pi^+ \rangle \end{aligned} \quad (D.1)$$

Consequently, the possible dipion states have $I = 0, 1, 2$. The $I = 2$ states are

$$\begin{aligned} |2; -2\rangle &= |\pi^- \pi^- \rangle \\ |2; -1\rangle &= \frac{1}{\sqrt{2}} [|\pi^0 \pi^- \rangle + |\pi^- \pi^0 \rangle] \\ |2; 0\rangle &= \frac{1}{\sqrt{6}} [|\pi^+ \pi^- \rangle + 2|\pi^0 \pi^0 \rangle + |\pi^- \pi^+ \rangle] \\ |2; 1\rangle &= \frac{1}{\sqrt{2}} [|\pi^+ \pi^0 \rangle + |\pi^0 \pi^+ \rangle] \\ |2; 2\rangle &= |\pi^+ \pi^+ \rangle \end{aligned} \quad (D.2)$$

The $I = 1$ states are

$$\begin{aligned} |1; -1\rangle &= \frac{1}{\sqrt{2}} [|\pi^0 \pi^- \rangle - |\pi^- \pi^0 \rangle] \\ |1; 0\rangle &= \frac{1}{\sqrt{2}} [|\pi^+ \pi^- \rangle - |\pi^- \pi^+ \rangle] \\ |1; 1\rangle &= \frac{1}{\sqrt{2}} [|\pi^+ \pi^0 \rangle - |\pi^0 \pi^+ \rangle] \end{aligned} \quad (D.3)$$

and finally the $I = 0$ state is

$$|0; 0\rangle = \frac{1}{\sqrt{3}}[|\pi^+\pi^-\rangle - |\pi^0\pi^0\rangle + |\pi^-\pi^+\rangle] \quad (\text{D.4})$$

For the neutral Ke4 decay, only the states $|2; \pm 1\rangle$ and $|1; \pm 1\rangle$ are relevant. Obviously they are symmetric ($I = 2$ case) and anti-symmetric ($I = 1$ case) under exchange of the two pions. Also note the normalization factors $1/\sqrt{2}$.

Bose-symmetry requires the dipion state to be symmetric in the whole, which requires a corresponding angular momentum L of the dipion:

$$(-1)^{I+L} = 1 \quad (\text{D.5})$$

At leading order, I can only change by $1/2$ during the Ke4 decay ($\Delta I = 1/2$ -rule)¹. Since the kaon has $I = 1/2$, the dipion dominantly has to be in the $I = 1$ state, which requires odd L in order to comply with Bose symmetry.

Thus the important conclusion can be made that for neutral Ke4, the dipion for symmetry reasons is expected to be suppressed in the s -wave mode ($L = 0$), and be dominantly in the p -wave ($L = 1$). This is a remarkable difference to charged Ke4 decays, where the s -wave dominates.

As far as the form factors are concerned, this symmetry constraint is taken care of by anti-symmetrizing the matrix element with respect to the pions and weighting with factor $\sqrt{2}$:

$$\begin{aligned} |1; 1\rangle &= \frac{1}{\sqrt{2}}[|\pi^+\pi^0\rangle - |\pi^0\pi^+\rangle] \\ T(\pi\pi) &= \sqrt{2} \cdot \frac{1}{2}[T(\pi^+\pi^0) - T(\pi^0\pi^+)] \end{aligned} \quad (\text{D.6})$$

As the form factors in the matrix element are multiplied with 4-momenta that in the case of G and H are already antisymmetric under pion exchange, these two form factors have to be symmetric, while F and R have to be antisymmetric:

$$\begin{aligned} F(\pi\pi) &= \sqrt{2} \cdot \frac{1}{2}[F(\pi^+\pi^0) - F(\pi^0\pi^+)] \\ G(\pi\pi) &= \sqrt{2} \cdot \frac{1}{2}[G(\pi^+\pi^0) + G(\pi^0\pi^+)] \\ R(\pi\pi) &= \sqrt{2} \cdot \frac{1}{2}[R(\pi^+\pi^0) - R(\pi^0\pi^+)] \\ H(\pi\pi) &= \sqrt{2} \cdot \frac{1}{2}[H(\pi^+\pi^0) + H(\pi^0\pi^+)] \end{aligned}$$

¹ $s \rightarrow uW^-$ is a $1/2$ change

Appendix E

Unitarity Corrections

The unitarity corrections consists of contributions from different loop integrals; we will also include a renormalization scale dependent term in our discussion. The formulae are taken from [35] and [16], but we will treat them a little bit different than in [16] (and [32], who used these results). The reason for this will be discussed below.

E.1 Formulae

The unitarity corrections for F and G are given by¹

$$\begin{aligned} U_F &= \frac{2G_0^3}{m^2}[a_F(t) - a_F(u)] \\ U_G &= \frac{2G_0^3}{m^2}[\Delta_1(m_{\pi\pi}^2) + a_G(t) + a_G(u)] \end{aligned}$$

with the Mandelstam variables

$$\begin{aligned} t &:= (p_{\pi^+} - p)^2 = \frac{1}{4}m^2c_\pi + \frac{1}{2}(m^2 + m_{e\nu}^2 - m_{\pi\pi}^2) - \sigma_\pi\sigma \cos\theta_{\pi\pi} \\ u &:= (p_{\pi^0} - p)^2 = \frac{1}{4}m^2c_\pi + \frac{1}{2}(m^2 + m_{e\nu}^2 - m_{\pi\pi}^2) + \sigma_\pi\sigma \cos\theta_{\pi\pi} \end{aligned}$$

where

$$\frac{1}{4}m^2c_\pi \approx m_{\pi^+}^2 \approx m_{\pi^0}^2 =: m_\pi^2 \quad (\text{E.1})$$

Δ_1 contains loop contributions from $\pi\pi$, KK and $\eta\eta$:

$$\Delta_1(m_{\pi\pi}^2) = 2m_{\pi\pi}^2[M_{\pi\pi}^r(m_{\pi\pi}^2) + \frac{1}{2}M_{KK}^r(m_{\pi\pi}^2)] \quad (\text{E.2})$$

¹as R and H do not contribute significantly to the Ke4 decay, and will not be discussed

a_F and a_G contain loop contributions from $K\pi$ and ηK :

$$\begin{aligned}
 a_F(x) &= \frac{1}{32} \left((14m^2 + 14m_\pi^2 - 19x)J_{K\pi}^r(x) + (2m^2 + 2m_\pi^2 - 3x)J_{\eta K}^r(x) \right) + \\
 &\quad + \frac{1}{16} \left((3m^2 - 7m_\pi^2 + 5x)K_{K\pi}(x) + (m^2 - 5m_\pi^2 + 3x)K_{\eta K}(x) \right) - \\
 &\quad - \frac{1}{8} \left(9(L_{K\pi}(x) + L_{\eta K}(x)) + (3m^2 - 3m_\pi^2 - 9x)(M_{K\pi}^r(x) + M_{\eta K}^r(x)) \right) - \\
 &\quad - \frac{1}{2} \left((m^2 + m_\pi^2 - x)J_{K\pi}^r(x) \right) \\
 a_G(x) &= \frac{1}{32} \left((2m^2 + 2m_\pi^2 + 3x)J_{K\pi}^r(x) - (2m^2 + 2m_\pi^2 - 3x)J_{\eta K}^r(x) \right) - \\
 &\quad - \frac{1}{16} \left((3m^2 - 7m_\pi^2 + 5x)K_{K\pi}(x) + (m^2 - 5m_\pi^2 + 3x)K_{\eta K}(x) \right) - \\
 &\quad - \frac{3}{8} \left(L_{K\pi}(x) + L_{\eta K}(x) - (m^2 - m_\pi^2 + x)(M_{K\pi}^r(x) + M_{\eta K}^r(x)) \right) -
 \end{aligned}$$

The needed loop integrals J, K, L, M are given by

$$\begin{aligned}
 J_{ij}(x) &= \frac{1}{32\pi^2} \left[2 - \frac{\Delta}{x} \ln \frac{m_i^2}{m_j^2} + \frac{\Sigma}{\Delta} \ln \frac{m_i^2}{m_j^2} - \right. \\
 &\quad \left. - \frac{\sqrt{\lambda}}{x} \left(\ln \frac{\sqrt{\lambda} + (x + \Delta)}{\sqrt{\lambda} - (x + \Delta)} + \ln \frac{\sqrt{\lambda} + (x - \Delta)}{\sqrt{\lambda} - (x - \Delta)} \right) \right] \\
 J_{ij}^r(x) &= J_{ij}(x) - \frac{1}{16\pi^2\Delta} \left(m_i^2 \ln \frac{m_i^2}{\mu^2} - m_j^2 \ln \frac{m_j^2}{\mu^2} \right) \\
 M_{ij}^r(x) &= \frac{(x - \Sigma)^2 + 4\Delta^2 - \Sigma^2}{12x^2} J_{ij}(x) + \frac{1}{288\pi^2} - \\
 &\quad - \frac{1}{96\pi^2\Delta} \left(m_i^2 \ln \frac{m_i^2}{\mu^2} - m_j^2 \ln \frac{m_j^2}{\mu^2} - 2m_i^2 m_j^2 \ln \frac{m_i^2}{m_j^2} + \frac{\Sigma\Delta}{x} \right) \\
 L_{ij}(x) &= \frac{\Delta^2}{4x} J_{ij}(x) \\
 K_{ij}(x) &= \frac{\Delta}{2x} J_{ij}(x)
 \end{aligned}$$

with the short notations

$$\begin{aligned}
 \Sigma &:= m_i^2 + m_j^2 \\
 \Delta &:= m_i^2 - m_j^2 \\
 \lambda &:= (x + \Delta)^2 - 4xm_i^2 = (x - (m_i + m_j)^2)(x - (m_i - m_j)^2)
 \end{aligned}$$

The case $m_i = m_j$ requires some care when calculating the limes, we therefore do not combine the logarithms of line 2 (see discussion below). To

U_G one has to add also a renormalization scale dependent term (it cancels for U_F):

$$U_G \rightarrow U_G + C_G = U_G - \frac{1}{256\pi^2} \left[5m_\pi^2 \ln \frac{m_\pi^2}{\mu^2} - 2m^2 \ln \frac{m^2}{\mu^2} - 3m_\eta^2 \ln \frac{m_\eta^2}{\mu^2} \right] \quad (\text{E.3})$$

it is important to fix μ at a certain value, since U_G depends significantly on μ (see graphical discussion below) - only together with the also μ dependent χPT coupling constants L_i^r , the μ dependence cancels out; consequently, also the values of L_i^r depend significantly on the choice of μ . To fix μ , we will use the usual choice of [34]

$$\mu = m_\eta = 547.30 \pm 0.12 \text{ MeV} \quad (\text{E.4})$$

This also leads to small unitarity corrections (in comparison with the L_i^r dependent contributions).

E.2 The Imaginary Part of the Unitarity Correction

Unitarity requires the corrections to also contain imaginary parts; a closer look at the integrals above reveals that imaginary parts only can enter via the last term of $J_{ij}(x)$. It is also important to consider the (implicit) integration rule $i\epsilon$. Formally, one has (up to irrelevant factors) two terms like

$$\sqrt{\lambda} \ln \frac{\sqrt{\lambda} + a + i\epsilon}{\sqrt{\lambda} - a + i\epsilon} \quad (\text{E.5})$$

with $a = x \pm \Delta$. Concentrating first only on the logarithm gives

$$\ln \frac{\sqrt{\lambda} + a + i\epsilon}{\sqrt{\lambda} - a + i\epsilon} = \ln \frac{\lambda - a^2 - ia\epsilon}{(\sqrt{\lambda} - a)^2} = \ln \left(\frac{\sqrt{\lambda} + a}{\sqrt{\lambda} - a} - ia\epsilon \right) \quad (\text{E.6})$$

note that the sign of a determines the direction of the shift $i\epsilon$. This logarithm can be imaginary when either the fraction has an imaginary part (in this case the $i\epsilon$ does not matter and can be omitted), or when the fraction gets negative. The first case requires $\lambda < 0$ - this leads to a purely imaginary value for the logarithm; as the logarithm has to be multiplied by $\sqrt{\lambda}$ (E.5), the result would be real again.

The latter case requires $\lambda \geq 0$ and $|a| > \sqrt{\lambda}$. The second relation is always true, the first one is equivalent to

$$(I) \quad x > (m_i + m_j)^2 \quad \text{or} \quad (II) \quad x < (m_i - m_j)^2 \quad (\text{E.7})$$

Looking at the sign for a one finds for the two cases:

$$\begin{aligned}
 (I) \quad & a > 0 \\
 (II) \quad & \begin{cases} a > 0 & a = x + \Delta \\ a < 0 & a = x - \Delta \end{cases}
 \end{aligned} \tag{E.8}$$

where we assumed $\Delta > 0^2$.

To determine the imaginary part we use

$$\ln(-1 \pm i\epsilon) = \pm i\pi \tag{E.9}$$

to find

$$\ln \frac{\sqrt{\lambda} + a + i\epsilon}{\sqrt{\lambda} - a - i\epsilon} = \ln \frac{a + \sqrt{\lambda}}{a - \sqrt{\lambda}} - i\pi \operatorname{sgn}(a) \tag{E.10}$$

As the signs of a are opposite for both terms in case (II), the imaginary part cancels in the whole. Thus, an imaginary part remains only for case (I).

To summarize, the result is

$$\ln \frac{\sqrt{\lambda} + (x + \Delta)}{\sqrt{\lambda} - (x + \Delta)} + \ln \frac{\sqrt{\lambda} + (x - \Delta)}{\sqrt{\lambda} - (x - \Delta)} = \ln \frac{(x + \sqrt{\lambda})^2 - \Delta^2}{x - \sqrt{\lambda})^2 - \Delta^2} - 2\pi i \cdot \theta(x - (m_i + m_j)^2) \tag{E.11}$$

Applying this to the unitarity corrections, one finds that only $\Delta_1(m_{\pi\pi}^2)$ has a phase. The imaginary parts of the unitarity corrections are thus given by:

$$\begin{aligned}
 \operatorname{Im} U_F &= 0 \\
 \operatorname{Im} U_G &= \frac{1}{96\pi} m_{\pi\pi}^2 \sigma_\pi^3 = \delta_1^1
 \end{aligned} \tag{E.12}$$

where δ_1^1 is the $\pi\pi$ scattering phase shift [17].

E.3 Graphical Discussion of the Unitarity Corrections

For a graphical discussion of the unitarity corrections, the corrections have been calculated for a flat distribution in $m_{\pi\pi}$, $m_{e\nu}$ and $\cos\theta_{\pi\pi}$. As expected, only the correction to G has a phase. Fig. E.1 shows the distributions for the norm of the corrections in comparison with the corresponding total form factor (at 1-loop level).

²the result must not depend on this assumption, because the original integral is symmetric in m_i and m_j

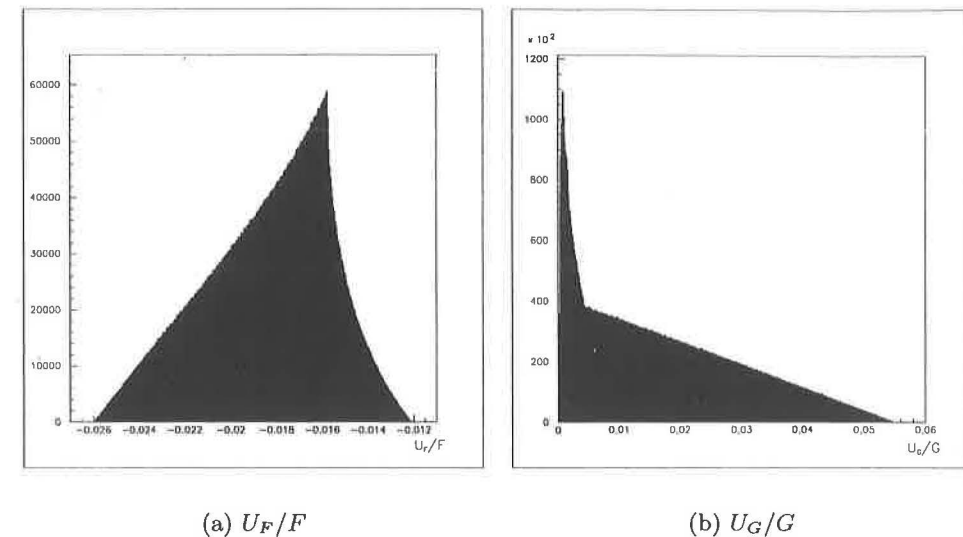


Figure E.1: ratio of the unitarity corrections and the total form factor

As the overall contribution of F to the decay rate is already small (see main text), a $\leq 3\%$ contribution of U_F to F can safely be neglected, while U_G can contribute up to almost 6% to G and has to be considered in the BR prediction.

The scatter plots of U_F versus the three Cabibbo variables are shown in fig. E.2 – obviously, there is a complicated dependence on all three variables.

The dependence of U_G on the Cabibbo variables is much simpler (fig. E.3) – it is mainly just a function of $m_{\pi\pi}$ with a slight dependence on $m_{e\nu}$, the dependence on $\cos\theta_{\pi\pi}$ is negligible.

In [16], U_G is parametrized as

$$U_G = G(0)\lambda_G q^2 \tag{E.13}$$

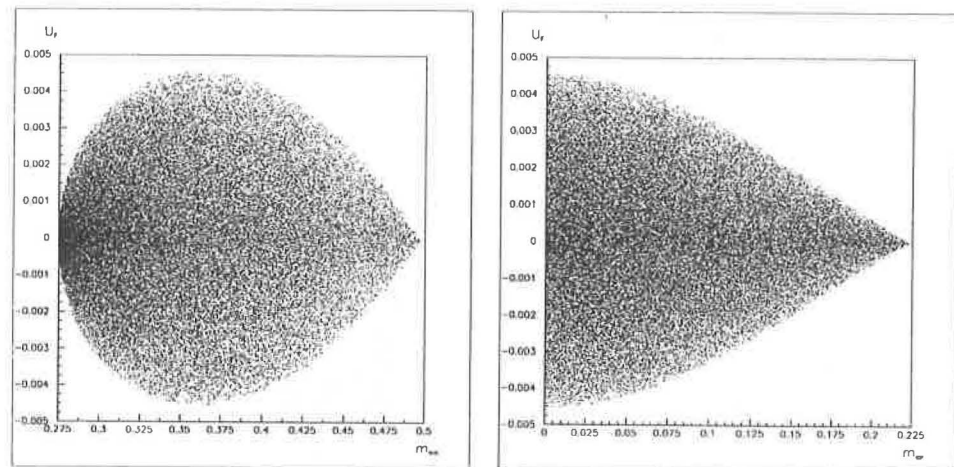
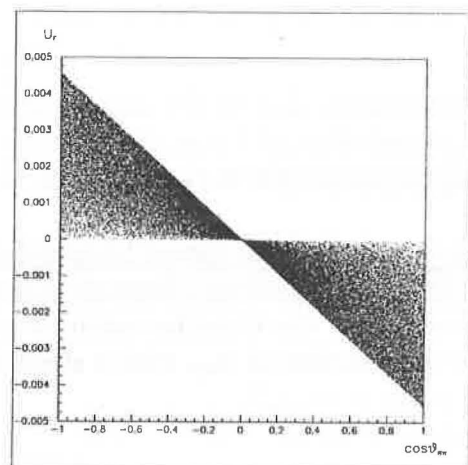
with

$$q^2 := \frac{m_{\pi\pi}^2 - 4m_\pi^2}{4m_\pi^2} \tag{E.14}$$

For the imaginary part, this is a good approximation, but the real part obviously rather depends linearly on $m_{\pi\pi}$ instead on q^2 . We therefore will not use this expansion, but a different one presented below³.

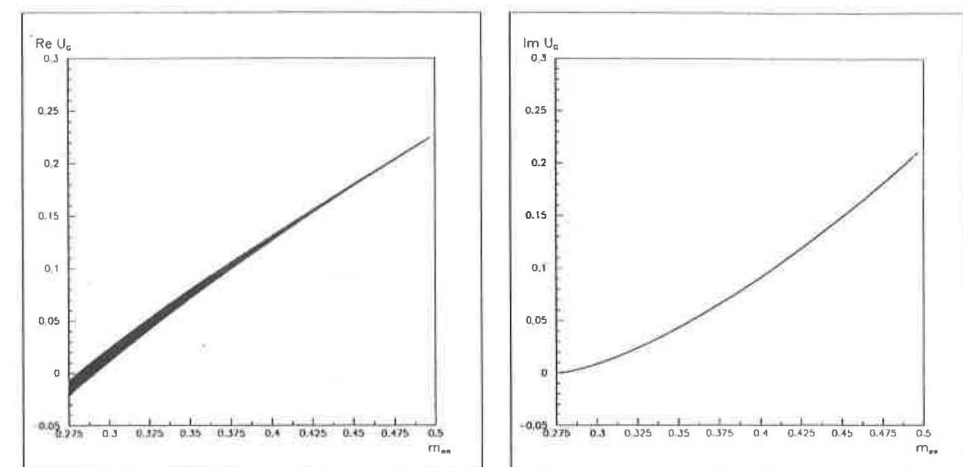
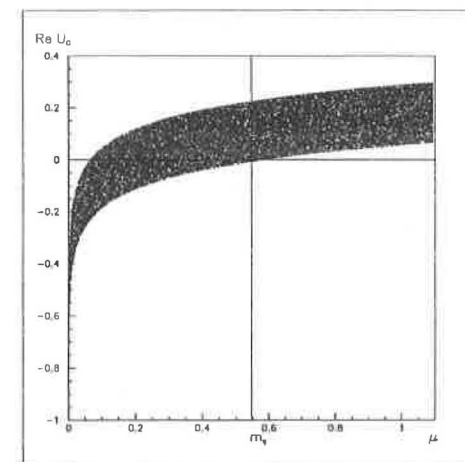
Above we mentioned the importance of the renormalization scale μ – fig. E.4 shows the dependence of $\operatorname{Re} U_G$ on μ ($\operatorname{Im} U_G$ and U_F do not depend

³the difference in the predicted BR is $\approx 1.5\%$, which is well below the experimental error, but the suggested approximation is not much more complicated, and at the same time much more accurate

(a) U_F vs. $m_{\pi\pi}$ (b) U_F vs. $m_{e\nu}$ (c) U_F vs. $\cos \theta_{\pi\pi}$ Figure E.2: scatter plot of the unitarity correction U_F as a function of three Cabibbo variables

on μ). As can be seen from this plot, the unitarity correction depends non-negligibly on the renormalization point⁴; the choice $\mu = m_\eta$ leads to vanishing corrections at $m_{\pi\pi}$ threshold.

⁴of course, physical observables are independent on μ , this μ dependence is absorbed by the also μ dependent couplings L_i^r (and higher order terms)

(a) $Re U_G$ (b) $Im U_G$ Figure E.3: scatter plot of the unitarity correction U_G as a function of $m_{\pi\pi}$ Figure E.4: $Re U_G$ as a function of the renormalization scale μ (the band width corresponds to the $m_{\pi\pi}$ dependence of $Re U_G$)

E.4 A Simple Approximation for the Unitarity Corrections

In the case of U_F , it is sufficient to give an upper bound (as the contribution to the BR turns out to be negligible):

$$\left| \frac{U_F}{\cos \theta_{\pi\pi}} \right| = -\frac{U_F}{\cos \theta_{\pi\pi}} < u_F \quad u_F \approx 0.005 \quad (\text{E.15})$$

The rather clean dependence of U_G on only $m_{\pi\pi}$ suggests to look for a simple approximation $\bar{U}_G(m_{\pi\pi})$. The imaginary part is already known (E.12):

$$\text{Im}U_G = G_0^3 \frac{1}{48\pi} \frac{\sigma_x^3}{x} \approx 0.37 \cdot \frac{\sigma_x^3}{x} \quad (\text{E.16})$$

For the real part, we use the approximation

$$\text{Re}U_G = u_G \frac{\sigma_x^2}{x} + c_G \quad u_G = 0.34 \quad c_G = -0.018 \quad (\text{E.17})$$

Fig. E.5 shows the comparison of these approximations with the true values, fig. E.6 compares the approximation for $\text{Re}U_G$ with that of [16].

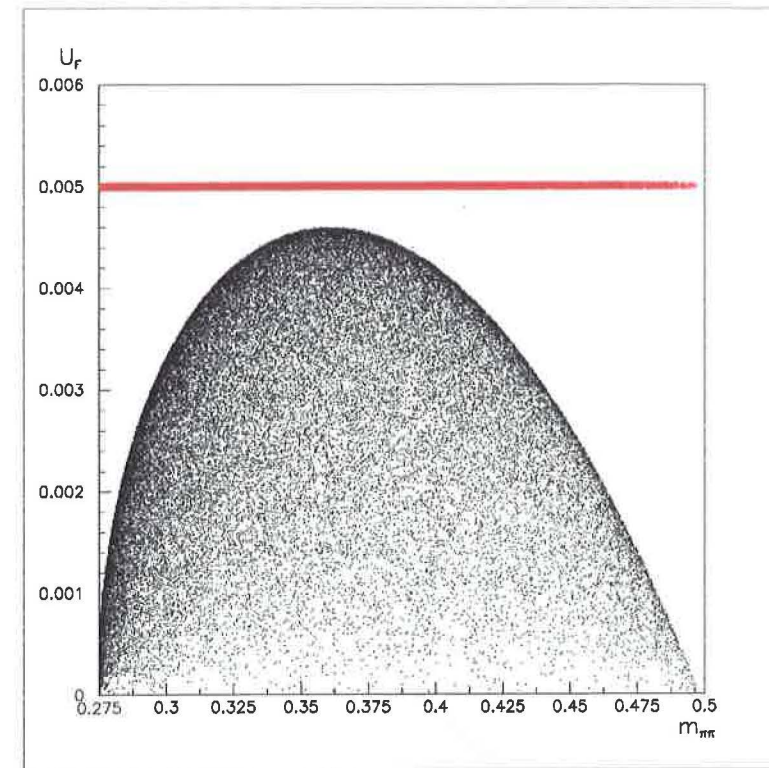
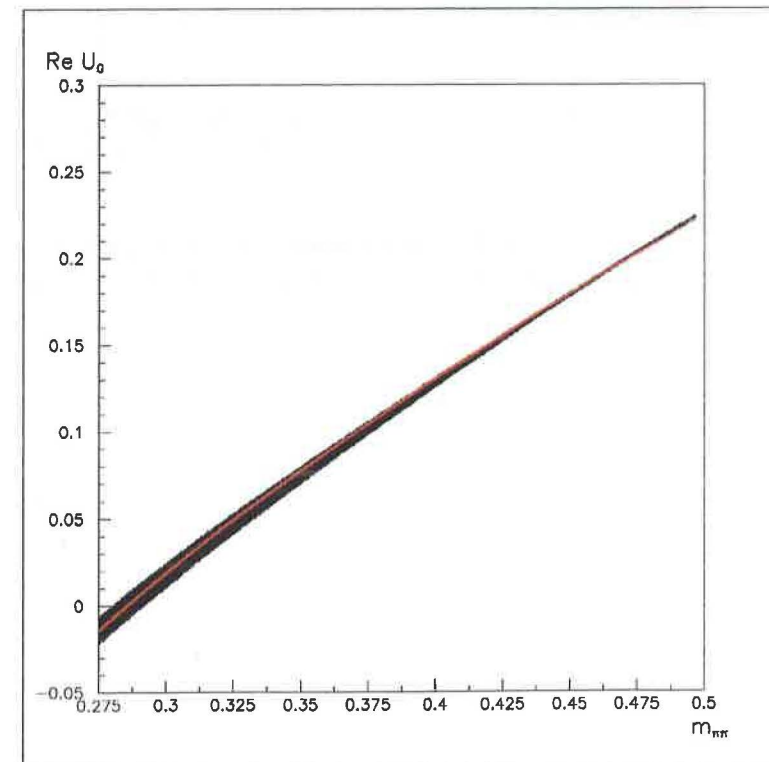
(a) U_F (b) $\text{Re}U_G$

Figure E.5: exact unitarity corrections (black) and the approximation (light gray) as a function of $m_{\pi\pi}$

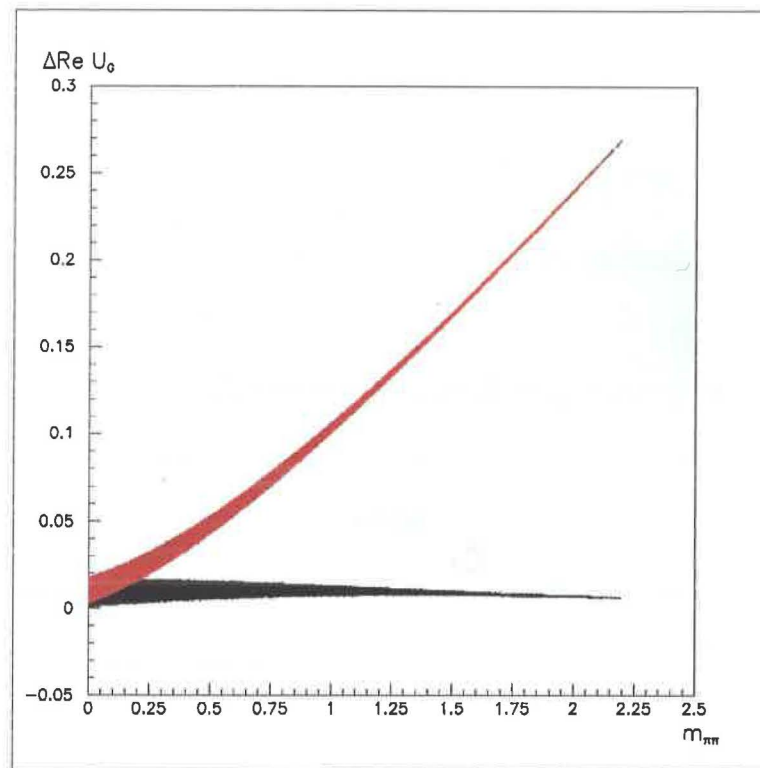


Figure E.6: the difference $\Delta \text{Re} U_G$ to the exact correction for the suggested approximation (black) and the linear expansion in q^2 (light gray) as a function of q^2

Appendix F

The Neutrino Ambiguity

F.1 The 2 Solutions

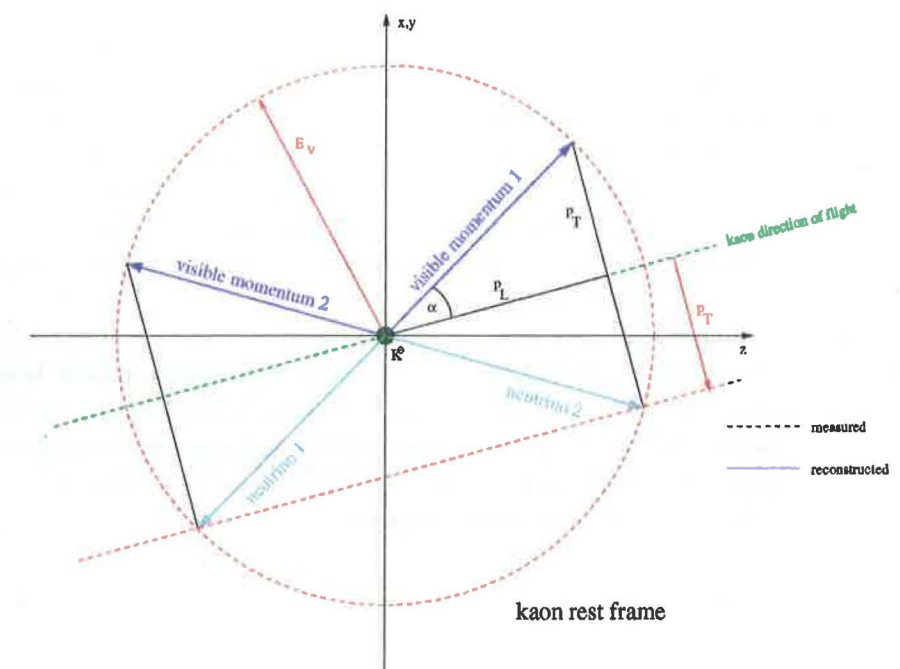


Figure F.1: the two physical solutions for the neutrino in a K^0_4 decay

Since the neutrino cannot be seen in the detector, its momentum has to be reconstructed according to the conservation rules. Three unknown momentum components and one unknown kaon energy¹ stand against four equations (4-momentum conservation), so one expects to find a solution.

¹the direction of the kaon can within errors assumed to be exactly in beam direction = z-axis

In the kaon rest frame, there is

$$\begin{aligned} E_\nu + E_{vis} &= m \\ \vec{p}_\nu + \vec{p}_{vis} &= 0 \end{aligned}$$

where p_{vis} is the total 4-momentum of the three observable particles (π^+ , π^0 , e). Since the neutrino is (at least to a very good approximation) massless, there is $E_\nu = |\vec{p}_\nu| = |\vec{p}_{vis}|$. Inserting this in energy conservation instantly gives

$$E_\nu = \frac{m^2 - m_{vis}^2}{2m} \Rightarrow E_{vis} = \frac{m^2 + m_{vis}^2}{2m} \quad (\text{F.1})$$

On the right side are only measurable quantities, therefore E_ν is also fixed. However, the orientation of the visible momentum $\vec{p}_{vis}$ with respect to the z-axis (= kaon direction of flight in the detector frame of reference), although measured in the detector's frame of reference, is not completely known in the kaon rest frame, simply because the kaon energy is not measured² - all known is the component transversal to the kaon direction of flight (p_T), since this component is invariant under any boosts in this direction.

Fig. F.1 shows the situation in the kaon rest frame³; E_ν restricts the neutrino 3-momentum to a circle, the line of a given p_T intersects this circle at two points. Both points correspond to a physical solution for the neutrino, but differ by the resulting kaon energy⁴. The solutions differ by the orientation of the neutrino with respect to the kaon direction of flight - there is a forward and a backward solution.

If the kaon energy was measured, or was known to be in a narrow range, a decision could be taken. In NA48, though, there is a wide range of kaon energies ($\approx 50 - 150\text{GeV}$), and all that can be done is a weighting of the solutions corresponding to the known shape of the kaon energy distribution. To get an estimation for the influence of wrong solutions, their dependence on the right solutions is studied in this appendix.

F.2 The Shift S_μ

For this purpose, we define the 4-momentum shift S_μ as the difference between the two solutions:

²in NA48, the kaon energy has a range of $\approx 50 - 150\text{GeV}$

³the coordinate system is chosen such that the z-axis points to the beam direction in the detector's frame of reference, while the second axis point in the direction of the visible transversal momentum; it is indicated that generally the direction of a single kaon need not be exactly in the beam direction, but is exaggerated in scale (this difference will be neglected)

⁴since p_L is measured in the detector frame of reference, and has two possible values in the kaon rest frame, there are also two different boosts that connect these values, resulting in two different kaon energies

$$S_\mu := p_\mu^{\nu_2} - p_\mu^{\nu_1} \quad (\text{F.2})$$

It is important to realize that S is not simply given by $2p_L$ as fig. F.1 might suggest, as the two solutions also correspond to two different frames of reference. The correct way to determine S is the following: first, one determines the boost that transforms the longitudinal component of the visible particle's momentum p_L to $-p_L$; this is the frame where the second solution for the neutrino momentum is chosen. This momentum is then transformed back to the original frame of reference, where it can now be compared to the first solution.

The equation to find the right boost is

$$\begin{aligned} p'_L &= -p_L \\ \beta\gamma E_{vis} + \gamma p_L &= -p_L \\ \beta\gamma \frac{b}{2m} + \gamma \frac{a}{2m} \cos \alpha &= -\frac{a}{2m} \cos \alpha \end{aligned}$$

where α is the angle between the visible momentum and the kaon direction of flight (as shown in fig. F.1) and with the definition of

$$a := m^2 - m_{vis}^2 \quad b := m^2 + m_{vis}^2 \quad (\text{F.3})$$

Solving for γ and $\beta\gamma$ gives

$$\gamma = \frac{b^2 + a^2 \cos^2 \alpha}{b^2 - a^2 \cos^2 \alpha} \quad \beta\gamma = -\frac{2ab \cos \alpha}{b^2 - a^2 \cos^2 \alpha} \quad (\text{F.4})$$

The energy for the second neutrino solution is given by F.1 in above frame where $p'_L = -p_L$, so consequently the energy in the original frame is given by⁵

$$E_{\nu_2} = \gamma E_\nu - \beta\gamma E_\nu \cos \alpha = E_\nu + \frac{2ma^2 \cos^2 \alpha}{b^2 - a^2 \cos^2 \alpha} \quad (\text{F.5})$$

Analogously, the longitudinal component of the second solution is given by

$$p_{L,\nu_2} = -\beta\gamma E_\nu + \gamma E_\nu \cos \alpha = -p_{L,\nu} + \frac{a \cos^2 \alpha + b}{b^2 - a^2 \cos^2 \alpha} \cdot \frac{a^2}{m} \cos \alpha \quad (\text{F.6})$$

Since the transversal component is the same, S results in

⁵mind the minus before $\beta\gamma$ due to the back transformation

$$\begin{aligned} S_0 &= \frac{2ma^2 \cos^2 \alpha}{b^2 - a^2 \cos^2 \alpha} \\ S_L &= \frac{2mab \cos \alpha}{b^2 - a^2 \cos^2 \alpha} \\ S_T &= 0 \end{aligned}$$

for $S^2 = S_\mu S^\mu$ one has

$$S^2 = -2mS_0 \quad (\text{F.7})$$

It is now important to realize that the orientation of $\vec{p}_{vis}$ with respect to the beam direction is completely unrelated to the physics of the Ke4 decay; it follows that $\cos \alpha$ is flatly distributed, and one may calculate an *effective* S^{eff} as the average:

$$S^{\text{eff}} := \frac{1}{2} \int_{-1}^1 d(\cos \alpha) S(\cos \alpha) \quad (\text{F.8})$$

It is also important to remark that by averaging over all angles $0 - \pi$ we already consider both solutions (with equal weight), since $\cos \alpha_{\nu_2} = -\cos \alpha_{\nu_1}$.

To average the vector part, one has to fix a reference coordinate system; it is convenient to choose a system where the z-Axis is in parallel with $\vec{p}_{vis}$, then the vector $\vec{S}$ is given by

$$\vec{S} = S_L \begin{pmatrix} \sin \alpha \cos \chi \\ \sin \alpha \sin \chi \\ \cos \alpha \end{pmatrix} = \frac{mab}{b^2 - a^2 \cos^2 \alpha} \begin{pmatrix} \sin 2\alpha \cos \chi \\ \sin 2\alpha \sin \chi \\ 2 \cos^2 \alpha \end{pmatrix} \quad (\text{F.9})$$

where χ is flatly distributed in $[-\pi, \pi]$. With the integral

$$\frac{1}{2} \int_{-1}^1 dx \frac{x^2}{c^2 - x^2} = \frac{c}{2} \ln \frac{c+1}{c-1} - 1 \quad (\text{F.10})$$

one now finds for the averages (we henceforth omit the index "eff")

$$\begin{aligned} S_0 &= -\frac{2m^2b}{a^2} \Delta p_\nu^0 + \frac{2m^3}{a} \Delta \\ \vec{S} &= -\frac{2m^2b}{a^2} \Delta \vec{p}_\nu \\ S^2 &= -2m^2 \Delta \\ \Delta &:= \frac{v+1}{v-1} \ln v - 2 \quad v := \frac{m^2}{m_{vis}^2}, \quad \Delta \approx 0 - 0.2 \end{aligned}$$

with $p_\nu = P_{e\nu} - Q_{e\nu}$. As a consequence, the (average) product with a 4-vector P_μ independent of α is given by

$$P_\mu S^\mu = \frac{m^2 \Delta}{a^2} [2amP_0 + bPQ_{e\nu} - bPP_{e\nu}] \quad (\text{F.11})$$

where P_0 has to be evaluated in the kaon rest frame.

Note that Δ itself is invariant under a shift S (simply because m_{vis} does not depend on the neutrino solution).

F.3 Cabibbo-Relations between the Two Solutions

We can now use the shift S to calculate the relations between the two sets of Cabibbo variables defined by the two solutions. Since S depends on m_{vis} which is not a Cabibbo-variable, we first have to express this by Cabibbo variables:

Obviously, one has

$$p_{vis} = P_{\pi\pi} + \frac{1}{2} [P_{e\nu} + Q_{e\nu}] \quad (\text{F.12})$$

which gives using (2.5)

$$m_{vis}^2 = m_{\pi\pi}^2 + \bar{p} + \sigma \cos \theta_{e\nu} \quad (\text{F.13})$$

One can now use the same (2.5) again to find the relations for the Cabibbo variables if one replaces p_ν by $p_\nu + S$, that is

$$P_{e\nu} \rightarrow P_{e\nu} + S \quad Q_{e\nu} \rightarrow Q_{e\nu} - S \quad (\text{F.14})$$

For the Cabibbo variables one now finds (using results from appendix B, page 163 for P^0, Q^0)

$$\begin{aligned} m_{\pi\pi}^2 &\rightarrow m_{\pi\pi}^2 \\ m_{e\nu}^2 &\rightarrow m_{e\nu}^2 \cdot [1 + \Delta \cdot f_{m_{e\nu}}(m_{\pi\pi}, m_{e\nu}, \cos \theta_{e\nu})] \\ \cos \theta_{\pi\pi} &\rightarrow \cos \theta_{\pi\pi} \cdot [1 + \Delta \cdot f_{\cos \theta_{e\nu}}(m_{\pi\pi}, m_{e\nu}, \cos \theta_{\pi\pi}, \cos \theta_{e\nu}, \phi)] \\ \cos \theta_{e\nu} &\rightarrow \cos \theta_{e\nu} \cdot [1 + \Delta \cdot f_{\cos \theta_{e\nu}}(m_{\pi\pi}, m_{e\nu}, \cos \theta_{\pi\pi}, \cos \theta_{e\nu}, \phi)] \\ \phi &\rightarrow \phi \cdot [1 + \Delta \cdot f_\phi(m_{\pi\pi}, m_{e\nu}, \cos \theta_{\pi\pi}, \cos \theta_{e\nu}, \phi)] \end{aligned}$$

As the functions f_x are rather complicated, they are not given here; important is only the general structure of the dependence between the two neutrino solutions: the mean value of the second solution is always shifted proportional to Δ with respect to the first one (see also MC results below).

There is one Cabibbo variable that does not depend on the neutrino solution: $m_{\pi\pi}$. This is of course obvious, but nevertheless important for the analysis, since distributions in this variable can be studied without worrying for the correct solution.

F.4 A Method to get the Mean Correct Solution from a Shifted Mean of Both Solutions

Since the value for the second solution is always proportional to the original value, in principle it would be possible to calculate a shifted mean of both solutions that results in the correct value⁶:

Generalizing the results from above, one has for one Cabibbo variable C_i :

$$C_{i,2} = C_{i,1} \cdot [1 + f_i(C_{j,1})] \quad (\text{F.15})$$

We now assume that $C_{i,1}$ is the correct value, while $C_{i,2}$ is the wrong one. If one shifts each solution for a Cabibbo variable⁷

$$\begin{aligned} C_{i,1}^S &= C_{i,1} + g_i(C_{j,1}) \\ C_{i,2}^S &= C_{i,2} + g_i(C_{j,2}) \end{aligned} \quad (\text{F.16})$$

and defines the shifted mean as

$$\bar{C}_i^S := \frac{1}{2} [C_{i,1}^S + C_{i,2}^S] \quad (\text{F.17})$$

one has

$$\bar{C}_i^S = C_{i,1} + \frac{1}{2} [g_i(C_{j,1}) + C_{i,1}f_i(C_{j,1}) + g_i(C_{i,1}(1 + f_i(C_{j,1})))] \quad (\text{F.18})$$

if one now finds a $g_i(C_{j,1})$ such that

$$g_i(C_{j,1}) + g_i(C_{i,1}(1 + f_i(C_{j,1}))) = -C_{i,1}f_i(C_{j,1}) \quad (\text{F.19})$$

the shifted mean would give the correct result:

$$\bar{C}_i^S = C_{i,1} \quad (\text{F.20})$$

Of course, (F.19) is a functional equation that is hard to solve in general, but the point is that it is possible to correct for the effect of the wrong solution.

This might seem like a paradox, since the same could have been derived assuming that the first solution is the wrong one. But it is important to

⁶please note that this is only true for the *mean* value; of course, it is impossible in principle to determine the correct solution for a single event

⁷note that the arguments of g_i are the Cabibbo variables of the specific solution, so this shift can be done without knowing the correct solution

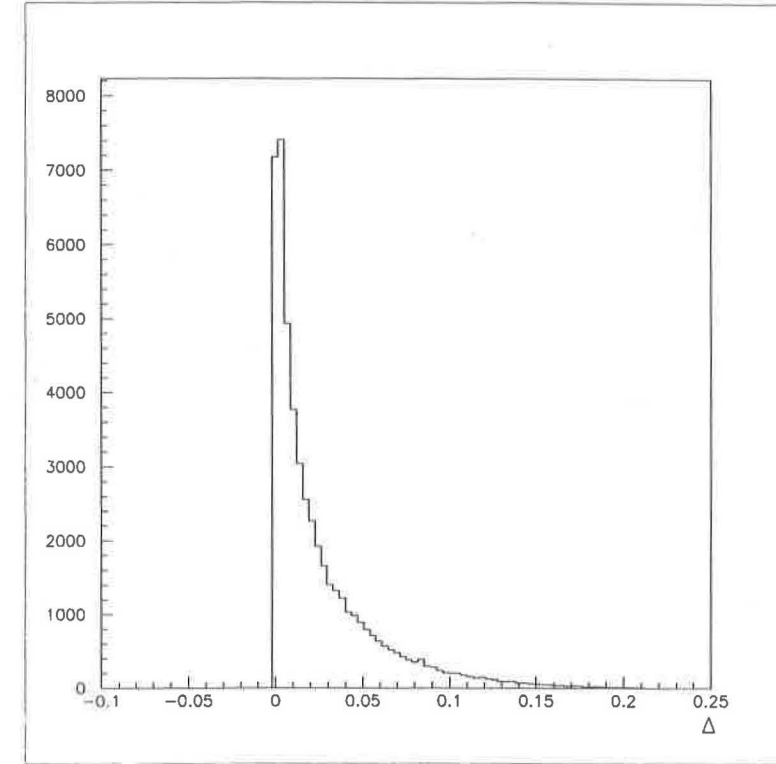


Figure F.2: the distribution of Δ for simulated Ke4 decays

remember that the mean is taken for constant solution 1, that is, always for the same true solution, while the wrong solutions vary. This makes sense, because one expects the same distribution in the Cabibbo variables for each Ke4 decay; at the same time, the distribution of the wrong variables depends on the detector frame of reference, and may be therefore different for each Ke4 decay.

F.5 MC Results

To illustrate the results from above, 50 Million of Ke4 decays with a flat distribution in the Cabibbo variables⁸ have been generated.

Fig. F.2 shows the distribution of Δ .

Fig. F.3 shows the means of the wrong solutions against the right solutions.

Fig. F.4 shows the distribution of the wrong solution for a gaussian distributed right solution. Obviously, the wrong solution also peaks at the

⁸as the invariant masses are correlated, flat in this case is meant with respect to the two dimensional distribution of both variables, which results in a sawtooth distribution for the 1d distributions

correct value, but has wide tails.

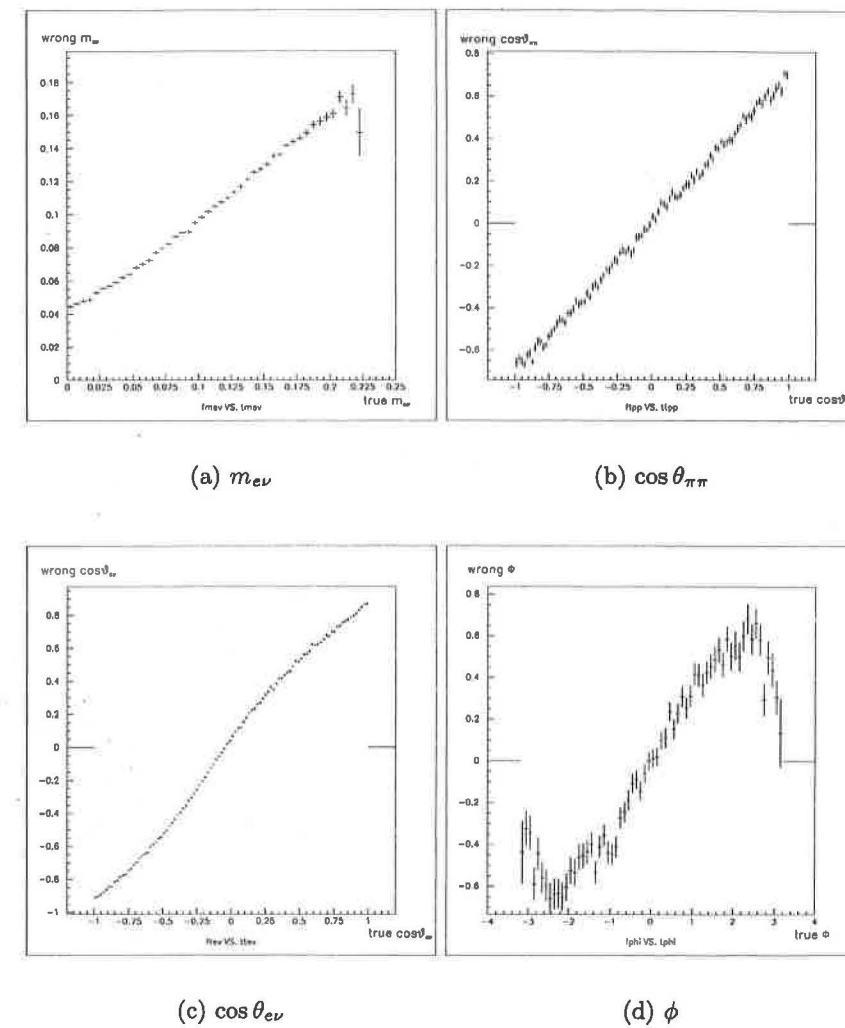


Figure F.3: the mean wrong solution as a function of the right solution for different Cabibbo variables

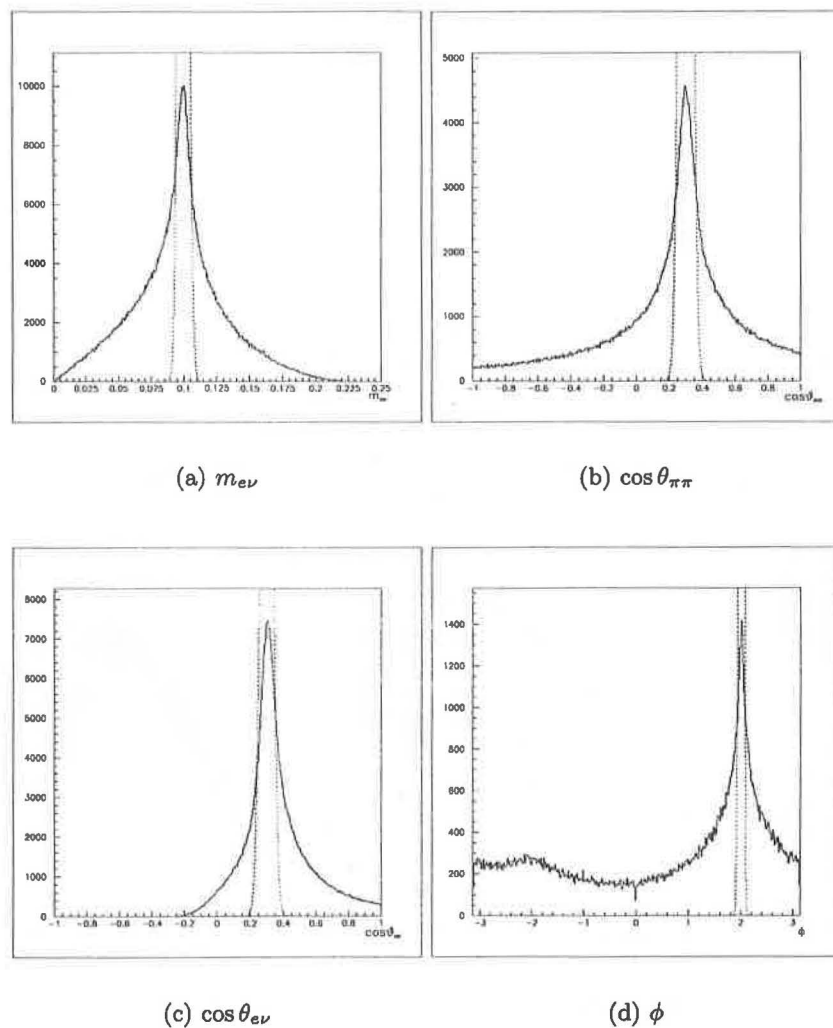


Figure F.4: the distributions of the wrong solution (solid) with a gaussian distributed true solution (dashed)

If one now uses both solutions with a weight of 50 percent each⁹, the peak for the right value dominates and the wrong values give a rather small background (see fig. F.5).

Finally, the differences between taking both solutions with equal weight and the true solution is shown for typical distributions in the Cabibbo variables (fig. F.6)

⁹this is the most "ignorant" weighting; due to acceptance differences (see MC study), and making use of the experimentally known kaon spectrum, it is possible to apply a more realistic weighting

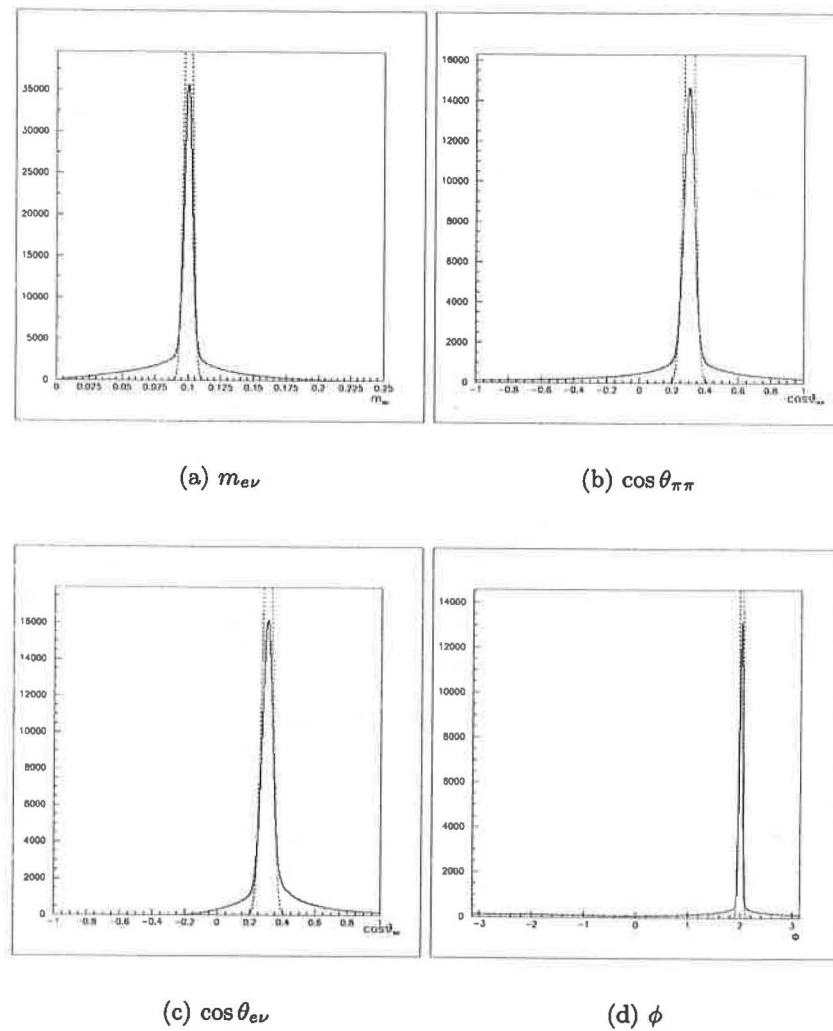


Figure F.5: the distributions of both solutions, equally weighted (solid) with a gaussian distributed true solution (dashed)

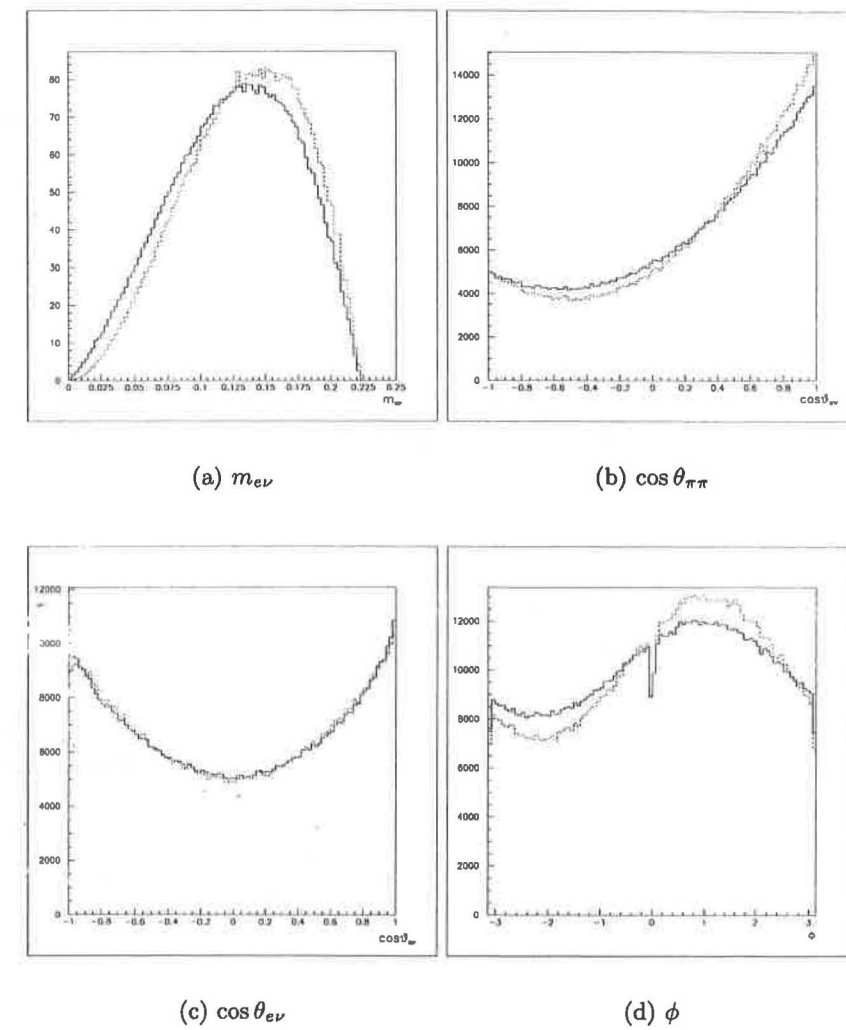


Figure F.6: both solutions with equal weight (solid) in comparison with the right solution (dashed) for some typical distributions in the Cabibbo variables

F.6 Conclusion

The main conclusion is that $m_{\pi\pi}$ is the most reliable Cabibbo variable, because it does not depend on the neutrino solution. This is the reason why mainly the distribution with respect to this variable is used to study the $Ke4$ decay. For the other variables, the best way is to use both solutions with a weight that sums up to 1; this weight can e.g. be chosen to reflect the probability for the corresponding kaon energy with respect to the (experimentally known) kaon spectrum.

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